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Edge-Odd Graceful Graphs Related to Cycles

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Abstract : Let G be a graph consisting of the vertex set $V(G)$ and the edge set $E(G)$ such that $|E(G)| = q$. An *edge-odd graceful labeling* is a bijection function $f : E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ such that for each $v \in V(G)$, $f^*(v) = \sum_{uv \in E(G)} f(uv) \pmod{2q}$ are all distinct. In this article, edge-odd graceful labelings for graphs related to cycles, $(n, 1)$-kite and $(n, 2)$-kite where n is an integer such that $n \geq 3$, the graph $P_2 \cdot nK_1$ where n is a positive integer and the cartesian product $C_n \Box P_3$ and $C_3 \Box P_k$ where $n \geq 3$ and $k \geq 4$ are obtained. Moreover, we show that if a graph G is edge-odd graceful and each vertex has odd degree, then the union of even copies of G is edge-odd graceful.

Keywords : edge-odd graceful; the cartesian product of a graph; the union of graphs.
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1 Introduction and Preliminaries

Thoughtout this paper, for a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A graph labeling which is an assignment of integers to the vertices or edges, or both, subject to certain condition was introduced by Rosa [1] in the late 1960s. Rosa called a function f a *β-valuation* (which well-known in the term *graceful labeling*) of a graph G with q edges if f is an injection from the vertex set to $\{0, 1, 2, \ldots, q\}$ such that each edge xy of G is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be *graceful* if G admits a graceful labeling. In 1991, Gnanajothi [2] defined a graph G with q edges to be *odd-graceful* if there is an injection f from the vertex set to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy of G is assigned the label $|f(x) - f(y)|$, the

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resulting edges labels are in $\{1, 3, 5, \ldots, 2q-1\}$. Later, Solairaju and Chithra [3] introduced a new type of labeling which can be regarded as an inverse problem of labeling defined by Gnanajothi as follows.

An *edge-odd graceful labeling* of a graph G with q edges is a bijection function f from $E(G)$ to $\{1, 3, 5, \ldots, 2q-1\}$ so that the induced mapping f^* from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ given by, for each $v \in V(G)$, $f^*(v) = \sum_{uv \in E(G)} f(uv) \pmod{2q}$. The edge labels and vertex labels are distinct. A graph which has an edge-odd graceful labeling is called *edge-odd graceful*.

Not all graphs are edge-odd graceful. For example, the star $K_{1,3}$, a graph with one vertex (called the center) joining to three vertices, is not edge-odd graceful. Without loss of generality, we label each edge by $1, 3$ and 5. Then, the center is labeled by $(1 + 3 + 5) \pmod 6 = 3$, which is the same as one of the three vertices.

The edge-odd graceful labelings of some graphs related to paths are shown in [3]. Later, Singhun [4] began to show edge-odd graceful labelings of $SF(n, m)$ where $n \geq 3$ is odd and m is even and $n|m$ and a wheel graph W_n, where n is even. In 2015, the authors in [5] also showed edge-odd graceful labelings of some prisms and prism-like graphs, Prism(S_n) when $n \geq 3$, Prism$_3(S_n)$ when $n \geq 3$ and $n \equiv 2 \pmod 6$, Prism(W_n) when $n \geq 3$ and $2|n$. Recently, Daoud [6] constructed several edge-odd graceful labelings for friend ship graphs $F_{r_n}^{(3)}, F_{r_n}^{(4)}$ and $\bar{F}_{r_n}^{(3)}$, wheel graph W_n, helm graph H_n, web graph Wb_n, double web graph $W_{n,n}$, fan graph F_n, double fan graph $F_{2,n}$, gear graph G_n, half gear graph HG_n and polar grid graph $P_{m,n}$.

In this article, we continue to show edge-odd graceful labelings of graphs related to cycles, (n, k)-kites when $n \geq 3$ is an integer and $k = 1, 2$, the graph $P_2 \cdot nK_1$ for all integer $n \geq 2$. Next, edge-odd graceful labelings of the cartesian product of a cycle and a path, $C_n \square P_3$ for all $n \geq 3$ and $C_3 \square P_k$ for all $k \geq 4$ are shown. Moreover, we show that if an edge-odd graceful graph G has all odd degree vertices, then the union of even copies of G is edge-odd graceful.

2 An (n, k)-kite

Let $n \geq 3$. A graph (n, k)-*kite* is a graph obtained from a cycle $v_1 v_2 v_3 \cdots v_n v_1$ and a path $v_{n+1} v_{n+2} v_{n+3} \cdots v_{n+k}$ by joining v_1 and v_{n+1}. Then, $V((n, k)\text{-kite}) = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}\}$ and $E((n, k)\text{-kite}) = \{v_i v_{i+1} \mid i \in \{1, 2, \ldots, n-1, n+1, \ldots, n+k-1\}\} \cup \{v_n v_1, v_1 v_{n+1}\}$. Edge-odd graceful labelings for $(n, 1)$-kite and $(n, 2)$-kite are shown in Algorithm 2.1 and 2.2.

Algorithm 2.1. *Let G be the graph $(n, 1)$-kite where $n \geq 3$. Define the edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 2n+1\}$ for G as follow.*

If n is odd, let

- $f(v_i v_{i+1}) = n + i + 1$ *for* $i \in \{1, 3, 5, \ldots, n-2\}$;
- $f(v_i v_{i+1}) = i + 1$ *for* $i \in \{2, 4, 6, \ldots, n-1\}$;
- $f(v_1 v_n) = 1$; *and*
- $f(v_1 v_{n+1}) = 2n + 1$.

If n is even, let

- $f(v_i v_{i+1}) = 2i + 1$ *for* $i \in \{1, 2, 3, \ldots, n-1\}$;
- $f(v_n v_1) = 2n + 1$; *and*
- $f(v_1 v_{n+1}) = 1$.

Theorem 2.1. *Let n be an integer such that $n \geq 3$. The graph $(n, 1)$-kite is edge-odd graceful.*

Proof. Let G be the graph $(n, 1)$-kite. By Algorithm 2.1, it is easy to see that f is a bijection from $E(G)$ to $\{1, 3, 5, \ldots, 2n+1\}$. It suffices to show that the induced mapping f^* on vertices of G are distinct.

Case 1 : n is odd. By Algorithm 2.1, (i) $f^*(v_1) = n + 2$, (ii) $f^*(v_i) = (n + 2i + 1) \pmod{2n+2}$ for $i \in \{2, 3, 4, \ldots, n-1\}$, (iii) $f^*(v_n) = n + 1$ and (iv) $f^*(v_{n+1}) = 2n + 1$.

Then, $f^*(v) \in \{1, 2, 3, \ldots, 2n+1\}$ for each $v \in V(G)$. By (i) and (iv), $f^*(v_1) = n+2$ and $f^*(v_{n+1}) = 2n+1$ are odd and distinct. By (ii) and (iii), $f^*(v_i) = (n+2i+1) \pmod{2n+2}$ for $i \in \{2, 3, 4, \ldots, n-1\}$ and $f^*(v_n) = n+1$ are even. It suffices to show that for $i \in \{2, 3, 4, \ldots, n-1\}$, $(n+2i+1) \pmod{2n+2}$ are distinct and also different from $n+1$.

Suppose that $n+2i+1 \equiv n+2j+1 \pmod{2n+2}$ for some $i, j \in \{2, 3, 4, \ldots, n-1\}$ and $i > j$. Then, $(n+1)|(i-j)$. However, $1 \leq i-j \leq n-3$, which is a contradiction.

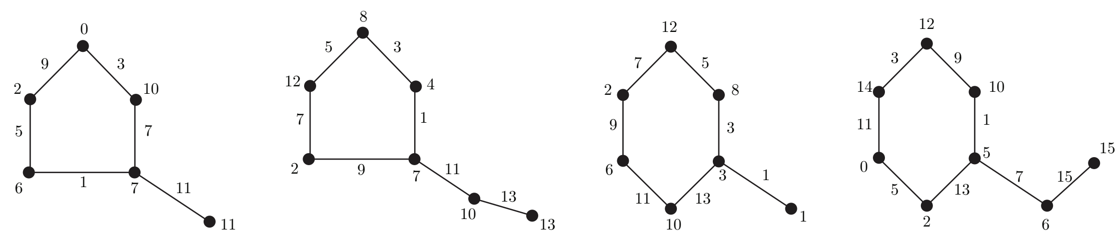
Suppose that $n+2i+1 \equiv n+1 \pmod{2n+2}$ for some $i \in \{2, 3, 4, \ldots, n-1\}$. Then, $i \equiv 0 \pmod{n+1}$. Thus, $(n+1)|i$, which is a contradiction.

Case 2 : n is even. By Algorithm 2.1, (i) $f^*(v_1) = 3$, (ii) $f^*(v_i) = 4i \pmod{2n+2}$ for $i \in \{2, 3, 4, \ldots, n\}$, and (iii) $f^*(v_{n+1}) = 1$.

Then, $f^*(v) \in \{1, 2, 3, \ldots, 2n+1\}$ for each $v \in V(G)$. By (i) and (iii), $f^*(v_1) = 3$ and $f^*(v_{n+1}) = 1$ are odd and distinct. By (ii), $f^*(v_i) = 4i \pmod{2n+2}$ for $i \in \{2, 3, 4, \ldots, n\}$ are all even. It suffices to show that for $i \in \{2, 3, 4, \ldots, n\}$, $f^*(v_i) = 4i \pmod{2n+2}$ are all distinct.

Suppose that $f^*(v_i) \equiv f^*(v_j) \pmod{2n+2}$ for some $i, j \in \{2, 3, 4, \ldots, n\}$ and $i > j$. Then, $2(i-j) = (n+1)t$ for some positive integer t. Since $n+1$ is odd, $(n+1)|(i-j)$. However, $1 \leq i-j \leq n-2$, which is a contradiction.

Therefore, the edge labelings obtained from Algorithm 2.1 are edge-odd graceful labelings. Hence, G is edge-odd graceful. □



(a) (b) (c) (d)

Figure 1: (a) $(5,1)$-kite, (b) $(5,2)$-kite, (c) $(6,1)$-kite and (d) $(6,2)$-kite

Algorithm 2.2. *Let G be the graph $(n, 2)$-kite where $n \geq 3$. Define the edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 2n+3\}$ for G as follow.*

If n is odd, let

- $f(v_i v_{i+1}) = 2i - 1$ *for* $i \in \{1, 2, 3, \ldots, n-1\}$;
- $f(v_1 v_n) = 2n - 1$;
- $f(v_1 v_{n+1}) = 2n + 1$; *and*
- $f(v_{n+1} v_{n+2}) = 2n + 3$.

If n is even, let

- $f(v_i v_{i+1}) = i$ *for* $i \in \{1, 3, 5, \ldots, n-1\}$;
- $f(v_i v_{i+1}) = n + 1 + i$ *for* $i \in \{2, 4, 6, \ldots, n-2\}$;
- $f(v_1 v_n) = 2n + 1$;
- $f(v_1 v_{n+1}) = n + 1$; *and*
- $f(v_{n+1} v_{n+2}) = 2n + 3$.

Theorem 2.2. *Let n be an integer such that $n \geq 3$. The graph $(n, 2)$-kite is edge-odd graceful.*

Proof. Let G be the graph $(n, 2)$-kite. By Algorithm 2.2, it is easy to see that f is a bijection from $E(G)$ to $\{1, 3, 5, \ldots, 2n + 1\}$. It suffices to show that the induced mapping f^* on vertices of G are distinct.

Case 1 : n is odd. We see from Algorithm 2.2 that (i) $f^*(v_1) = 2n-3$, (ii) $f^*(v_i) = (4i-4) \pmod{2n+4}$ for $i \in \{2, 3, 4, \ldots, n\}$, (iii) $f^*(v_{n+1}) = 2n$, and (iv) $f^*(v_{n+2}) = 2n + 3$.

Then, $f^*(v) \in \{1, 2, 3, \ldots, 2n+4\}$ for each $v \in V(G)$. By (i) and (iv), $f^*(v_1) = 2n-3$ and $f^*(v_{n+2}) = 2n + 3$ are odd and distinct. By (ii) and (iii), for $i \in \{2, 3, 4, \ldots, n\}$, $f^*(v_i) = (4i - 4) \pmod{2n + 4}$ and $f^*(v_{n+1}) = 2n$ are even. It suffices to show that, for $i \in \{2, 3, 4, \ldots, n\}$, $(4i - 4) \pmod{2n + 4}$ are all distinct and also different from $2n$.

Suppose that $4i - 4 \equiv 4j - 4 \pmod{2n + 4}$ for some $i, j \in \{2, 3, 4, \ldots, n\}$ and $i < j$. Then, $(n + 2)|2(j - i)$. Since n is odd, $n + 2$ is odd and $(n + 2)|(j - i)$. However, $1 \leq j - i \leq n - 2$, which is a contradiction.

Suppose that $4i - 4 \equiv 2n \pmod{2n + 4}$ for some $i \in \{2, 3, 4, \ldots, n\}$. Then, $2i \equiv 0 \pmod{n + 2}$. Thus, $(n + 2)|2i$. Since n is odd, $n + 2$ is odd and $(n + 2)|i$. However $2 \leq i \leq n$, it is a contradiction.

Case 2 : n is even. We see from Algorithm 2.2 that (i) $f^*(v_1) = n - 1$, (ii) $f^*(v_i) = (n + 2i) \bmod (2n + 4)$ for $i \in \{2, 3, 4, \ldots, n\}$, (iii) $f^*(v_{n+1}) = n$, and (iv) $f^*(v_{n+2}) = 2n + 3$.

Then, $f^*(v) \in \{1, 2, 3, \ldots, 2n+4\}$ for each $v \in V(G)$. By (i) and (iv), $f^*(v_1) = n-1$ and $f^*(v_{n+2}) = 2n + 3$ are odd and distinct. By (ii) and (iii), $f^*(v_i) = (n + 2i) \pmod{2n + 4}$ for $i \in \{2, 3, 4, \ldots, n\}$ and $f^*(v_{n+1}) = n$ are even. It suffices to show that $(n + 2i) \pmod{2n + 4}$ are all distinct for all $i \in \{2, 3, 4, \ldots, n\}$ and also different from n.

Suppose that $n+2i \equiv n+2j \pmod{2n+4}$ for some $i, j \in \{2, 3, 4, \ldots, n\}$ and $i > j$. Then, $(n+2)|(i-j)$. However, $1 \leq i - j \leq n - 2$, it is a contradiction.

Suppose that $n + 2i \equiv n \pmod{2n + 4}$ for some $i \in \{2, 3, 4, \ldots, n\}$. Then, $(n + 2)|i$. However, $2 \leq i \leq n$, it is a contradiction.

Therefore, the edge labelings obtained from Algorithm 2.2 are edge-odd graceful labelings. Hence, G is edge-odd graceful. □

3 $P_2 \cdot nK_1$

Let n be a positive integer. The graph $P_2 \cdot nK_1$ is a graph obtained from a path $P_2 : v_1v_2$ and n independent vertices, $u_1, u_2, \ldots, u_n$, by joining each vertex u_i for $i \in \{1, 2, 3, \ldots, n\}$ with v_j for $j \in \{1, 2\}$. In the case that $n = 1$, the graph is denoted by $P_2 \cdot K_1$. Then, $V(P_2 \cdot nK_1) = \{v_1, v_2, u_1, u_2, \ldots, u_n\}$ and $E(P_2 \cdot nK_1) = \{v_1v_2\} \cup \{u_iv_j \mid i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{1, 2\}\}$. It can be seen that $P_2 \cdot nK_1$ contains $n + 2$ vertices and $2n + 1$ edges. For $n = 1$, $P_2 \cdot K_1$ is a cycle C_3 or the fan F_2 and it is obviously edge-odd graceful. Edge-odd graceful labelings for $P_2 \cdot nK_1$ where $n \geq 2$ are shown.

Algorithm 3.1. *Let n be an integer such that $n \geq 2$. Let G be the graph $P_2 \cdot nK_2$. Define the edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 4n + 1\}$ for G as follow.*

If $n \equiv 0$ or $2 \pmod 4$, let

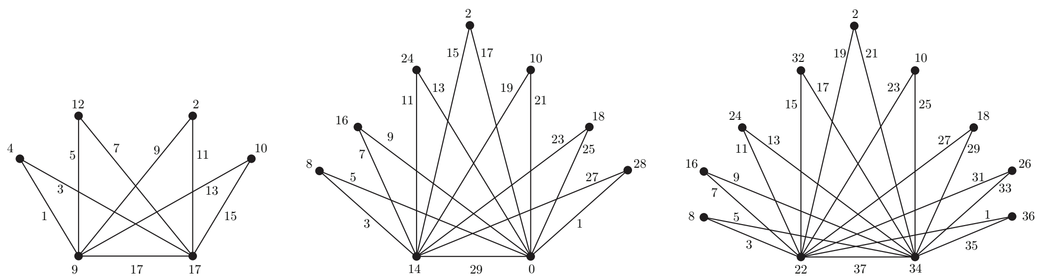
- $f(v_1v_2) = 4n + 1$;
- $f(u_iv_1) = 4i - 3$ *for* $i \in \{1, 2, 3, \ldots, n\}$; *and*
- $f(u_iv_2) = 4i - 1$ *for* $i \in \{1, 2, 3, \ldots, n\}$.

If $n \equiv 1 \pmod 4$, let

- $f(u_iv_1) = 4i - 1$ *for* $i \in \{1, 2, 3, \ldots, n - 1\}$;
- $f(u_nv_1) = 1$;
- $f(u_iv_2) = 4i + 1$ *for* $i \in \{1, 2, 3, \ldots, n - 1\}$;
- $f(u_nv_2) = 4n - 1$; *and*
- $f(v_1v_2) = 4n + 1$.

If $n \equiv 3$ *(mod 4), let*

- $f(u_iv_1) = 4i - 1$ *for* $i \in \{1, 2, 3, \ldots, n\}$*;*
- $f(u_iv_2) = 4i + 1$ *for* $i \in \{1, 2, 3, \ldots, n - 1\}$*;*
- $f(u_nv_2) = 1$*; and*
- $f(v_1v_2) = 4n + 1$.



(a) (b) (c)

Figure 2: (a) $P_2 \cdot 4K_1$, (b) $P_2 \cdot 7K_1$ and (c) $P_2 \cdot 9K_1$

Theorem 3.1. *Let* n *be an integer such that* $n \geq 2$*. The graph* $P_2 \cdot nK_1$ *is edge-odd graceful.*

Proof. Case 1 : $n \equiv 0$ or 2 (mod 4). We see from Algorithm 3.1 that (i) $f^*(u_i) = (8i - 4) \pmod{4n + 2}$ for $i \in \{1, 2, 3, \ldots, n\}$, (ii) $f^*(v_1) = (2n^2 + 3n + 1) \pmod{4n + 2}$, and (iii) $f^*(v_2) = (2n^2 + 5n + 1) \pmod{4n + 2}$. Then, $f^*(v) \in \{1, 2, 3, \ldots, 4n + 2\}$ for each $v \in V(G)$.

Next, we shall show that, for all $i \in \{1, 2, 3, \ldots, n\}$, $f^*(u_i)$ are all distinct, and also different from $f^*(v_1)$ and $f^*(v_2)$, and $f^*(v_1)$ is different from $f^*(v_2)$.

Suppose that $f^*(u_i) \equiv f^*(u_j) \pmod{4n + 2}$ for some $i, j \in \{1, 2, 3, \ldots, n\}$ and $i > j$. Then, $(4n + 2)|(i - j)$. However, $1 \leq i - j \leq n - 1$, which is a contradiction.

Suppose that $f^*(u_i) \equiv f^*(v_1) \pmod{4n + 2}$ for some $i \in \{1, 2, 3, \ldots, n\}$. Thus, $2n^2 + 3n + 5 - 8i = 2(2n + 1)t$ for some integer t. Then, $2|(2n^2 + 3n + 5 - 8i)$. Thus, $2|(3n + 5)$. However, $3n + 5$ is odd, which is a contradiction.

Suppose that $f^*(u_i) \equiv f^*(v_2) \pmod{4n + 2}$ for some $i \in \{1, 2, 3, \ldots, n\}$. Then, $2n^2 + 5n + 5 - 8i = 2(2n + 1)t$ for some integer t. Then, $2|(2n^2 + 5n + 5 - 8i)$. Thus, $2|(5n + 5)$. However, $5n + 5$ is odd, which is a contradictions.

Suppose that $f^*(v_1) \equiv f^*(v_2) \pmod{4n + 2}$. Then, $n \equiv 0 \pmod{2n + 1}$. Thus, $(2n + 1)|n$, which is a contradiction.

Case 2 : $n \equiv 1$ (mod 4). Then, $n = 4t + 1$ for some positive integer t. We see from Algorithm 3.1 that (i) $f^*(u_i) = 8i \pmod{16t + 6}$ for $i \in \{1, 2, 3, \ldots, 4t\}$, (ii) $f^*(u_{4t+1}) = 16t + 4$, (iii) $f^*(v_1) = (32t^2 + 20t + 6) \pmod{16t + 6}$, and (iv) $f^*(v_2) = (32t^2 + 44t + 8) \pmod{16t + 6}$. Then, $f^*(v) \in \{1, 2, 3, \ldots, 4n + 2\}$ for each $v \in V(G)$.

Next, we show that, for $i \in \{1, 2, 3, \ldots, 4t + 1\}$, $f^*(u_i)$ are all distinct, also different from $f^*(v_1)$ and $f^*(v_2)$, and $f^*(v_1)$ is different from $f^*(v_2)$.

Suppose that $f^*(u_i) \equiv f^*(u_j) \pmod{16t + 6}$ for some $i, j \in \{1, 2, 3, \ldots, 4t\}$ and $i > j$. Then, $4(i - j) = (8t + 3)k$ for some positive integer k. Thus, $4|(3k)$ and $4|k$. Then, $k = 4s$ for some positive integer s. That is, $(8t + 3)k \in \{32t + 12, 64t + 24, 96t + 36, \ldots\}$. However, $4 \leq 4(i - j) = (8t + 3)k \leq 16t - 4 < 32t + 12$, it is a contradiction.

Suppose that $f^*(u_i) \equiv f^*(u_{4t+1}) \pmod{16t + 6}$ for some $i \in \{1, 2, 3, \ldots, 4t\}$. Then, $4i + 1 = (8t + 3)k \in \{8t + 3, 16t + 6, 24t + 9, \ldots\}$. However, $4i + 1 \in \{5, 9, 13, \ldots, 8t + 1, 8t + 5, \ldots, 16t + 1\}$, it is a contradiction.

Suppose that $f^*(u_i) \equiv f^*(v_1) \pmod{16t+6}$ for some $i \in \{1,2,3,\ldots,4t\}$. Then, $4i \equiv 16t^2+10t+3 \pmod{8t+3}$. Since $16t^2+10t+3 \equiv 4t+3 \pmod{8t+3}$, $4i = (4t+3)k+(8t+3) \in \{12t+6, 16t+9, 20t+12, \ldots\}$. However, $4i \in \{4,8,12,\ldots,12t+4,12t+8,\ldots,16t\}$, it is a contradiction.

Since $f^*(v_1) = (32t^2+20t+6) \pmod{16t+6} = 8t+6$, $f^*(v_1)$ and $f^*(u_{4t+1})$ are distinct.

Suppose that $f^*(u_i) \equiv f^*(v_2) \pmod{16t+6}$ for some $i \in \{1,2,3,\ldots,4t\}$. Then, $4i \equiv 16t^2+22t+4 \pmod{8t+3}$. Since $16t^2+22t+4 \equiv 8t+1 \pmod{8t+3}$, $4i = (8t+3)k+(8t+1) \in \{16t+4, 24t+7, 32t+10, \ldots\}$. However, $4i \in \{4,8,12,\ldots,16t\}$, it is a contradiction.

Since $f^*(v_2) = (32t^2+44t+8) \pmod{16t+6} = 16t+2$, $f^*(v_2)$ and $f^*(u_{4t+1})$ are distinct.

Since $f^*(v_1) = 32t^2+20t+6 \pmod{16t+6} = 8t+6$ and $f^*(v_2) = (32t^2+44t+8) \pmod{16t+6} = 6t+2$, $f^*(v_1)$ and $f^*(v_2)$ are distinct.

Case 3 : $n \equiv 3 \pmod 4$. Then, $n = 4t+3$ for some positive integer t. We see from Algorithm 3.1 that (i) $f^*(u_i) = 8i \pmod{16t+14}$ for $i \in \{1,2,3,\ldots,4t+2\}$, (ii) $f^*(u_{4t+3}) = 16t+12$, (iii) $f^*(v_1) = (32t^2+68t+34) \pmod{16t+14}$, and (iv) $f^*(v_2) = (32t^2+60t+28) \pmod{16t+14}$. Then, $f^*(v) \in \{1,2,3,\ldots,4n+2\}$ for each $v \in V(G)$.

Next, we show that, for all $i \in \{1,2,3,\ldots,4t+3\}$, $f^*(u_i)$ are all distinct, and also different from $f^*(v_1)$ and $f^*(v_2)$, and $f^*(v_1)$ and $f^*(v_2)$ are distinct.

Suppose that $f^*(u_i) \equiv f^*(u_j) \pmod{16t+14}$ for some $i,j \in \{1,2,3,\ldots,4t+2\}$ and $i > j$. Then, $(8t+7)|4(i-j)$. Since $(8t+7) \nmid 4$, $(8t+7)|(i-j)$. However, $1 \le i-j \le 4t+1$, which is a contradiction.

Suppose that $f^*(u_i) \equiv f^*(u_{4t+3}) \pmod{16t+14}$ for some $i \in \{1,2,3,\ldots,4t+2\}$. Then, $4i = (8t+7)k+(8t+6) \in \{16t+13, 24t+20, 32t+29, \ldots\}$. However, $4i \in \{4,8,12,\ldots,16t+8\}$ which is a contradiction.

Suppose that $f^*(u_i) \equiv f^*(v_1) \pmod{16t+14}$ for some $i \in \{1,2,3,\ldots,4t+2\}$. Then, $4i \equiv 16t^2+34t+17 \equiv 4t+3 \pmod{8t+7}$. Then, $4i = (8t+7)k+(4t+3) \in \{12t+10, 20t+17, 28t+24, \ldots\}$. However, $4i \in \{4,8,12,\ldots,12t+8,12t+12,\ldots,16t+8\}$, which is a contradiction.

Since $f^*(v_1) = (32t^2+68t+34) \pmod{16t+14} = 8t+6$ and $f^*(u_{4t+3}) = 16t+12$, $f^*(v_1)$ and $f^*(u_{4t+3})$ are distinct.

Suppose that $f^*(u_i) \equiv f^*(v_2) \pmod{16t+14}$ for some $i \in \{1,2,3,\ldots,4t+2\}$. Then, $4i \equiv 16t^2+30t+14 \pmod{8t+7}$. Since $16t^2+30t+14 \equiv 0 \pmod{8t+7}$, $4i \equiv 0 \pmod{8t+7}$. Then, $4i = (8t+7)k \in \{8t+7, 16t+14, 24t+21, \ldots\}$. However, $4i \in \{4,8,12,\ldots,8t+4,8t+8,\ldots,16t+8\}$, which is a contradiction.

Since $f^*(v_2) = (32t^2+60t+28) \pmod{16t+14} = 0$ and $f^*(u_{4t+3}) = 16t+12$, $f^*(v_2)$ and $f^*(u_{4t+3})$ are distinct.

Since $f^*(v_1) = (32t^2+68t+34) \pmod{16t+14} = 8t+6$ and $f^*(v_2) = (32t^2+60t+28) \pmod{16t+14} = 0$, $f^*(v_1)$ and $f^*(v_2)$ are distinct.

Therefore, the edge labelings obtained from Algorithm 3.1 are edge-odd graceful labelings. Hence, G is edge-odd graceful. □

4 The cartesian product of $C_n \square P_3$ and $C_3 \square P_k$

The cartesian product of C_n and P_k, denoted by $C_n \square P_k$, is a graph obtained from k copies of C_n by joining the same vertex of the j-th copy to the $(j+1)$-th copy for $j \in \{1,2,3,\ldots,k-1\}$. We let $\{u_i^j | i \in \{1,2,3,\ldots,n\}\}$ be the set of vertices of the j-th copy of C_n where $j \in \{1,2,3,\ldots,k\}$ and the edge set is the set $\{u_i^j u_i^{j+1} | i \in \{1,2,3,\ldots,n\}$ and $j \in \{1,2,3,\ldots,k-1\}\} \cup \{u_i^j u_{i+1}^j | i \in \{1,2,3,\ldots,n-1\}$ and $j \in \{1,2,3,\ldots,k-1\}\} \cup \{u_1^j u_n^j | j \in \{1,2,3,\ldots,k-1\}\}$. Then, $|E(C_n \square P_k)| = kn + (k-1)n = (2k-1)n$ Note that, we name the vertices of each cycle in the counterclockwise direction.

Algorithm 4.1. *Let G be the graph $C_n \square P_3$ and $n \ge 3$. Define the edge labeling $f : E(G) \to \{1,3,5,\ldots,10n-1\}$ for G by*

- $f(u_1^1 u_1^2) = 1$;
- $f(u_1^2 u_1^3) = 3$;

- $f(u_i^1 u_i^2) = 4n - 4i + 5$, *for* $i \in \{2, 3, 4, \ldots, n\}$;
- $f(u_i^2 u_i^3) = 4n - 4i + 7$, *for* $i \in \{2, 3, 4, \ldots, n\}$;
- $f(u_i^2 u_{i+1}^2) = 4n + 2i - 1$, *for* $i \in \{1, 2, 3, \ldots, n-1\}$;
- $f(u_1^2 u_n^2) = 6n - 1$;
- $f(u_i^3 u_{i+1}^3) = 6n + 4i - 3$, *for* $i \in \{1, 2, 3, \ldots, n-1\}$;
- $f(u_1^3 u_n^3) = 10n - 3$;
- $f(u_i^1 u_{i+1}^1) = 6n + 4i - 1$, *for* $i \in \{1, 2, 3, \ldots, n-1\}$; *and*
- $f(u_1^1 u_n^1) = 10n - 1$.

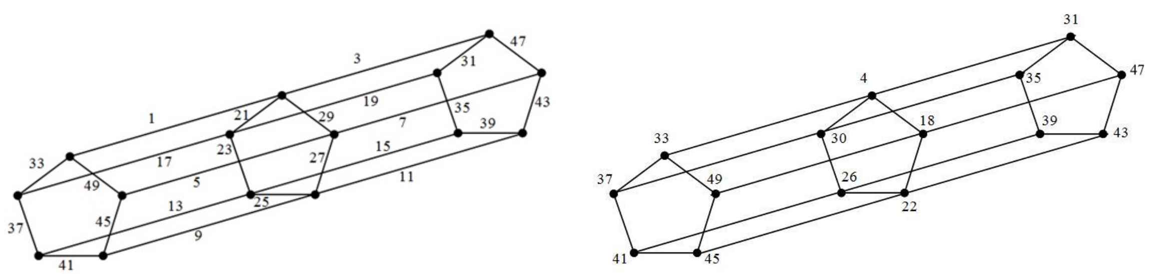


Figure 3: Edge-label and induced vertex-labels for $C_5 \square P_3$

Theorem 4.1. *Let n be an integer such that $n \geq 3$. The graph $C_n \square P_3$ is edge-odd graceful.*

Proof. Let G be the graph $C_n \square P_3$. By Algorithm 4.1, we see that (i) $f^*(u_1^1) = 6n+3$, (ii) $f^*(u_n^1) = 10n-1$, (iii) $f^*(u_i^1) = 6n + 4i - 1$ for $i \in \{2, 3, 4, \ldots, n-1\}$, (iv) $f^*(u_1^2) = 4$, (v) $f^*(u_n^2) = 2n + 8$, (vi) $f^*(u_i^2) = 6n - 4i + 8$ for $i \in \{2, 3, 4, \ldots, n-1\}$, (vii) $f^*(u_1^3) = 6n + 1$, (viii) $f^*(u_n^3) = 10n - 3$, and (ix) $f^*(u_i^3) = 6n + 4i - 3$ for $i \in \{2, 3, 4, \ldots, n-1\}$.

Next, we show that $f^*(v) \in \{1, 2, 3, \ldots, 10n\}$ for each $v \in V(G)$ and they are distinct.

Case 1 : $n = 3$. The assertion is true by direct calculation.

Case 2 : $n \geq 4$. Let $A_1 = \{f^*(u_1^1), f^*(u_n^1), f^*(u_1^2), f^*(u_n^2), f^*(u_1^3), f^*(u_n^3)\}$, $A_2 = \{6n + 4i - 1 | i \in \{2, 3, 4, ..., n-1\}\}$, $A_3 = \{6n - 4i + 8 | i \in \{2, 3, 4, ..., n-1\}\}$ and $A_4 = \{6n + 4i - 3 | i \in \{2, 3, 4, ..., n-1\}\}$. We can see that $f^*(u_1^1), f^*(u_n^1), f^*(u_1^2), f^*(u_n^2), f^*(u_1^3)$ and $f^*(u_n^3)$ are all distinct.

Since $4 < 2n + 8 < 2n + 12 (= \min A_3) < 6n (= \max A_3) < 6n + 1 < 6n + 3 < 6n + 5 (= \min A_4) < 6n + 7 (= \min A_2) < 10n - 7 (= \max A_4) < 10n - 5 (= \max A_2) < 10n - 3 < 10n - 1$, $A_1 \cap A_j = \varnothing$ for $j \in \{2, 3, 4\}$.

Next, we notice that all elements in A_3 are even, while all elements in A_2 and A_4 are odd. Thus, $A_2 \cap A_3$ and $A_3 \cap A_4$ are empty. Finally, let $a \in A_2$ and $b \in A_4$, i.e., $a = 6n + 4i - 1$ and $b = 6n + 4l - 3$ for some i and $l \in \{2, 3, 4, ..., n-1\}$. Assume that $a = b$. We have $4(l - i) = 2$, which leads to a contradiction. That is, $A_2 \cap A_4 = \varnothing$.

Therefore, f defined by Algorithm 4.1 is an edge-odd graceful labeling for $C_n \square P_3$. □

Algorithm 4.2. *Let G denote the graph $C_3 \square P_k$ where $k \geq 4$. Define the edge labeling $f : E(C_3 \square P_k) \to \{1, 3, 5, \ldots, 12k - 7\}$ for G as follow.*

If $k \equiv 0 \pmod 4$, let

- $f(u_1^i u_1^{i+1}) = 2i - 1$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_2^i u_2^{i+1}) = 2k + 2i - 3$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_3^i u_3^{i+1}) = 4k + 2i - 5$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;

- $f(u_1^i u_2^i) = 6k + 2i - 7$, *for* $i \in \{1, 2, 3, \ldots, k\}$;
- $f(u_2^i u_3^i) = 8k + 2i - 7$, *for* $i \in \{1, 2, 3, \ldots, k\}$; *and*
- $f(u_1^i u_3^i) = 10k + 2i - 7$, *for* $i \in \{1, 2, 3, \ldots, k\}$.

If $k \equiv 1 \pmod 4$, *let*

- $f(u_1^i u_1^{i+1}) = 6i - 5$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_2^i u_2^{i+1}) = 6i - 3$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_3^i u_3^{i+1}) = 6i - 1$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_1^i u_2^i) = 8k - 2i - 5$, *for* $i \in \{1, 2, 3, \ldots, k\}$;
- $f(u_2^i u_3^i) = 12k - 2i - 5$, *for* $i \in \{1, 2, 3, \ldots, k\}$; *and*
- $f(u_1^i u_3^i) = 10k - 2i - 5$, *for* $i \in \{1, 2, 3, \ldots, k\}$.

If $k \equiv 2 \pmod 4$, *let*

- $f(u_1^i u_1^{i+1}) = 6k + 2i - 1$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_2^i u_2^{i+1}) = 8k + 2i - 3$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_3^i u_3^{i+1}) = 10k + 2i - 5$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_1^i u_2^i) = 6i - 5$, *for* $i \in \{1, 2, 3, \ldots, k\}$;
- $f(u_2^i u_3^i) = 6i - 1$, *for* $i \in \{1, 2, 3, \ldots, k\}$; *and*
- $f(u_1^i u_3^i) = 6i - 3$, *for* $i \in \{1, 2, 3, \ldots, k\}$.

If $k \equiv 3 \pmod 4$, *let*

- $f(u_1^i u_1^{i+1}) = 2i - 1$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_2^i u_2^{i+1}) = 2k + 2i - 3$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_3^i u_3^{i+1}) = 4k + 2i - 5$, *for* $i \in \{1, 2, 3, \ldots, k-1\}$;
- $f(u_1^i u_2^i) = 12k - 6i - 5$, *for* $i \in \{1, 2, 3, \ldots, k\}$;
- $f(u_2^i u_3^i) = 12k - 6i - 1$, *for* $i \in \{1, 2, 3, \ldots, k\}$; *and*
- $f(u_1^i u_3^i) = 12k - 6i - 3$, *for* $i \in \{1, 2, 3, \ldots, k\}$.

Theorem 4.2. *Let* k *be an integer such that* $k \geq 4$. *The graph* $C_3 \square P_k$ *is edge-odd graceful.*

Proof. Let G denote the graph $C_3 \square P_k$.

Case 1 : $k \equiv 0 \pmod 4$. By Algorithm 4.2, we have (i) $f^*(u_1^1) = 4k - 3$; (ii) $f^*(u_2^1) = 4k - 5$; (iii) $f^*(u_3^1) = 10k - 7$; (iv) $f^*(u_1^k) = 10k - 11$; (v) $f^*(u_2^k) = 10k - 13$; (vi) $f^*(u_3^k) = 4k - 9$; (vii) $f^*(u_1^i) = 4k + 8i - 12$ for $i \in \{2, 3, 4, \ldots, k-1\}$; (viii) $f^*(u_2^i) = 6k + 8i - 16 \pmod{12k - 6}$ for $i \in \{2, 3, 4, \ldots, k-1\}$; and (ix) $f^*(u_3^i) = 2k + 8i - 14$ for $i \in \{2, 3, 4, \ldots, k-1\}$.

Next, we show that $f^*(v) \in \{1, 2, 3, \ldots, 12k - 6\}$ for each $v \in V(G)$ and they are distinct.

Let $B_1 = \{f^*(u_1^1), f^*(u_2^1), f^*(u_3^1), f^*(u_1^k), f^*(u_2^k), f^*(u_3^k)\}$, $B_2 = \{4k + 8i - 12 | i \in \{2, 3, 4, ..., k-1\}\}$, $B_3 = \{(6k + 8i - 16) \pmod{12k - 6} \, | i \in \{2, 3, 4, ..., k-1\}\}$ and $B_4 = \{2k + 8i - 14 | i \in \{2, 3, 4, ..., k-1\}\}$. We can see that $f^*(u_1^1)$, $f^*(u_2^1)$, $f^*(u_3^1)$, $f^*(u_1^k)$, $f^*(u_2^k)$ and $f^*(u_3^k)$ are all distinct.

We notice that all elements in B_1 are odd, while all elements in B_2, B_3 and B_4 are even, we conclude that $B_1 \cap B_j = \varnothing$ for $j \in \{2, 3, 4\}$. Since $k \equiv 0 \pmod 4$, $k = 4m$ for some $m \in \mathbb{N}$. Then, $B_2 = \{8(2m + i - 2) + 4 | i \in \{2, 3, 4, \ldots, 4m - 1\}\}$ and $B_4 = \{8(m + i - 2) + 2 | i \in \{2, 3, 4, ..., 4m - 1\}\}$. Consider $B_3 = \{(24m + 8i - 16) \pmod{48m - 6} | i \in \{2, 3, 4, \ldots, 4m - 1\}\} = \{8(3m + i - 2) | i \in \{2, 3, 4, \ldots, 3m + 1\}\} \cup \{(24m + 8i - 16) \pmod{48m - 6} | i \in \{3m + 2, 3m + 3, 3m + 4, \ldots, 4m - 1\}\} = B_{31} \cup \{6, 8(1) + 6, 8(2) + 6, \ldots, 8(m - 5) + 6, 8(m - 4) + 6, 8(m - 3) + 6\} = B_{31} \cup B_{32}$.

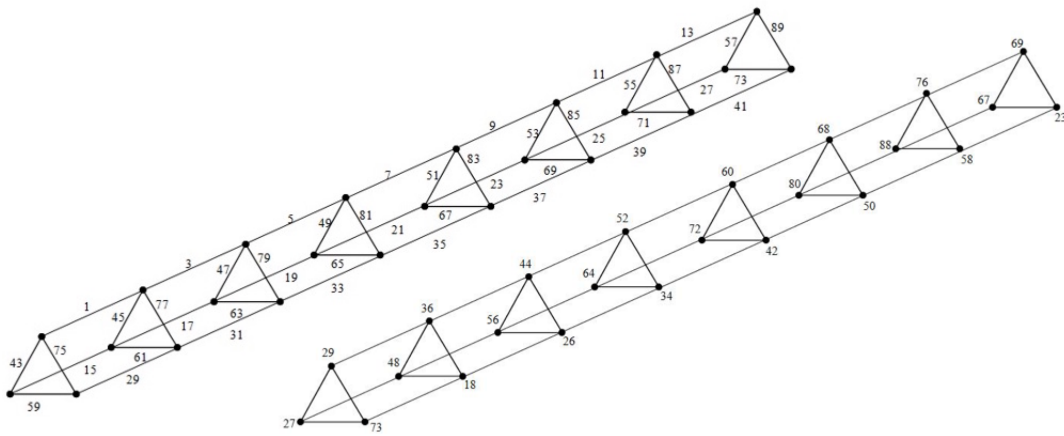


Figure 4: Edge-label and induced vertex-labels for $C_3 \square P_8$

Notice that elements in B_2, B_{31}, B_{32} and B_4 are arithmetic progression with common difference 8, we also can see that each element in B_2, B_{31}, B_{32} and B_4 is congruent to $4, 0, 6$ and 2 modulo 8, respectively. Thus, B_2, B_3 and B_4 are distinct.

Case 2 : $k \equiv 1 \pmod 4$. By Algorithm 4.2, $f^*(u_1^1) = 6k - 7$; $f^*(u_2^1) = 8k - 5$; $f^*(u_3^1) = 10k - 3$; $f^*(u_1^k) = 8k - 15$; $f^*(u_2^k) = 10k - 13$; $f^*(u_3^k) = 12k - 11$; $f^*(u_1^i) = (6k + 8i - 20) \pmod{12k - 6}$; $f^*(u_2^i) = (8k+8i-16) \pmod{12k-6}$; and $f^*(u_3^i) = (10k+8i-12) \pmod{12k-6}$ for $i \in \{2, 3, 4, ..., k-1\}$.

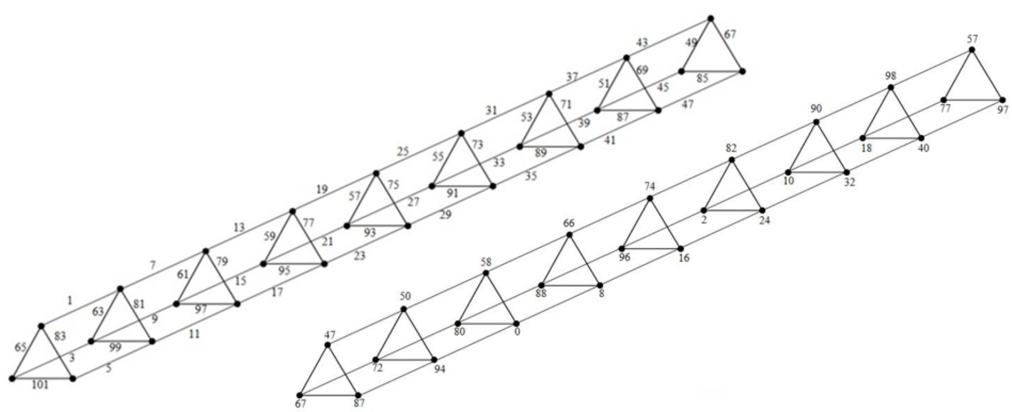


Figure 5: Edge-label and induced vertex-labels for $C_3 \square P_9$

Next, we show that $f^*(v) \in \{1, 2, 3, \ldots, 12k - 6\}$ for each $v \in V(G)$ and they are distinct. Let $B_1 = \{f^*(u_1^1), f^*(u_2^1), f^*(u_3^1), f^*(u_1^k), f^*(u_2^k), f^*(u_3^k)\}$, $B_2 = \{(6k+8i-20) \pmod{12k-6} | i \in \{2, 3, 4, ..., k-1\}\}$, $B_3 = \{(8k + 8i - 16) \pmod{12k - 6} | i \in \{2, 3, 4, ..., k - 1\}\}$ and $B_4 = \{(10k + 8i - 12) \pmod{12k - 6} | i \in \{2, 3, 4, ..., k - 1\}\}$. We can see that $f^*(u_1^1)$, $f^*(u_2^1)$, $f^*(u_3^1)$, $f^*(u_1^k)$, $f^*(u_2^k)$ and $f^*(u_3^k)$ are all distinct.

We notice that all elements in B_1 are odd, while all elements in B_2, B_3 and B_4 are even, we conclude that $B_1 \cap B_j = \varnothing$ for $j \in \{2, 3, 4\}$. Since $k \equiv 1 \pmod 4$, $k = 4m + 1$ for some $m \in \mathbb{N}$.

If $m = 1$, then $B_2 = \{26, 34, 42\}, B_3 = \{40, 48, 2\}$ and $B_4 = \{0, 8, 16\}$.

If $m = 2$, then $B_2 = \{50, 58, 66, 74, 82, 90, 98\}, B_3 = \{72, 80, 88, 96, 2, 10, 18\}$ and $B_4 = \{94, 0, 8, 16, 24, 32, 40\}$.

If $m \geq 3$, then $B_2 = \{8(3m+i-2)+2 | i \in \{2,3,4,\dots,3m+2\}\} \cup \{(24m+8i-14) \pmod{48m+6} | i \in \{3m+3, 3m+4, 3m+5, \dots, 4m\}\} = B_{21} \cup \{4, 8(1)+4, 8(2)+4, \dots, 8(m-5)+4, 8(m-4)+4, 8(m-3)+4\} = B_{21} \cup B_{22}$. $B_3 = \{8(4m+i-1) | i \in \{2,3,4,...,2m+1\}\} \cup \{(32m+8i-8) \pmod{48m+6} | i \in \{2m+2, 2m+3, 2m+4, \dots, 4m\}\} = B_{31} \cup \{2, 8(1)+2, 8(2)+2, \dots, 8(2m-4)+2, 8(2m-3)+2, 8(2m-2)+2\} = B_{31} \cup B_{32}$. $B_4 = \{8(5m+i-1)+6 | i \in \{2,3,4,\dots,m\}\} \cup \{(40m+8i-2) \pmod{48m+6} | i \in \{m+1, m+2, m+3, \dots, 4m\}\} = B_{41} \cup \{0, 8(1), 8(2), \dots, 8(3m-3), 8(3m-2), 8(3m-1)\} = B_{41} \cup B_{42}$.

Notice that each element in B_{21} and B_{32} is congruent to 2 modulo 8. However, $\min B_{21} = 24m+2 > 16m-14 = \max B_{32}$. Similarly, each element in B_{31} and B_{42} is congruent to 0 modulo 8. However, $\min B_{31} = 32m+8 > 24m-8 = \max B_{42}$. Finally, each element in B_{22} and B_{41} is congruent to 4 and 6 modulo 8, respectively.

Thus, for all $m \geq 1$, B_2, B_3 and B_4 are all distinct.

Case 3 : $k \equiv 2 \pmod 4$. By Algorithm 4.2, $f^*(u_1^1) = 6k+5$; $f^*(u_2^1) = 8k+5$; $f^*(u_3^1) = 10k+5$; $f^*(u_1^k) = 8k-5$; $f^*(u_2^k) = 10k-5$; $f^*(u_3^k) = 1$; $f^*(u_1^i) = (16i-6) \pmod{12k-6}$; $f^*(u_2^i) = (4k+16i-8) \pmod{12k-6}$; $f^*(u_3^i) = (8k+16i-10) \pmod{12k-6}$ for $i \in \{2,3,4,\dots,k-1\}$.

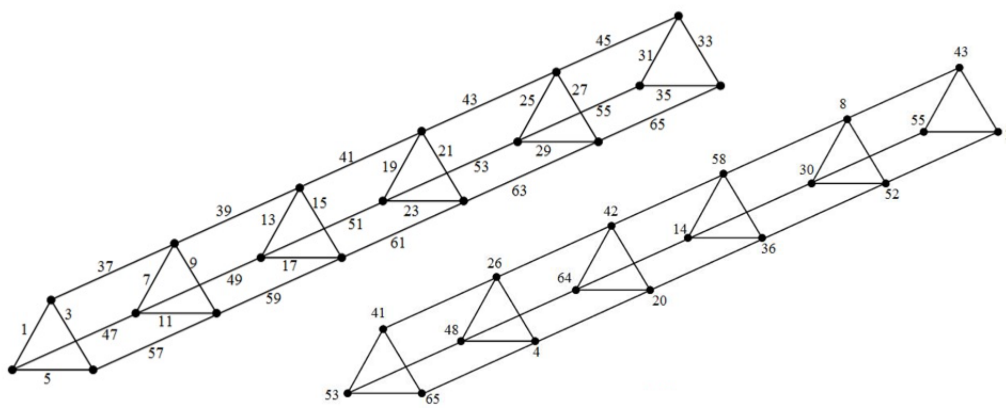


Figure 6: Edge-label and induced vertex-labels for $C_3 \square P_6$

Next, we show that $f^*(v) \in \{1,2,3,\dots,12k-6\}$ for each $v \in V(G)$ and they are distinct. Let $B_1 = \{f^*(u_1^1), f^*(u_2^1), f^*(u_3^1), f^*(u_1^k), f^*(u_2^k), f^*(u_3^k)\}$, $B_2 = \{(16i-6) \pmod{12k-6} | i \in \{2,3,4,...,k-1\}\}$, $B_3 = \{(4k+16i-8) \pmod{12k-6} | i \in \{2,3,4,\dots,k-1\}\}$ and $B_4 = \{(8k+16i-10) \pmod{12k-6} | i \in \{2,3,4,\dots,k-1\}\}$. Then, $f^*(u_1^1), f^*(u_2^1), f^*(u_3^1), f^*(u_1^k), f^*(u_2^k)$ and $f^*(u_3^k)$ are all distinct.

We notice that all elements in B_1 are odd, while all elements in B_2, B_3 and B_4 are even, we conclude that $B_1 \cap B_j = \varnothing$ for $j \in \{2,3,4\}$. Since $k \equiv 2 \pmod 4$, $k = 4m+2$ for some $m \in \mathbb{N}$.

If $m = 1$, then $B_2 = \{26, 42, 58, 8\}$, $B_3 = \{48, 64, 14, 30\}$ and $B_4 = \{4, 20, 36, 52\}$.

If $m \geq 2$, then $B_2 = \{(16i-6) \pmod{48m+18} | i \in \{2,3,4,\dots,4m+1\}\} = \{8(2i-1)+2 | i \in \{2,3,4,\dots,3m+1\}\} \cup \{(16i-6) \pmod{48m+18} | i \in \{3m+2, 3m+3, 3m+4, \dots, 4m+1\}\} = B_{21} \cup \{8(1), 8(3), 8(5), \dots, 8(2m-5), 8(2m-3), 8(2m-1)\} = B_{21} \cup B_{22}$. $B_3 = \{(16m+16i) \pmod{48m+18} | i \in \{2,3,4,\dots,4m+1\}\} = \{8(2m+2i) | i \in \{2,3,4,\dots,2m+1\}\} \cup \{(16m+16i) \pmod{48m+18} | i \in \{2m+2, 2m+3, 2m+4, \dots, 4m+1\}\} = B_{31} \cup \{8(1)+6, 8(3)+6, 8(5)+6, \dots, 8(4m-5)+6, 8(4m-3)+6, 8(4m-1)+6\} = B_{31} \cup B_{32}$. $B_4 = \{(32m+16i+6) \pmod{48m+18} | i \in \{2,3,4,...,4m+1\}\} = \{8(4m+2i)+6 | i \in \{2,3,4,\dots,m\}\} \cup \{(32m+16i+6) \pmod{48m+18} | i \in \{m+1, m+2, m+3, \dots, 4m+1\}\} = B_{41} \cup \{4, 8(2)+4, 8(4)+4, \dots, 8(6m-4)+4, 8(6m-2)+4, 8(6m)+4\} = B_{41} \cup B_{42}$.

Notice that each element in B_{22} and B_{31} is congruent to 0 modulo 8. However, $\max B_{22} = 16m-8 < 16m+32 = \min B_{31}$. Similarly, each element in B_{32} and B_{41} is congruent to 6 modulo 8. However, $\max B_{32} = 32m-2 < 32m+38 = \min B_{41}$. Finally, each element in B_{21} and B_{42} is congruent to 2 and 4 modulo 8, respectively.

Thus, for all $m \geq 1, B_2, B_3$ and B_4 are distinct.

Case 4 : $k \equiv 3 \pmod 4$. By Algorithm 4.2, $f^*(u_1^1) = 12k - 13$; $f^*(u_2^1) = 2k - 7$; $f^*(u_3^1) = 4k - 7$; $f^*(u_1^k) = 2k - 5$; $f^*(u_2^k) = 4k - 5$; $f^*(u_3^k) = 6k - 5$; $f^*(u_1^i) = 12k - 8i - 6$; $f^*(u_2^i) = (16k - 8i - 8) \pmod{12k - 6}$; $f^*(u_3^i) = 8k - 8i - 4$ for $i \in \{2, 3, 4, \ldots, k - 1\}$.

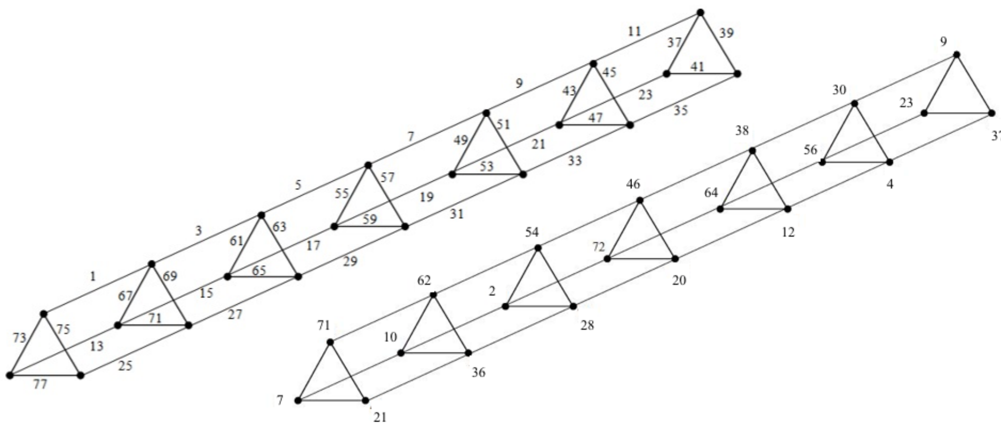


Figure 7: Edge-label and induced vertex-labels for $C_3 \square P_7$

Next, we show that $f^*(v) \in \{1, 2, 3, \ldots, 12k - 6\}$ for each $v \in V(G)$ and they are distinct. Let $B_1 = \{f^*(u_1^1), f^*(u_2^1), f^*(u_3^1), f^*(u_1^k), f^*(u_2^k), f^*(u_3^k)\}, B_2 = \{12k - 8i - 6 | i \in \{2, 3, 4, \ldots, k - 1\}\}, B_3 = \{(16k - 8i - 8) (\mathrm{mod}\ 12k - 6) | i \in \{2, 3, 4, \ldots, k - 1\}\}$ and $B_4 = \{8k - 8i - 4 | i \in \{2, 3, 4, \ldots, k - 1\}\}$. We can see that $f^*(u_1^1)$, $f^*(u_2^1)$, $f^*(u_3^1)$, $f^*(u_1^k)$, $f^*(u_2^k)$ and $f^*(u_3^k)$ are all distinct.

We notice that all elements in B_1 are odd, while all elements in B_2, B_3 and B_4 are even, we conclude that $B_1 \cap B_j = \varnothing$ for $j \in \{2, 3, 4\}$. Since $k \equiv 3 (\mathrm{mod}\ 4)$, $k = 4m + 3$ for some $m \in \mathbb{N}$. Then, $B_2 = \{8(6m - i + 3) + 6 | i \in \{2, 3, 4, \ldots, 4m + 2\}\}$ and $B_4 = \{8(4m - i + 2) + 4 | i \in \{2, 3, 4, \ldots, 4m + 2\}\}$. Consider $B_3 = \{(64m - 8i + 40) (\mathrm{mod}\ 48m + 30) | i \in \{2, 3, 4, \ldots, 4m + 2\}\} = \{(64m - 8i + 40) (\mathrm{mod}\ 48m + 30) | i \in \{2, 3, 4, \ldots, 2m + 1\}\} \cup \{8(8m - i + 5) | i \in \{2m + 2, 2m + 3, 2m + 4, \ldots, 4m + 2\}\} = \{8(2m - 1) + 2, 8(2m - 2) + 2, 8(2m - 3) + 2, \ldots, 8(2) + 2, 8(1) + 2, 2\} \cup B_{32} = B_{31} \cup B_{32}$.

Notice that elements in B_2, B_{31}, B_{32} and B_4 are arithmetic progression with common difference 8. We also can see that each element in B_2, B_{31}, B_{32} and B_4 is congruent to 6, 2, 0 and 4 modulo 8, respectively. Thus, B_2, B_3 and B_4 are distinct.

Therefore, f defined by Algorithm 4.2 are edge-odd graceful labelings for $C_3 \square P_k$ for any integer k such that $k \geq 4$. □

5 Union of even copies of edge-odd graceful graphs

In this section, we recall that the graph nG is a graph consisting of n copies of G.

Theorem 5.1. *Let $k \geq 1$ and G be an edge-odd graceful graph such that deg v is odd for all $v \in V(G)$. Then, the graph $2kG$ is edge-odd graceful.*

Proof. Let us denote a graph G of q edges by G_1 and $G_2, G_3, G_4, \ldots, G_{2k}$ be its copy.

Suppose that G_1 is edge-odd graceful with edge-odd graceful labeling f together with its induced mapping f^*. Define $g : E(2kG) \to \{1, 3, 5, \ldots, 4kq - 1\}$ by

$g(e) = f(e) + (2i - 2)q$ for $e \in E(G_i)$ and $i \in \{1, 2, 3, \ldots, 2k\}$.

Since f is an injection and $f(E(G_1)) = \{1, 3, 5, \ldots, 2q - 1\}$, g is an injection and $g(E(2kG)) = \{1, 3, 5, \ldots, 4kq - 1\}$. Thus, g is a bijection. Next, consider the induced mapping g^* of g which is

$g^*(v) = f^*(v) + (2i - 2)q \deg v \pmod{4kq}$ for $v \in V(G_i)$ and $i \in \{1, 2, 3, \ldots, 2k\}$.

Let v_1 and v_2 be two vertices of $2kG$.

Case 1 : v_1 and v_2 are in the same copy. Assume that they are in G_i for some $i \in \{1, 2, 3, \ldots, 2k\}$ and $g^*(v_1) = g^*(v_2)$. Then, $f^*(v_1) + (2i-2)q \deg v_1 \equiv f^*(v_2) + (2i-2)q \deg v_2 \pmod{4kq}$. Since $2q|4kq$, we have $2q|(f^*(v_1) - f^*(v_2) + (2i-2)q(\deg v_1 - \deg v_2))$. Since $2q|(2i-2)q$ for $1 \leq i \leq 2k$, $f^*(v_1) \equiv f^*(v_2) \pmod{2q}$ which contradicts with the property of f^*.

Case 2 : v_1 and v_2 are in the different copy. Assume that $g^*(v_1) = g^*(v_2)$. Then, without loss of generality, let $v_1 \in V(G_i)$ and $v_2 \in V(G_j)$ for some $1 \leq i < j \leq 2k$.

If v_2 is the copy of v_1, then $f^*(v_1) + (2i-2)q \deg v_1 \equiv f^*(v_1) + (2j-2)q \deg v_1 \pmod{4kq}$. Hence, $4kq|2(j-i)q \deg v_1$. Since $j - i < 2k$, $2|\deg v_1$, which is a contradiction.

If v_2 is not the copy of v_1, then $f^*(v_1) + (2i-2)q \deg v_1 \equiv f^*(v_2) + (2j-2)q \deg v_2 \pmod{4kq}$. Hence, there is an integer t such that $f^*(v_1) - f^*(v_2) = 2q(2kt + (j-i)(\deg v_2 - \deg v_1)$. That is, $f^*(v_1) \equiv f^*(v_2) \pmod{2q}$, which is a contradiction.

Therefore, from all the cases, we can conclude that for each $v \in V(2kG)$, the induced mapping g^* of g are all distinct. Hence, $2kG$ is edge-odd graceful. □

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