Thai **J**ournal of **M**athematics : (2020) 35–44
Special Issue: The 14^{th} IMT-GT ICMSA 2018
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

The Number of Squares Reachable in k Moves with $(1, b)$-Knight's Move where $b \in \{3, 5, 7\}$

Sirirat Singhun[†,1] **, Ratree Theprod**[†] **and Ratinan Boonklurb**[‡]

[†]Department of Mathematics, Faculty of Science,
Ramkhamhaeng University, Bangkok 10240, Thailand
e-mail : sin_sirirat@ru.ac.th and ratree150@gmail.com
[‡] Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University,
Bangkok 10330, Thailand
e-mail : ratinan.b@chula.ac.th

Abstract : The $m \times n$ *chessboard* is an array with squares arranged in m rows and n columns. If $m, n \to \infty$, then it is called an *infinite chessboard*. An (a, b)*-knight move* is the move with a squares vertically or a squares horizontally and then b squares move at 90 degrees angle. If $a = 1$ and $b = 2$, then it is a legal knight's move. There are many way to discuss on the chessboard. Such as finding the way to land on each square exactly once and then return to the starting square (shown by many researchers) or counting the number of squares that the knight can move to in k moves. In this paper, we focus on the latter. The formula of number of squares reachable by a knight with $(1, b)$-knight's move in k moves where $b \in \{3, 5, 7\}$ are obtained. Moreover, the cumulative number of squares are also obtained.

Keywords : knight's move; $m \times n$ chessboard; infinite chessboard.
2010 Mathematics Subject Classification : 05C99.

1 Introduction and Preliminaries

The $m \times n$ chessboard is an array with m rows and n columns such that each square is colored by black and white alternatively. The standard chessboard is the 8×8 chessboard. The *legal knight's move* is a move one square vertically or one square horizontally and then two squares move at 90 degrees angle. Several problems concerning knight's move on the $m \times n$ chessboard are posted. One, postd by Schwenk [1] in 1991, is the characterization of the $m \times n$ chessboard that the knight can move to each square and return to the starting square. Such move is called a *closed knight's tour*.

Theorem 1.1. [1] *The $m \times n$ chessboard with $m \le n$ admits a closed knight's tour unless one or more of the following conditions hold:*
(i) m and n are both odd; or
(ii) $m = 1$ or 2 or 4; or
(iii) $m = 3$ and $n = 4$ or 6 or 8.

[1]Corresponding author.

In 2003, Chia et al. [2] generalized the knight's move to a squares vertically or a squares horizontally and then b squares move at 90 degrees angle. Such move is called an (a, b)- *knight's move.* Then, the legal knight's move is an $(1, 2)$-knight's move. The problem of an (a, b)-knight's move are considered in similar way. That is, finding a closed (a, b)-knight's tour on the $m \times n$ chessboard. Another problem was posed by Miller et al. [3] in 2013. They found formula for the number of squares reachable by a knight's with an $(1, 2)$-knight's move on an infinite chessboard in a minimum number of k moves. Note that the infinite chessboard is an $m \times n$ chessboard where $m, n \to \infty$. Moreover, the cumulative number of squares that the knight can reach in k moves is obtained.

Theorem 1.2. [3] *The number of squares that required exactly k moves in order to be reached by a sole knight from its initial position on an infinite chessboard are* $1, 8, 32, 68$ *and* 96 *for* $k = 0, 1, 2, 3$ *and* 4, *respectively, and* $28k - 20$ *for* $k \geq 5$.

Corollary 1.3. [3] *The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position on an infinite chessboard are* $1, 9, 41$ *and* 109 *for* $k = 0, 1, 2$ *and* 3, *respectively, and* $14k^2 - 6k + 5$ *for* $k \geq 4$.

In 2017, the authors in [4] obtained formula for the number of squares reachable by a knight with an $(1, 4)$-knight's move on an infinite chessboard in a minimum number of k moves. Also, the cumulative number of squares that the knight can reach in k or fewer moves is obtained.

Theorem 1.4. [4] *Assume that the knight moves with the* $(1, 4)$-*knight's move. Then, the number of squares that require exactly k moves in order to be reached by a sole knight from its initial position* $(0, 0)$ *on an infinite chessboard are* $1, 8, 32, 88, 204, 324, 448, 548$ *and* 620 *for* $k = 0, 1, 2, 3, 4, 5, 6, 7$ *and* 8, *respectively, and* $92k - 132$ *for* $k \geq 9$.

Corollary 1.5. [4] *Assume that the knight moves with the* $(1, 4)$-*knight's move. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position* $(0, 0)$ *on an infinite chessboard are* $1, 9, 41, 129, 333, 657, 105, 653$ *and* 2273 *for* $k = 0, 1, 2, 3, 4, 5, 6, 7$ *and* 8, *respectively, and* $46k^2 - 86k + 17$ *for* $k \geq 9$.

In this paper, we consider an $(1, b)$-knight's move on an infinite chessboard where $b \in \{3, 5, 7\}$. We obtain formula for the number of squares reachable by a knight with an $(1, b)$-knight's move and then the cumulative number of squares that the knight can reach in k or fewer moves is obtained.

2 Main Results

For an infinite chessboard, let (i, j) denote the square of the chessboard where i is the row coordinate and j is the column coordinate of the square which is plotted in an $x - y$ coordinated system. If the knight lands on (i, j), then for an $(1, b)-$knight's move, the knight can move to eight squares, $(i \pm 1, j \pm b)$ and $(i \pm b, j \pm 1)$. Assume that the knight starts at the square $(0, 0)$. Then, eight squares are $(\pm 1, \pm b)$ and $(\pm b, \pm 1)$. We see that these eight positions are symmetric with the row 0 and the column 0. Note that, we obtain eight squares with in one move.

We separate the infinite chessboard into four parts. Part 1 is from the row 1 upward and the column 1 to the right, part 2 is from the row 1 upward and the column -1 to the left, part 3 is from the row -1 downward and the column -1 to the left and part 4 is from the row -1 downward and the column 1 to the right.

To obtain the minimum number of squares with two moves, the knight must move from those eight squares $(\pm 1, \pm b)$ and $(\pm b, \pm 1)$. Also, the minimum number of squares reachable in three moves is the number of squares that move from the squares reachable in two moves. Then, the minimum number of squares reachable in k moves is the number of squares that move from the squares reachable in $k - 1$ moves. Since the infinite chessboard is symmetric with the row 0 and the column 0, it suffices to only count squares in the row 0, the column 0 and in part 1. For convenience to count squares, we label each square with the number k when the knight can reach to that square with in k moves and we color the starting square $(0, 0)$ with black.

2.1 $(1, 3)$-Knight's Move

In this section, we consider $(1, 3)$-knight's move on the infinite chessboard. We first count the number k's where $k \in \{0, 1, 2, 3, 4\}$ around the square $(0, 0)$ in Lemma 2.1. Next, the number k's where $k > 4$ are counted by separating whether k is even or odd in Lemma 2.2. Then, we conclude all k's in Theorem 2.3. Finally, the cumulative number of squares reachable in k or fewer moves is shown in Corollary 2.4.

Lemma 2.1. *The number of squares that require exactly k $(1, 3)$-knight's moves in order to be reached by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $1, 8, 32, 68$ and 80 for $k = 0, 1, 2, 3$ and 4, respectively.*

Proof. From Figure 1, we obtain each number by direct counting. $\square$

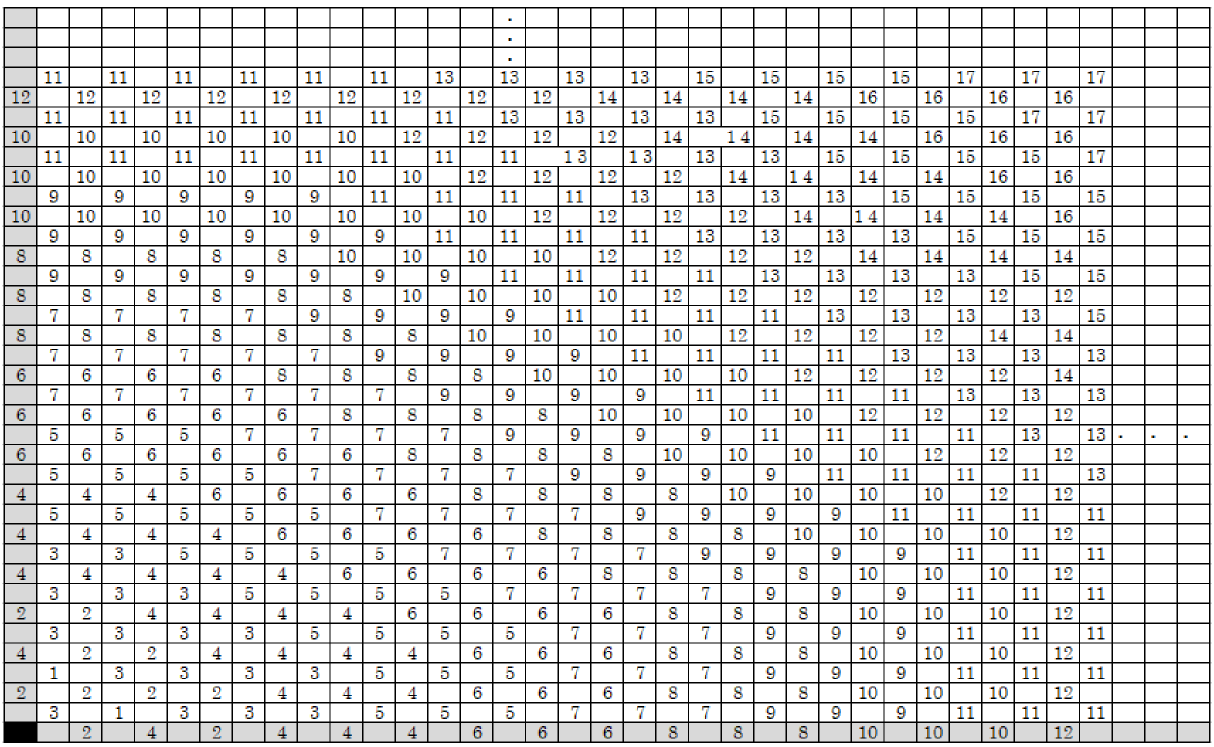

Figure 1: The number of squares with k moves where $k = 0, 1, 2, 3, 4$ in part 1

Lemma 2.2. *The number of squares that require exactly k $(1, 3)$-knight's moves in order to be reached by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $56t + 8$ and $56t - 20$ for $k = 2t + 1$ where $t \in \{2, 3, 4, \dots\}$ and $k = 2t$ where $t \in \{3, 4, 5 \dots\}$, respectively.*

Proof. Since the board is symmetric with the row 0 and the column 0, it suffices to count on the part 1 together with squares on row 0 and then multiply by 4. From Figure 1, for $k = 5, 7, 9$ and 11, the number of squares that the knight moves by $(1, 3)$-knight's move from the square $(0, 0)$ with k moves are $120, 176, 232$ and 288, respectively. For $k = 6, 8, 10$ and 12, the numbers of squares that the knight moves from the square $(0, 0)$ with k moves are $148, 204, 260$ and 316, respectively.

Case 1 : $k = 2t + 1$ where $t \in \{2, 3, 4, \dots\}$. In part 1, the 5's appear in 8 columns. We count 5's in each column. Then, we obtain $(3 + 3) + (4 + 4 + 4) + 5 + 4 + 3 = 30$ squares. The 7's appear in 11 columns,we obtain $(3+3+3)+(4+4+4+4+4+4)+6+5+4 = 44$ squares. The 9's appears in 14 columns, we obtain $(3+3+3+3)+(4+4+4+4+4+4+4+4)+7+6+5 = 58$ squares. With the same algorithm, the

k's appear in $3t+2$ columns. We obtain $\underbrace{3+3+3+\cdots+3}_{t \text{ terms}}+\underbrace{4+4+4+\cdots+4}_{2t-1 \text{ terms}}+(t+3)+(t+2)+(t+1)$ and the arithmetic sequence of squares is $30, 44, 58, \ldots$. Then, the general term is of the form $14t+2$ for $k = 2t+1$ where $t \in \{2, 3, 4, \ldots\}$.

Since the k's where k is odd do not appear in the row 0 and column 0, the number of squares that require exactly k $(1, 3)$-knight's moves is $4(14t + 2) = 56t + 8$ for $k = 2t + 1$ where $t \in \{2, 3, 4, \ldots\}$.

Case 2 : $k = 2t$ where $t \in \{3, 4, 5, \ldots\}$. In part 1, the 6's appears in 9 columns. Counting 6's in each column, we obtained $(3+3)+(4+4+4+4)+5+4+3 = 34$ squares. The 8's appears in 12 columns, we obtained $(3+3+3)+(4+4+4+4+4+4)+6+5+4 = 48$ squares. The 10's appears in 15 columns, we obtained $(3+3+3+3)+(4+4+4+4+4+4+4+4)+7+6+5 = 62$ squares. With the same algorithm, the k's appear in $3t$ columns. We obtain $\underbrace{3+3+3+\cdots+3}_{t-1 \text{ terms}}+\underbrace{4+4+4+\cdots+4}_{2t-2 \text{ terms}}+(t+2)+(t+1)+t$ and the arithmetic sequence of squares is $34, 48, 62, \ldots$. Then, the general term is in the form $14t - 8$ for $k = 2t$ where $t \in \{3, 4, 5 \ldots\}$.

In row 0, we obtain the k's where $k (\geq 6)$ is even in 3 columns and it is similar for the column 0. Then, for $k = 2t$ where $t \in \{3, 4, 5 \ldots\}$, the number of squares that require exactly k $(1, 3)$-knight's moves is $4(14t - 8) + 4(3) = 56t - 20$ for $k = 2t$ where $t \in \{3, 4, 5 \ldots\}$. □

Lemmas 2.1 and 2.2 are concluded in the following theorem.

Theorem 2.3. *Assume that the knight moves with the $(1, 3)$-knight's move. Then, the number of squares that require exactly k moves in order to be reached by a sole knight from its initial position $(0, 0)$ on the infinite chessboard are*

(i) $1, 8, 32, 68$ and 80 for $k = 0, 1, 2, 3$ and 4, respectively,

(ii) $56t + 8$ for $k = 2t + 1$ where $t \in \{2, 3, 4, \ldots\}$, and

(iii) $56t - 20$ for $k = 2t$ where $t \in \{3, 4, 5, \ldots\}$.

Corollary 2.4. *Assume that the knight moves with the $(1, 3)$-knight's move. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $1, 9, 41, 109, 189, 309$ and 457 for $k = 0, 1, 2, 3, 4, 5$ and 6, and $56t^2 + 44t - 3$ for $k = 2t + 1$ where $t \in \{3, 4, 5, \ldots\}$ and $56t^2 - 12t - 11$ for $k = 2t$ where $t \in \{4, 5, 6, \ldots\}$.*

Proof. For $k = 0, 1, 2, 3, 4, 5$ and 6, it easy to see that the cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $1, 9, 41, 109, 189, 309$ and 457, respectively. We assume that $k > 6$.

Case (i) : $k = 2t + 1$ where $t \in \{3, 4, 5, \ldots\}$. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0, 0)$ on an infinite chessboard is of the form $457 + \sum_{i=3}^{t}(56i + 8) + \sum_{i=4}^{t}(56i - 20) = 56t^2 + 44t - 3$.

Case (ii) : $k = 2t$ where $t \in \{4, 5, 6, \ldots\}$. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0, 0)$ on an infinite chessboard is of the form $457 + \sum_{i=3}^{t-1}(56i + 8) + \sum_{i=4}^{t}(56i - 20) = 56t^2 - 12t - 11$. □

2.2 $(1, 5)$-Knight's Move

In this section, we consider $(1, 5)-$ knight's move on the infinite chessboard. We first count the number k's where $k \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ around the square $(0, 0)$ in Lemma 2.5. Next, the number k's where $k > 7$ are counted by separating whether k is even or odd in Lemma 2.6. Then, we conclude all k's in Theorem 2.7. Finally, the cumulative number of squares reachable in k or fewer moves is shown in Corollary 2.8.

Lemma 2.5. *The number of squares that require exactly k $(1, 5)$-knight's moves in order to be reached by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $1, 8, 32, 88, 192, 304, 360$ and 416 for $k = 0, 1, 2, 3, 4, 5, 6$ and 7, respectively.*

Proof. From Figures 2 and 3, we obtain each number by direct counting. □

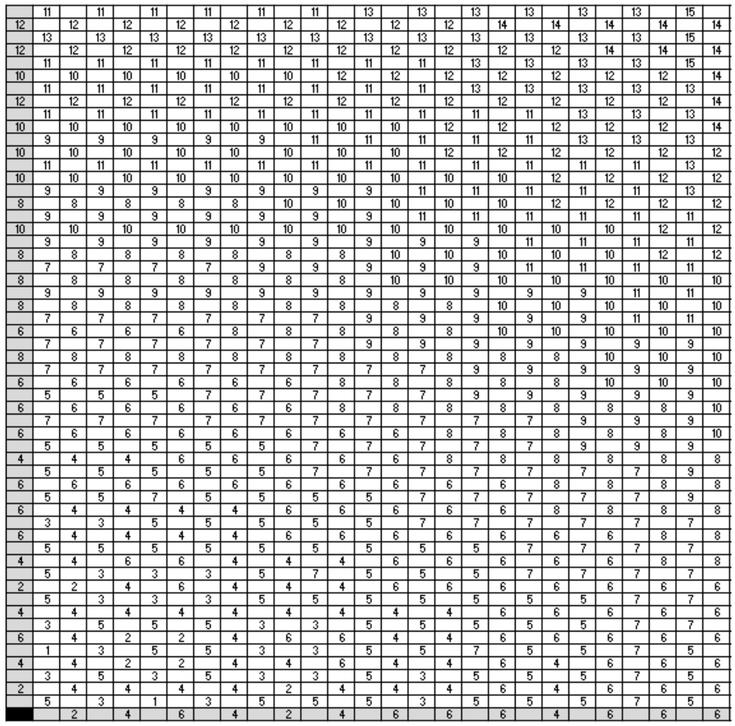

Figure 2: The number k's with $(1, 5)$-knight's move from column 0 to column 26

Lemma 2.6. *The number of squares that require exactly k $(1, 5)$-knight's moves in order to be reached by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $136t - 72$ and $136t - 4$ for $k = 2t$ where $t \in \{4, 5, 6, \dots\}$ and $k = 2t + 1$ where $t \in \{4, 5, 6, \dots\}$, respectively.*

Proof. Since the board is symmetric with the row 0 and the column 0, it suffices to count on part 1 together with squares on row 0 and then multiply by 4. From Figure 2, we count numbers $472, 608, 744$ and 880 for $k = 8, 10, 12$ and 14, respectively. The numbers of squares that the knight moves from the square $(0, 0)$ with k moves are $536, 672, 808$ and 944 for $k = 9, 11, 13$ and 15, respectively.

Case 1 : $k = 2t$ where $t \in \{4, 5, 6, \dots\}$. We shall find the formula of the k's in part 1 as follows. The 8's appear in 19 columns. We count 8's in each column. Then, we obtain $(5 + 5 + 5) + 6 + 5 + [(7 + 5) + (7 + 5) + (7 + 5) + (7 + 5)] + 10 + 5 + 8 + 6 + 6 + 4 = 113$ squares. The 10's appears in 24 columns, we obtain $(5 + 5 + 5 + 5) + 6 + 5 + [(7 + 5) + (7 + 5) + (7 + 5) + (7 + 5) + (7 + 5) + (7 + 5)] + 11 + 5 + 9 + 7 + 7 + 5 = 147$ squares. With the same algorithm, the k's appear in $5t - 1$ columns. We obtain $\underbrace{5 + 5 + 5 + \cdots + 5}_{t-1 \text{ terms}} + 6 + 5 + \underbrace{(7 + 5) + (7 + 5) + (7 + 5) + \cdots + (7 + 5)}_{2t-4 \text{ terms}} + (t + 6) + 5 + (t + 4) + 2(t + 2) + t$ and the arithmetic sequence of squares is $472, 608, 744, 880, \dots$. Then, the general term is of the form $34t - 23$ for $k = 2t$ where $t \in \{4, 5, 6 \dots\}$.

In row 0, we obtain the k's in 5 columns and it is similar for the column 0.

Then, the number of squares that require exactly k $(1, 5)$-knight's moves is $4(34t-23)+4(5) = 136t-72$ for $k = 2t$ where $t \in \{4, 5, 6 \dots\}$.

Case 2 : $k = 2t+1$ where $t \in \{4, 5, 6, \dots\}$ The 9's appears in 22 columns. Counting 9's in each column, we obtain $(5+5+5+5)+6+5+[(7+5)+(7+5)+(7+5)+(7+5)+(7+5)]+11+5+9+7+7+5 = 135$ squares. The 11's appears in 27 columns. We obtain $(5+5+5+5+5)+6+5+[(7+5)+(7+5)+(7+5)+$

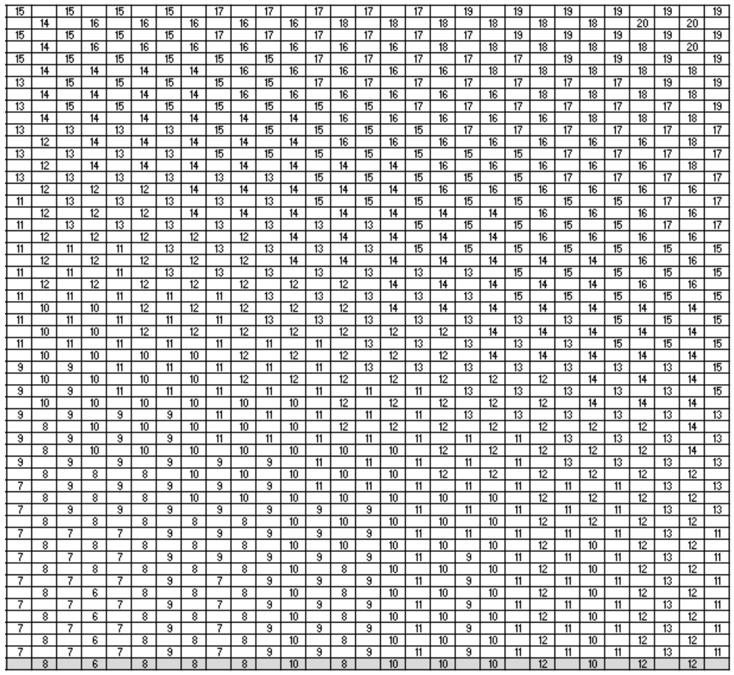

Figure 3: The number k's with $(1,5)$-knight's move from column 27 to column 55

$(7+5)+(7+5)+(7+5)+(7+5)]+12+5+10+8+8+6 = 169$ squares. With the same algorithm, the k's appear in $5t+2$ columns. We obtain $\underbrace{5+5+5+\cdots+5}_{t \text{ terms}}+6+5+\underbrace{(7+5)+(7+5)+\cdots+(7+5)}_{2t-3 \text{ terms}}+(t+7)+5+(t+5)+2(t+3)+(t+1)$ and the arithmetic sequence of squares is $540, 676, 812, \ldots$. Then, the general is of the form $34t-1$ for $k = 2t+1$ where $t \in \{4,5,6\ldots\}$.

Thus, the number of squares that require exactly k $(1,5)$-knight's moves is $4(34t-1) = 136t-4$ for $k = 2t+1$ where $t \in \{4,5,6\ldots\}$. $\square$

Lemmas 2.5 and 2.6 are concluded in the following theorem.

Theorem 2.7. *Assume that the knight moves with the $(1,5)$-knight's move. Then, the number of squares that require exactly k moves in order to be reached by a sole knight from its initial position $(0,0)$ on the infinite chessboard are*

(i) $1, 8, 32, 88, 192, 304, 360$ *and* 416 *for* $k = 0,1,2,3,4,5,6$ *and* 7*, respectively,*

(ii) $136t-72$ *for* $k = 2t$ *where* $t \in \{4,5,6,\ldots\}$*, and*

(iii) $136t-4$ *for* $k = 2t+1$ *where* $t \in \{4,5,6,\ldots\}$*.*

Corollary 2.8. *Assume that the knight moves with the $(1,5)-$knight's move. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard are $1, 9, 41, 129, 321, 625, 985, 1401$, for $k = 0,1,2,3,4,5,6,7$ and $136t^2-76t+1$ for $k = 2t$ where $t \in \{4,5,6,\ldots\}$ and $136t^2+60t-3$ for $k = 2t+1$ where $t \in \{4,5,6,\ldots\}$.*

Proof. For $k = 0,1,2,3,4,5,6$ and 7, it is easy to see that the cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard are $1, 9, 41, 129, 321, 625, 985$ and 1401, respectively. We assume that $k > 7$.

Case (i) : $k = 2t+1$ where $t \in \{4,5,6\ldots\}$. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard is of the form $1401 + \sum_{i=4}^{t}(136i-4) + \sum_{i=4}^{t}(136i-72) = 136t^2+60t-3$.

Case (ii) : $k = 2t$ where $t \in \{4,5,6,\dots\}$. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard is of the form $1401 + \sum_{i=4}^{t-1}(136i-4) + \sum_{i=4}^{t}(136i-72) = 136t^2 - 76t + 1$. □

2.3 $(1,7)$-Knight's Move

In this section, we consider $(1,7)$-knight's move on the infinite chessboard. We first count the number k's where $k \in \{0,1,2,3,4,5,6,7,8,9,10,11\}$ around the square $(0,0)$ in Lemma 2.9. Next, the number k's where $k > 7$ are counted by separating wether k is even or odd in Lemma 2.10. Then, we conclude all k's in Theorem 2.11. Finally, the cumulative number of squares reachable in k or fewer moves is shown in Corollary 2.12.

Lemma 2.9. *The number of squares that require exactly k $(1,7)$-knight's moves in order to be reached by a sole knight from its initial position $(0,0)$ on an infinite chessboard are* $1, 8, 32, 88, 192, 360, 608, 872, 960, 1040, 1112$ *and* 1208 *for* $k = 0,1,2,3,4,5,6,7,8,9,10$ *and* 11, *respectively.*

Proof. From Figures 4-9, we obtain each number by direct counting. □

Figure 4: The number k's with $(1,7)$-knight's move from column $0-49$ and row $66-98$

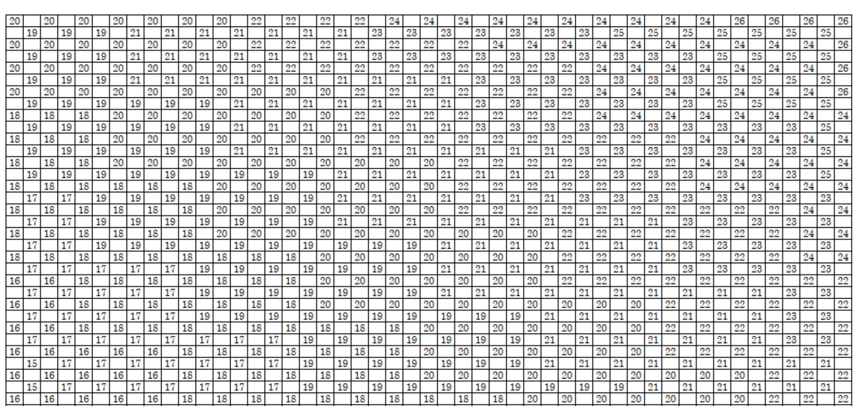

Figure 5: The number k's with $(1,7)$-knight's move from column $50-98$ and row $66-98$

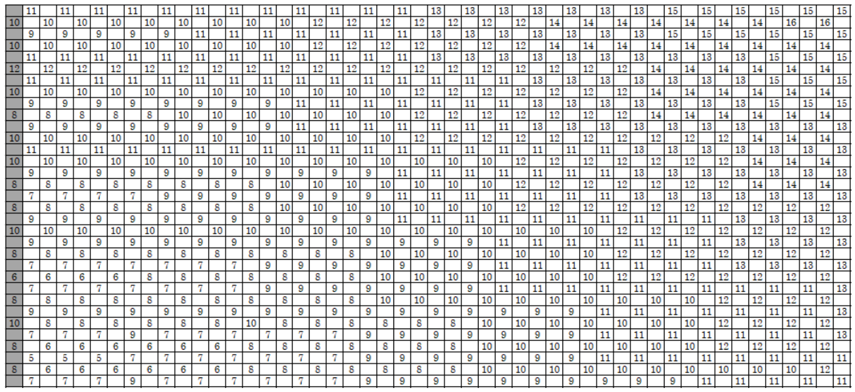

Figure 6: The number k's with $(1, 7)$-knight's move from column $0 - 49$ and row $33 - 65$

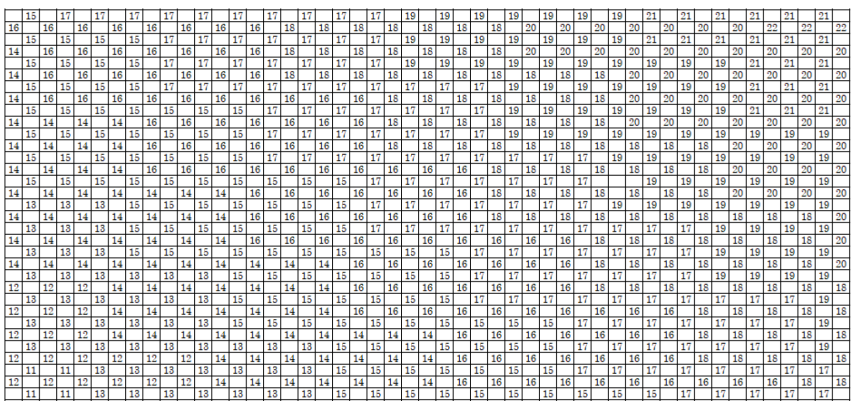

Figure 7: The number k's with $(1, 7)$-knight's move from column $50 - 98$ and row $33 - 65$

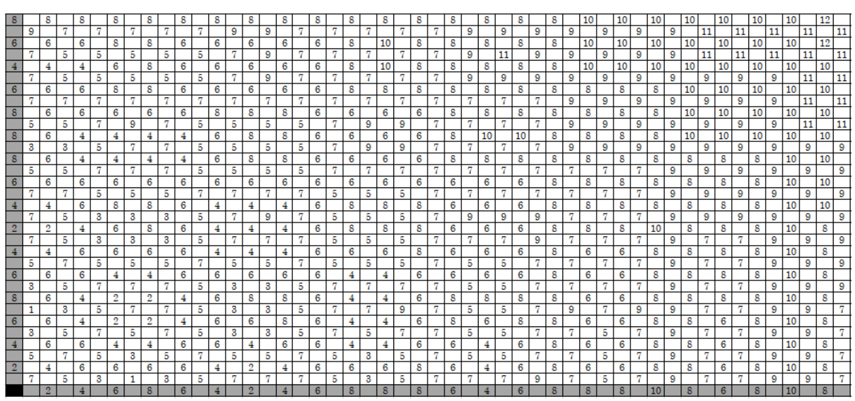

Figure 8: The number k's with $(1, 7)$-knight's move from column $0 - 49$ and row $0 - 32$

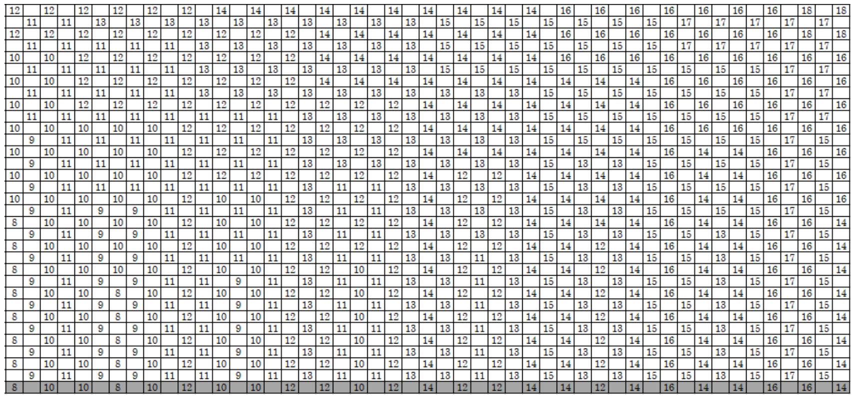

Figure 9: The number k's with $(1, 7)$-knight's move from column $50 - 98$ and row $0 - 32$

Lemma 2.10. *The number of squares that require exactly k $(1, 7)$-knight's moves in order to be reached by a sole knight from its initial position $(0, 0)$ on an infinite chessboard are $248t - 156$ for $k = 2t$ and $248t - 32$ for $k = 2t + 1$, where $t \geq 6$, respectively.*

Proof. Since the board is symmetric with the row 0 and the column 0, it suffices to count on part 1 together with squares on row 0 and then multiply by 4. From Figure 4-9, we count numbers $1332, 1580, 1828$ and 2076 for $k = 12, 14, 16$ and 18, respectively. The numbers of squares that the knight moves from the square $(0, 0)$ with k moves are $1456, 1704, 1952$ and 2200 for $k = 13, 15, 17$ and 19, respectively.

The 12's appears in 39 columns. Counting 12's in each column of each block, we obtained $(7+7+7+7+7)+8+7+7+9+[(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)]+7+7+18+7+7+15+12+7+12+9+9+6 = 326$ squares. The 14's appears in 46 columns. We obtain $(7+7+7+7+7+7)+8+7+7+9+[(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)]+7+7+19+7+7+16+13+7+13+10+10+7 = 388$ squares. With the same algorithm, the k's appear in $7t - 3$ columns. We obtain $\underbrace{7+7+7+\cdots+7}_{t-1 \text{ terms}}+8+7+7+9+\underbrace{(7+7+10)+(7+7+10)+(7+7+10)+\cdots+(7+7+10)}_{2t-6 \text{ terms}}+7+7+(t+12)+7+7+(t+9)+(t+6)+7+(t+6)+(t+3)+(t+3)+(t)$ and the arithmetic sequence of squares is $1332, 1572, 1620, \ldots$. Then, the general is in the form $62t - 46$ for $k = 2t$ where $t \in \{6, 7, 8 \ldots\}$. In row 0, we obtain the k's in 7 columns and it is similar for the column 0. Then, for $k = 2t$ where $t \in \{6, 7, 8 \ldots\}$, the number of squares that require exactly k $(1, 7)$-knight's moves is $4(62t - 46) + 4(7) = 248t - 156$ for $k = 2t$ where $t \in \{6, 7, 8 \ldots\}$.

The 13's appears in 43 columns. We obtain $(7+7+7+7+7+7)+8+7+7+9+[(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)]+7+7+19+7+7+16+13+7+13+10+10+7 = 364$ squares. The 15's appear in 50 columns. We obtain $(7+7+7+7+7+7+7)+8+7+7+9+[(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)+(7+7+10)]+7+7+20+7+7+17+14+7+14+11+11+8 = 422$ squares. With the same algorithm, the k's appear in $7t + 1$ columns. We obtain $\underbrace{7+7+7+\cdots+7}_{t \text{ terms}}+8+7+7+9+\underbrace{(7+7+10)+(7+7+10)+(7+7+10)+\cdots+(7+7+10)}_{2t-5 \text{ terms}}+7+7+(t+13)+7+7+(t+10)+(t+7)+7+(t+7)+(t+4)+(t+4)+(t+1)$ and the arithmetic sequence of squares is $1456, 1704, 1952, \ldots$. Then, the general terms is of the form $62t - 8$ for $k = 2t + 1$ where $t \in \{6, 7, 8 \ldots\}$. Then, for $k = 2t + 1$ where $t \in \{6, 7, 8, \ldots\}$, the number of squares that require exactly k $(1, 7)$-knight's moves is $4(62t - 8) = 248t - 32$ for $k = 2t + 1$ where $t \in \{6, 7, 8 \ldots\}$. Note that the odd numbers k do not appear in the row 0 and the column 0. $\square$

Lemmas 2.9 and 2.10 are concluded in the following theorem.

Theorem 2.11. *Assume that the knight moves with the* $(1,7)$*-knight's move. Then, the number of squares that require exactly* k *moves in order to be reached by a sole knight from its initial position* $(0,0)$ *on the infinite chessboard are*

(i) $1, 8, 32, 88, 192, 360, 608, 872, 960, 1040, 1112$ *and* 1208 *for* $k = 0,1,2,3,4,5,6,7,8,9,10$ *and* 11*, respectively,*

(ii) $248t - 156$ *for* $k = 2t$ *where* $t \in \{6, 7, 8 \dots\}$*, and*

(iii) $248t - 32$ *for* $k = 2t + 1$ *where* $t \in \{6, 7, 8 \dots\}$*.*

Corollary 2.12. *Assume that the knight moves with the* $(1,7)-$*knight's move. The cumulative number of squares reachable in* k *or fewer moves by a sole knight from its initial position* $(0,0)$ *on an infinite chessboard are* $1, 9, 41, 129, 321, 681, 1289, 2161, 3121, 4161, 5273$ *and* 6481 *for* $k = 0, 1, 2, \dots, 11$*,* $248t^2 - 188t + 13$ *for* $k = 2t$ *where* $t \in \{6, 7, 8, \dots\}$ *and* $248t^2 + 60t - 19$ *for* $k = 2t + 1$ *where* $t \in \{6, 7, 8, \dots\}$*.*

Proof. For $k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and 11, it is easy to see that the cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard are $1, 9, 41, 129, 321, 681, 1289, 2161, 3121, 4161, 5273$ and 6481, respectively. We assume that $k > 11$.

Case (i) : $k = 2t + 1$ where $t \in \{6, 7, 8 \dots\}$. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard is of the form $6481 + \sum_{i=6}^{t}(248i - 156) + \sum_{i=6}^{t}(248i - 32) = 248t^2 + 60t - 19$.

Case (ii) : $k = 2t$ where $t \in \{6, 7, 8, \dots\}$. The cumulative number of squares reachable in k or fewer moves by a sole knight from its initial position $(0,0)$ on an infinite chessboard is of the form $6481 + \sum_{i=6}^{t}(248i - 156) + \sum_{i=6}^{t-1}(248i - 32) = 248t^2 - 188t + 13$. □

Acknowledgement : We would like to thank the referee(s) for his comments and suggestions on the manuscript.

References

[1] A.L. Schwenk, Which rectangular chessboards have a knight's tour, Math. Magazine 64 (1991) 325-332.

[2] G.L. Chia and Siew-Hui Ong, *Generalized knight's tours on rectangular chessboards*, Discrete Applied Math. **150** (2005), 80–89.

[3] A.M. Miller and D.L. Farnsworth, Counting the number of Squares reachable in k knight's moves, Open J. of Discrete Math. 3 (2013) 151-154.

[4] R. Theprod, S. Singhun and R. Boonklurb, The number of squares reachable in k moves with the $(1,4)$-knight's move, Proc. of Annual Pure and Applied Math. Conference 2017, 241-248.

(Received 21 November 2018)
(Accepted 11 June 2019)