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Cops and Robbers Game on The Cartesian, Direct and Strong Products of Hypergraphs

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Abstract : The game of *cops and robbers* is a game that is usually played on a finite connected graph G with two players, cop and robber, according to the following rules: (i) cop chooses a vertex of G to begin and robber then chooses other vertex of G to begin and (ii) they alternatively move from their present vertices to adjacent vertices along edges of G where the first move is a turn of cop. However, they can also choose not to move from their positions at each of their turns as well. If cop catches some robber after finite moves by occupying the same vertex as robber, it is called cop wins and such a graph is called a *cop-win graph*; otherwise, it is called robber wins and such a graph is called a *robber-win graph*. Recently, the game of cops and robbers played on a hypergraph has been defined and some rules of the game have been changed; that is, they can move from their present vertex x to any vertex y which is in the same hyperedge as vertex x. A hypergraph which cop wins is called a *cop-win hypergraph*; otherwise, a *robber-win hypergraph*. Throughout this paper, we consider the game of cops and robbers on the products of cop-win hypergraphs. Then, we prove that their cartesian and minimal (maximal) rank preserving direct products are robber-win hypergraphs, and their standard (normal) strong product is still a cop-win hypergraph.

Keywords : cops and robbers; hypergraph; cop-win hypergraph.
2010 Mathematics Subject Classification : 05C65.

1 Introduction

Let G be a finite connected graph. A vertex-pursuit game of two players, *cop and robber*, played on a graph G was first introduced by Nowakowski and Winkler [1]. The rules of the game are defined as follows:

(i) First, the cop selects some vertex to begin and the robber then selects the other vertex to begin.

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(ii) In each round, the cop and the robber take altenatively moving from their present vertex to other vertices along edges. However, they can also choose not to move from their positions at each of their turns as well.

There are two winning strategies to finish the game such as cop can catch robber by occupying the same vertex as the robber after finite number of moves, or robber can run away. The graph which cop has the winning strategy is called a *cop-win graph*; otherwise, a *robber-win graph*. In [1], the cop-win graph are characterized and the products of cop-win graphs is a cop-win graph are proved.

Besides playing on graphs, cops and robbers game can be played on other stucture; that is, hypergraph.

Definition 1.1. [2] *The pair $H = (V, E)$ is called a* ***hypergraph*** *including* ***vertex set*** *V or $V(H)$ which is a finite set and* ***(hyper)edge set*** *E or $E(H)$ which is a family of subsets of V. A hypergraph in which all edges have the same size $r \geq 0$ is called r-**uniform***

In 2011, Baird [3] introduced the game of cops and robbers played on hypergraphs. Cop and robber can move from their present vertex x to any vertex y belonging to the same hyperedge as vertex x, which is slightly changed from the game played on graphs. A hypergraph which cop wins is called a *cop-win hypergraph* and a hypergraph which robber wins is called a *robber-win hypergraph.*

Definition 1.2. [3] *A hypergraph is* ***t-joined*** *if each intersection of hyperedges contains exactly t vertices. A* ***hyperpath*** *is a sequence of hyperedges $E_1, E_2, E_3, \ldots, E_k$, such that E_i and E_{i+1} are t-joined for some $t > 0$ and for $1 \leq i \leq k-1$ and $E_i \cap E_j = \emptyset$ when $j \neq i+1 (mod\ k)$. For an integer $k > 2$, a* ***k-hypercycle*** *is a collection of k hyperedges $E_1, E_2, E_3, \ldots, E_k$ with two hyperedges E_i and E_j incident if $i = j+1 (mod\ k)$.*

Baird [3] has proved that a path is a cop-win hypergraph and a cycle of length exceed 4 is a robber-win hypergraph. Throughout this paper, we consider the game of cops and robbers played on the products of hypergraphs, namely the cartesian product, the direct product and the strong product.

Definition 1.3. [4] *Let $H_1 = (V_1, E_2)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The* ***Cartesian product*** *$H = H_1 \square H_2$ of two hypergraphs H_1 and H_2 has the vertex set $V(H) = V_1 \times V_2$ and the edge set $E(H) = \{\{x_1\} \times e_2 \mid x_1 \in V_1, e_2 \in E_2\} \cup \{e_1 \times \{x_2\} \mid e_1 \in E_1, x_2 \in V_2\}$*

Example 1.4. *Let $H_1 = (V_1, E_1)$ where $V_1 = \{1, 2, 3\}$ and $E_1 = \{\{1, 2, 3\}\}$ and $H_2 = (V_2, E_2)$ where $V_2 = \{a, b\}$ and $E_1 = \{\{a, b\}\}$*

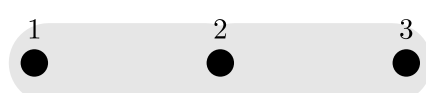

Figure 1: Hypergraph H_1

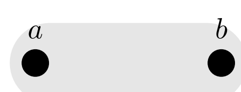

Figure 2: Hypergraph H_2

The vertex set $V_1 \times V_2 = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$. We use ij instead of (i, j) in the following hypergraph.

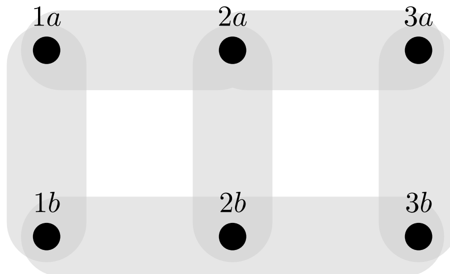

Figure 3: The Cartesian Product $H_1 \square H_2$

Definition 1.5. [4] *For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, their* ***minimal rank preserving direct product*** *$H_1 \times_1 H_2$ has the vertex set $V_1 \times V_2$. A subset of $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \times_1 H_2$ if and only if*

(i) *$\{x_1, x_2, x_3, \ldots, x_r\}$ is an edge in H_1 and $\{y_1, y_2, y_3, \ldots, y_r\}$ is a subset of an edge in H_2, or*

(ii) *$\{x_1, x_2, x_3, \ldots, x_r\}$ is a subset of an edge in H_1 and $\{y_1, y_2, y_3, \ldots, y_r\}$ is an edge in H_2.*

Example 1.6. *We use hypergraphs H_1 and H_2 in Example 1.4.The vertex set $V_1 \times V_2 = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$. We use ij instead of (i, j) in the following hypergraph.*

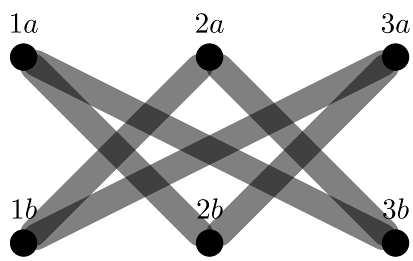

Figure 4: The Minimal Rank Preserving Direct Product $H_1 \times_1 H_2$

Definition 1.7. [4] *For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, their* ***maximal rank preserving direct product*** *$H_1 \times_2 H_2$ has the vertex set $V_1 \times V_2$. A subset of $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \times_2 H_2$ if and only if*

(i) *$\{x_1, x_2, x_3, \ldots, x_r\}$ is an edge in H_1 and there is an edge e_2 in E_2 such that $\{y_1, y_2, y_3, \ldots, y_r\}$ is a multiset of elements of e_2 and $e_2 \subseteq \{y_1, y_2, y_3, \ldots, y_r\}$, or*

(ii) *$\{y_1, y_2, y_3, \ldots, y_r\}$ is an edge in H_2 and there is an edge e_1 in E_1 such that $\{x_1, x_2, x_3, \ldots, x_r\}$ is a multiset of elements of e_1 and $e_1 \subseteq \{x_1, x_2, x_3, \ldots, x_r\}$.*

Example 1.8. *We use hypergraphs H_1 and H_2 in Example 1.4. The vertex set $V_1 \times V_2 = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$. We use ij instead of (i, j) in the following hypergraph.*

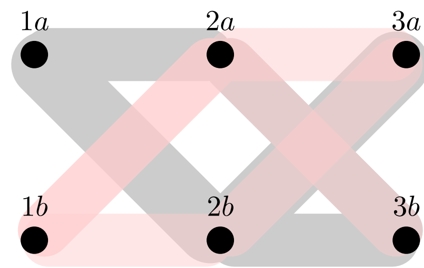

Figure 5: The Maximal Rank Preserving Direct Product $H_1 \times_2 H_2$

Notice that if H_1 and H_2 are r-uniform hypergraphs, then $H_1 \times_1 H_2 = H_1 \times_2 H_2$.

Definition 1.9. [4] *For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, their **normal (strong) product**. $H_1 \boxtimes_1 H_2$ has the vertex set $V_1 \times V_2$ and the edge set $E(H_1 \boxtimes_1 H_2) = E(H_1 \Box H_2) \cup E(H_1 \times_1 H_2)$.*

That is, a subset $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \boxtimes_1 H_2$ if and only if

(i) $\{x_1, x_2, x_3, \ldots, x_r\} \in E_1$ and $y_1 = y_2 = y_3 = \cdots = y_r \in V_2$, or

(ii) $\{y_1, y_2, y_3, \ldots, y_r\} \in E_2$ and $x_1 = x_2 = x_3 = \cdots = x_r \in V_1$, or

(iii) $\{x_1, x_2, x_3, \ldots, x_r\} \in E_1$ and $\{y_1, y_2, y_3, \ldots, y_r\}$ is a subset of an edge in H_2, or

(iv) $\{y_1, y_2, y_3, \ldots, y_r\} \in E_2$ and $\{x_1, x_2, x_3, \ldots, x_r\}$ is a subset of an edge in H_1.

Example 1.10. *We use hypergraphs H_1 and H_2 in Example 1.4. The vertex set $V_1 \times V_2 = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$. We use ij instead of (i, j) in the following hypergraph.*

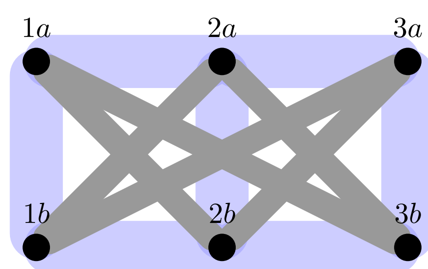

Figure 6: The Normal (Strong) Product $H_1 \boxtimes_1 H_2$

Definition 1.11. [4] *For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, their **standard strong product**. $H_1 \boxtimes_2 H_2$ has the vertex set $V_1 \times V_2$ and the edge set $E(H_1 \boxtimes_2 H_2) = E(H_1 \Box H_2) \cup E(H_1 \times_2 H_2)$.*

That is, a subset $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \boxtimes_2 H_2$ if and only if

(i) $\{x_1, x_2, x_3, \ldots, x_r\} \in E_1$ and $y_1 = y_2 = y_3 = \cdots = y_r \in V_2$, or

(ii) $\{y_1, y_2, y_3, \ldots, y_r\} \in E_2$ and $x_1 = x_2 = x_3 = \cdots = x_r \in V_1$, or

(iii) $\{x_1, x_2, x_3, \ldots, x_r\} \in E_1$ and there is an edge e_2 in E_2 such that $\{y_1, y_2, y_3, \ldots, y_r\}$ is a multiset of elements of e_2 and $e_2 \subseteq \{y_1, y_2, y_3, \ldots, y_r\}$, or

(iv) $\{y_1, y_2, y_3, \ldots, y_r\} \in E_2$ and there is an edge e_1 in E_1 such that $\{x_1, x_2, x_3, \ldots, x_r\}$ is a multiset of elements of e_1 and $e_1 \subseteq \{x_1, x_2, x_3, \ldots, x_r\}$

Example 1.12. *We use hypergraphs H_1 and H_2 in Example 1.4. The vertex set $V_1 \times V_2 = \{(1,a),(2,a),(3,a),(1,b),(2,b),(3,b)\}$. We use ij instead of (i,j) in the following hypergraph.*

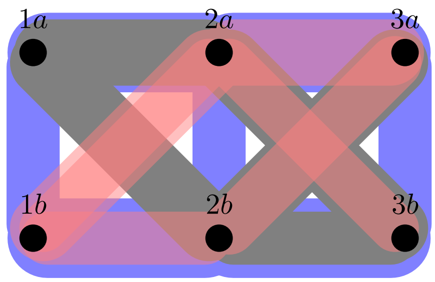

Figure 7: The Standard Strong Product $H_1 \boxtimes_2 H_2$

2 The Cartesian Product and Direct Product of Cop-Win Hypergraphs

According to the definition of the cartesian product, we see that a subset $\{(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots,(x_r,y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \square H_2$ if and only if

(i) $\{x_1, x_2, x_3, \ldots x_r\} \in E_1$ and $y_1 = y_2 = y_3 = \cdots = y_r \in V_2$, or

(ii) $\{y_1, y_2, y_3, \ldots y_r\} \in E_2$ and $x_1 = x_2 = x_3 = \cdots = x_r \in V_1$.

Theorem 2.1. *The cartesian product of cop-win hypergraphs is a robber-win hypergraph.*

Proof. Assume that $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are cop-win hypergraphs. We see that a subhypergraph H' with any four vertice of the form $(x_{i_1}, y_{j_1}), (x_{i_1}, y_{j_2}), (x_{i_2}, y_{j_1}), (x_{i_2}, y_{j_2})$ where $x_{i_1}, x_{i_2} \in V_1, y_{j_1}, y_{j_2} \in V_2, i_1 \neq i_2$ and $j_1 \neq j_2$ forms a cycle of length 4. By [3], such a cycle is a robber-win hypergraph, so is a subhypergraph H'. Thus, $H_1 \square H_2$ is a robber-win hypergraph. □

From Definitions 1.5 and 1.7, we observe that for each vertex (x_i, y_j) in $H_1 \times_* H_2$ where $*$ is 1 or 2, there are at least one vertex of the form $(x_i, y_{j'})$ and at least one vertex of the form $(x_{i'}, y_j)$ which are not adjacent to (x_i, y_j) where $i \neq i'$ and $j \neq j'$.

Theorem 2.2. *The minimal (maximal) rank preserving direct product of cop-win hypergraphs is a robber-win hypergraph.*

Proof. Let k and l be positive integers. Assume that $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are cop-win hypergraphs where $V_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ and $V_2 = \{y_1, y_2, y_3, \ldots, y_l\}$.

First of all, cop selects one vertex in $H_1 \times_* H_2$, say (x_{i_1}, y_{i_2}) where $1 \leq i_1 \leq k$ and $1 \leq i_2 \leq l$. Then, robber selects other vertices so that he can avoid cop at the beginning, say (x_{j_1}, y_{j_2}) where $1 \leq j_1 \leq k$ and $1 \leq j_2 \leq l$. Next, cop moves to one vertex which is in the same edge as (x_{i_1}, y_{i_2}) and (x_{j_1}, y_{j_2}), say (x', y'). By the previous observation, robber can move to one vertex which is in the same edge as (x_k, y_l), but not in the same edge as the vertex (x', y'), say (x'', y''). We know that there exists an edge containing (x', y') and (x'', y''). Then, cop moves along such an edge and stays at some vertices, say $(\overline{x}, \overline{y})$. However, robber can find the vertex which is in the same edge as (x'', y''), but not in the same edge as the vertex $(\overline{x}, \overline{y})$ and then stay at this vertex. Continue this process, we conclude that robber can escape from cop. □

3 The Strong Product of Cop-Win hypergraphs

Before showing the strong product of cop-win hypergraphs is a cop-win hypergraph, we prove the following lemma.

Lemma 3.1. *Let H_1 and H_2 be hypergraphs both having only one (hyper)edge, e_1 and e_2, respectively. Then, $H_1 \boxtimes_* H_2$ is a cop-win hypergraph.*

Proof. By [3], we know that a path is a cop-win hypergraph. Then, H_1 and H_2 are cop-win hypergraphs. Let k and l be positive integers. Let $e_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ and $e_2 = \{y_1, y_2, y_3, \ldots, y_l\}$. We see that cop can choose any vertex in each edge so that he can win the game. Without loss of generality, let (x_1, y_1) be the stating vertex of cop. There are three possible cases of the starting vertex of robber.

Case 1. Robber occupies the vertex (x_i, y_1) where $i \neq 1$. Since each vertex in H_1 is adjacent to each other, there exists an edge in $H_1 \Box H_2$ containing both x_1 and x_i. Then, cop moves along such a edge to catch robber.

Case 2. Robber occupies the vertex (x_1, y_j) where $j \neq 1$. Since each vertex in H_2 is adjacent to each other, there exists an edge in $H_1 \Box H_2$ containing both y_1 and y_j. Then, cop moves along such a edge to catch robber.

Case 3. Robber occupies the vertex (x_i, y_j) where $i, j \neq 1$. Since each vertex in H_1 is adjacent to each other and each vertex in H_2 is also adjacent to each other, there exists an edge in $H_1 \times_* H_2$ containing both (x_1, y_1) and (x_i, y_j). Then, cop moves along such edges to catch robber.

From the previous three cases, $H_1 \boxtimes_* H_2$ is a cop-win hypergraph. □

Theorem 3.2. *If H_1 and H_2 are cop-win hypergraphs, then $H_1 \boxtimes_* H_2$ is also a cop-win hypergraph.*

Proof. Let k and l be positive integers. Assume that $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are cop-win hypergraphs, where $V_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ and $V_2 = \{y_1, y_2, y_3, \ldots, y_l\}$.

To consider $H_1 \boxtimes_* H_2$, let i and j be positive integers and let $S_i = \{x_i\} \times E_2$ and $T_j = E_1 \times \{y_j\}$. We consider three possible cases of the present vertex of cop and the present vertex of robber.

Case 1. Cop chooses (x_i, y_{j_1}) to stay and robber chooses (x_i, y_{j_2}) to stay where $j_1 \neq j_2$. To catch robber, cop moves along some edges in S_i. If y_{j_1} and y_{j_2} are in the same edge in H_2, then cop can occupy the same vertex as robber in $H_1 \boxtimes_* H_2$. Otherwise, there are two different edges of H_2, one containing y_{j_1} and the other containing y_{j_2}, cop moves to the vertex (x_i, y_{j_3}) where y_{j_3} is the vertex which cop chooses in the next turn in H_2.

Case 2. Cop chooses (x_{i_1}, y_j) to stay and robber chooses (x_{i_2}, y_j) to stay where $i_1 \neq i_2$. To catch robber, cop moves along some edges in T_j. If x_{i_1} and x_{i_2} are in the same edge in H_1, then cop can occupy the same vertex as robber in $H_1 \boxtimes_* H_2$. Otherwise, there are two different edges of H_1, one containing x_{i_1} and the other containing x_{i_2}, cop moves to the vertex (x_{i_3}, y_j) where x_{i_3} is the vertex which cop chooses in the next turn in H_1.

Case 3. Cop chooses (x_{i_1}, y_{j_2}) to stay and robber chooses (x_{i_2}, y_{j_2}) to stay where $i_1 \neq i_2$ and $j_1 \neq j_2$. To catch robber, cop moves along some edges in $E(H_1 \times_* H_2)$. If both x_{i_1} and x_{i_2} are in the same edge in H_1, and both y_{j_1} and y_{j_2} are in the same edge in H_2, then cop can occupy the same vertex as robber in $H_1 \boxtimes_* H_2$. Otherwise, there are two different edges of H_1, one containing x_{i_1} and the other containing x_{i_2}, and there are two different edges of H_2, one containing y_{j_1} and the other containing y_{j_2}, cop moves to the vertex (x_{i_3}, y_{j_3}) where x_{i_3} is the vertex which cop chooses in the next turn in H_1 and y_{j_3} is the vertex which cop chooses in the next turn in H_2.

Following the three cases after finite moves, cop and robber stay at some vertices in the same $H' \boxtimes_* H''$ where H' and H'' are hypergraphs both having only one (hyper)edge $e \in E_1$ and $f \in E_2$, respectively. Then, by Lemma 3.1, cop can catch robber. □

Corollary 3.3. *Let $m \geq 2$ be a positive integer. If H is a collection of m cop-win hypergraphs, then the standard (normal) strong product of such m cop-win hypergraphs is a cop-win hypergraph.*

Proof. We prove by the mathematical induction on m. For $m = 2$, the corollary done by Theorem 3.2. Let $m > 2$. Assume that the standard (normal) strong product of $m - 1$ cop-win hypergraphs is a cop-win hypergraph. By induction hypothesis and Theorem 3.2, we obtain that the standard (normal) strong product of m cop-win hypergraphs is also a cop-win hypergraph. □

4 Conclusion an Discussion

According to the cartesian product and the minimal (maximal) rank preserving direct product of cop-win hypergraphs, we obtain that both products are not a cop-win hypergraph. However, their standard (normal) strong product whose edge set is the union of the edge set of two previous products. Thus, we observe that the edge set of the minimal (maximal) rank preserving direct product destroys a cycle of four vertices in a certain of the cartesian product and the edge set of cartesian product converts non-adjacent vertex to adjacent vertex in the minimal (maximal) rank preserving direct product, which causes the standard (normal) strong product of cop-win hypergraphs to be a cop-win hypergraph.

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