**T**hai **J**ournal of **M**athematics : (2020) 1–13
Special Issue: The $14^{th}$ IMT-GT ICMSA 2018
**http://thaijmath.in.cmu.ac.th**
Online ISSN 1686-0209

# Closed Knight's Tour on $(m, m, r)$-Ringboards

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**Abstract :** It is well known that a *legal knight's move* is the resulting of moving two squares horizontally or vertically on the board and then turning and moving one square in the perpendicular direction. That is, if we start at $(i, j)$, then the knight can move to one of eight squares: $(i \pm 2, j \pm 1)$ or $(i \pm 1, j \pm 2)$ (if exist). A *closed knight's tour* is a legal knight's move that visit every squares on a given board exactly once and return to its starting position. A closed knight's tour over a rectangular board or a three-dimensional cube have been studied widely. Some researchers turn their attention to investigate a closed knight's tour over a ring board of width $r$, $(m, n, r)$-ringboard. For $m, n > 2r$, the $(m, n, r)$-*ringboard* is defined to be an $m \times n$ chessboard with the middle part missing and the rim contains $r$ rows and $r$ columns. In this paper, we give necessary and sufficient conditions for the $(m, m, r)$-ringboard to have a closed knight's tour.

**Keywords :** knight's move; closed knight's tour; open knight's tour; Hamiltonian cycle.
**2010 Mathematics Subject Classification :** 05C45; 00A08.

## 1 Introduction and Preliminaries

The knight's tour problem on a chessboard is an interesting mathematical problem as you can see some of them listed in [1]. Each square of the $m \times n$ chessboard is labeled by $(i, j)$ in the matrix fashion. A *legal knight's move* is the resulting of moving two squares horizontally or vertically on the board and then turning and moving one square in the perpendicular direction. That is, if we start at $(i, j)$, then the knight can move to one of eight squares: $(i \pm 2, j \pm 1)$ or $(i \pm 1, j \pm 2)$ (if exist). A *closed knight's tour* is a legal knight's move that visit every squares on a given chessboard exactly once and return to its starting position. A closed knight's tour over a rectangular board or a three-dimensional cube have been studied widely.

In 1991, Schwenk [2] obtained necessary and sufficient conditions for the existence of a closed knight's tour for the $m \times n$ chessboard as follows.

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**Theorem 1.1.** [2] *An $m \times n$ chessboard with $m \leq n$ admits a closed knight's tour unless one or more of the following conditions holds:*

*(a)* $m$ *and* $n$ *are both odd;*

*(b)* $m = 1, 2$ *or* $4$*; or*

*(c)* $m = 3$ *and* $n = 4, 6$ *or* $8$.

Sometime, instead of finding a closed knight's tour, we can also find an open knight's tour on a chessboard. An *open knight's tour* is a legal knight's move that visit every squares on a given chessboard exactly once and the starting and terminating positions are different. In 2005, Chia and Ong [3] obtained necessary and sufficient conditions for the existence of an open knight's tour for the $m \times n$ chessboard as follows.

**Theorem 1.2.** [3] *An $m \times n$ chessboard with $m \leq n$ admits an open knight's tour unless one or more of the following conditions holds:*

*(a)* $m = 1$ *or* $2$*;*

*(b)* $m = 3$ *and* $n = 3, 5, 6$*; or*

*(c)* $m = 4$ *and* $n = 4$.

Some researchers turn their attention to investigate a closed knight's tour over a ring board of width $r$ or an $(m, n, r)$-ringboard. For $m, n > 2r$, the $(m, n, r)$-*ringboard* is defined to be an $m \times n$ chessboard with the middle part missing and the rim contains $r$ rows and $r$ columns. In 1996, Wiitala [4] showed that the $(m, m, 2)$-ringboard contains no closed knight's tour.

The knight's tour problem on the $(m, n, r)$-ringboard can be converted to a certain graph problem. If we regard each square of the $(m, n, r)$-ringboard as a vertex, then a graph $G$ represented the $(m, n, r)$-ringboard is a graph with $2r(m + n - 2r)$ vertices and two vertices are joined by an edge whenever the knight can be moved from one square to another. Then, a closed knight's tour is a Hamiltonian cycle in $G$.

In this paper, we extend the result of [4] by providing necessary and sufficient conditions for the $(m, m, r)$-ringboard to have a closed knight's tour where $m \geq 3$ and $m > 2r$.

## 2 Construction of open knight's tours

The open knight's tour of Chia and Ong [3] cannot be used directly with our construction of a closed knight's tour over the $(m, m, r)$-ringboard. Thus, this section gives our own construction of the open knight's tours that we can apply further.

**Lemma 2.1.** *(i) A $3 \times 4t$ chessboard contains an open knight's tour which begins with $(3, 1)$ and ends with $(1, 4t)$ when $t \in \mathbb{N}$.*
*(ii) A $3 \times n$ chessboard contains an open knight's tour which begins with $(2, 2)$ and ends with $(1, n)$ when $n$ is odd and $n \geq 7$.*

*Proof.* The required open knight's tours for $3 \times 7$ and $3 \times 9$ chessboards are shown in Figure 1.

Next, we construct an open knight's tour on the $3 \times 4$ chessboard which begins with $(3, 1)$ and ends with $(1, 4)$ as shown in Figure 2.

For $n = 4t$ where $t \geq 2$, the $3 \times n$ chessboard is obtained from $t$ copies of $3 \times 4$ chessboard. We can construct an open knight's tour for $3 \times 4t$ chessboard by joining $(1, 4)$ of the $i$th $3 \times 4$ chessboard to $(3, 1)$ of the $(i + 1)$th $3 \times 4$ chessboard, $i \in \{1, 2, 3, ..., t - 1\}$, as shown in Figure 3. Note that the obtained open knight's tour for the $3 \times 4t$ chessboard starts at $(3, 1)$ and ends at $(1, 4t)$.

Finally, for $n \geq 11$ and $n \equiv 2k + 1 \pmod 4$ where $k \in \{1, 2\}$, we connect the $3 \times (7 + 2(k - 1))$ chessboard with the $3 \times 4t$ chessboard, $t \geq 1$, and join $(1, 7 + 2(k - 1))$ of the $3 \times (7 + 2(k - 1))$ chessboard to $(3, 1)$ of the $3 \times 4t$ chessboard as shown in Figures 4 and 5.

□

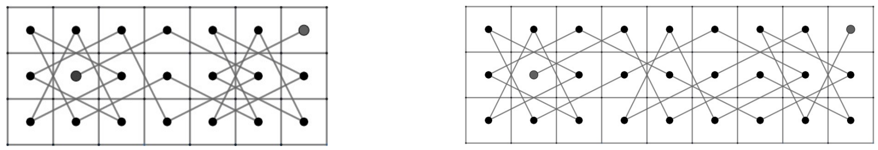


Figure 1: The required open knight's tours for $3 \times 7$ and $3 \times 9$ chessboards

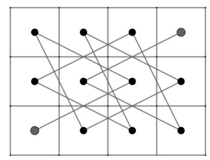


Figure 2: Our constructed open knight's tour on the $3 \times 4$ chessboard

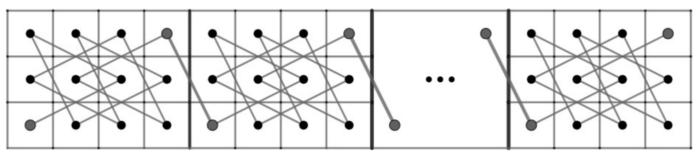


Figure 3: The obtained open knight's tour for $3 \times 4t$ chessboard

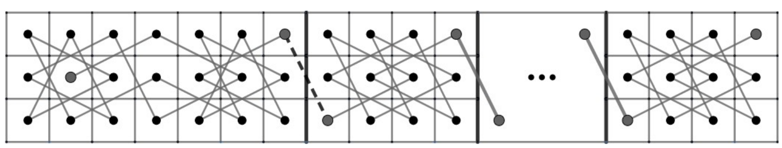


Figure 4: The required open knight's tour for $3 \times n$ chessboard where $n \equiv 3 \pmod 4$

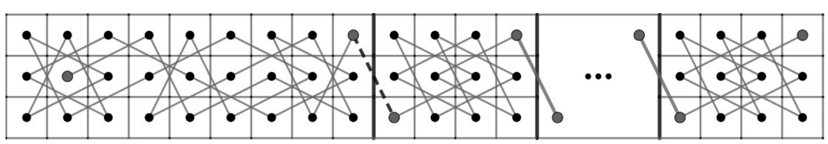


Figure 5: The required open knight's tour for $3 \times n$ chessboard where $n \equiv 1 \pmod 4$

**Lemma 2.2.** *A $4 \times n$ chessboard contains an open knight's tour which begins with $(4, 2)$ and ends with $(1, n-1)$ when $n$ is odd and $n \geq 3$.*

*Proof.* The required open knight's tours for $4 \times 3$, $4 \times 5$ and $4 \times 7$ chessboards are shown in Figure 6.

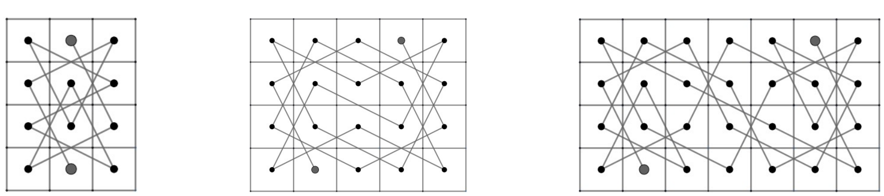


Figure 6: The required open knight's tours for $4 \times 3$, $4 \times 5$ and $4 \times 7$ chessboards

Next, we construct a path $P$ (dash line) and a cycle $C$ (solid line) on the $4 \times 6$ chessboard as shown in Figure 7.

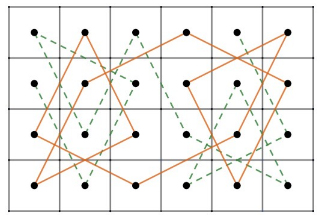


Figure 7: A path $P$ and a cycle $C$ on the $4 \times 6$ chessboard

Then, we construct a path $P'$ and a cycle $C'$ on the $4 \times n$ chessboard where $n \equiv 0 \pmod 6$ and $n \geq 12$ as follow.

For $n = 6t$, where $t \geq 2$, $4 \times n$ chessboard is obtained from $t$ copies of $4 \times 6$ chessboard. We can construct the path $P'$ on the $4 \times n$ chessboard by joining $(1,5)$ of the $i$th $4 \times 6$ chessboard to $(2,1)$ of the $(i+1)$th $4 \times 6$ chessboard for all $i \in \{1, 2, 3, ..., t-1\}$. Next, we can construct the cycle $C'$ on the $4 \times n$ chessboard by

(i) deleting the edge $(1,6) - (3,5)$ of the $i$th $4 \times 6$ chessboard and $(2,2) - (4,1)$ of the $(i+1)$th $4 \times 6$ chessboard and

(ii) joining $(1,6)$ of the $i$th $4 \times 6$ chessboard to $(2,2)$ of the $(i+1)$th $4 \times 6$ chessboard and joining $(3,5)$ of the $i$th $4 \times 6$ chessboard to $(4,1)$ of the $i$th $4 \times 6$ chessboard for all $i \in \{1, 2, 3, ..., t-1\}$.

The path $P'$ and cycle $C'$ on the $4 \times 6t$ chessboard is shown in Figure 8.

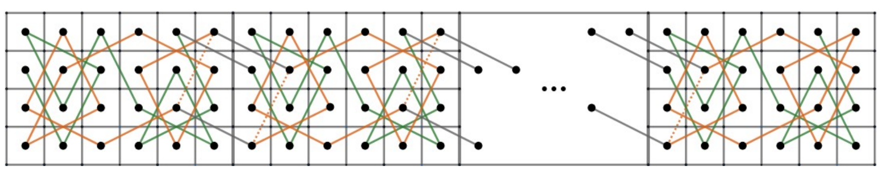


Figure 8: The path $P'$ and cycle $C'$ on the $4 \times 6t$ chessboard

Finally, for $n \geq 9$ and $n \equiv 2k + 1 \pmod 6$ where $k \in \{1, 2, 3\}$, we connect the $4 \times (3 + 2(k-1))$ chessboard with the $4 \times 6t$ chessboard, $t \geq 1$, and delete the edge $(1, 3+2(k-1)) - (3, 2+2(k-1))$ of the $4 \times (3+2(k-1))$ chessboard and $(2,2) - (4,1)$ of the $4 \times 6t$ chessboard and join $(1, 2+2(k-1))$, $(1, 3+2(k-1))$, $(3, 2+2(k-1))$ of the $4 \times (3+2(k-1))$ chessboard to $(2,1)$, $(2,2)$, $(4,1)$ of the $4 \times 6t$ chessboard, respectively, as shown in Figures 9, 10 and 11.

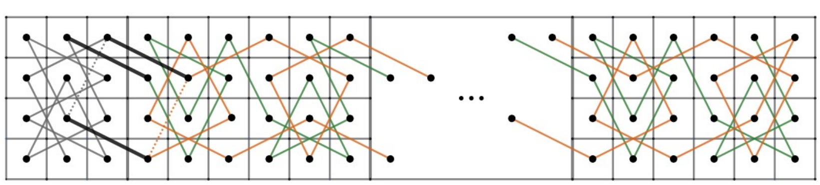


Figure 9: The required open knight's tour for $4 \times n$ chessboard where $n \equiv 3 \pmod 6$

□

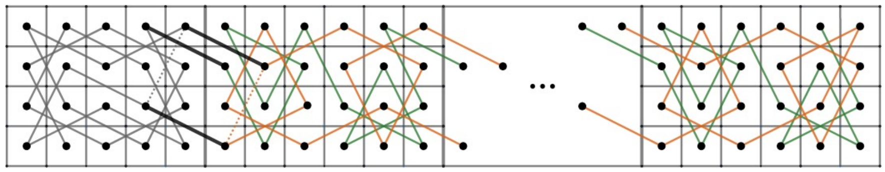


Figure 10: The required open knight's tour for $4 \times n$ chessboard where $n \equiv 5 \pmod 6$

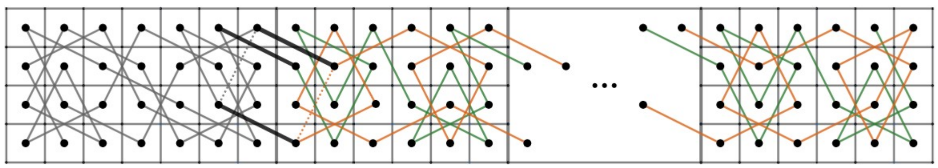


Figure 11: The required open knight's tour for $4 \times n$ chessboard where $n \equiv 1 \pmod 6$

**Lemma 2.3.** *An $4 \times n$ chessboard contains an open knight's tour which begins with $(4, 1)$ and ends with $(4, 2)$ when $n$ is even and $n \geq 6$.*

*Proof.* The required open knight's tours for $4 \times 6$, $4 \times 8$ and $4 \times 10$ chessboards are shown in Figure 12.

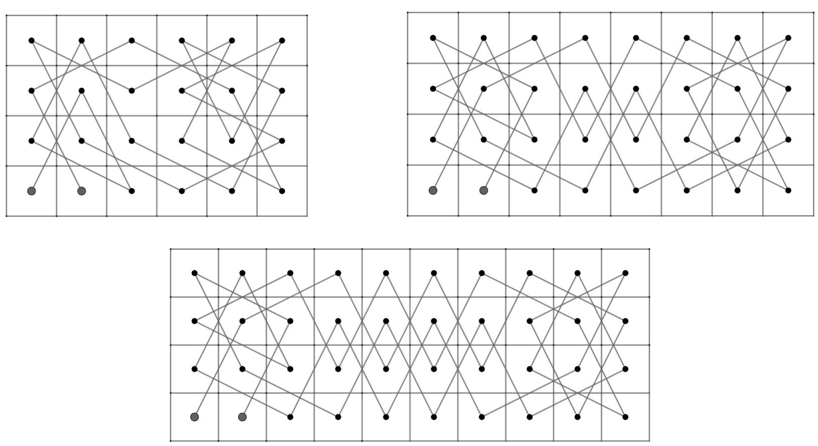


Figure 12: The required open knight's tours for $4 \times 6$, $4 \times 8$ and $4 \times 10$ chessboards

Next, we construct two cycles $C_1$ (dash line) and $C_2$ (solid line) on the $4 \times 6$ chessboard as shown in Figure 13.

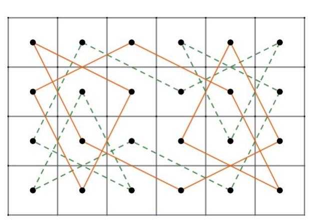


Figure 13: Two cycles $C_1$ and $C_2$ on the $4 \times 6$ chessboard

Then, we construct two cycles $C'_1$ and $C'_2$ on the $4 \times n$ chessboard where $n \equiv 0 \pmod 6$ and $n \geq 12$ as follow.

For $n = 6t$ where $t \geq 2$, the $4 \times n$ chessboard is obtained from $t$ copies of the $4 \times 6$ chessboard. We construct two cycles $C_1'$ and $C_2'$ on the $4 \times 6t$ chessboard by the following algorithm and they are shown in Figure 14.

(i) Delete the edges $(1, 6) - (3, 5)$ and $(2, 5) - (4, 6)$ of the $i$th $4 \times 6$ chessboard and delete the edges $(2, 2) - (4, 1)$ and $(1, 1) - (3, 2)$ of the $(i + 1)$th $4 \times 6$ chessboard, $i \in \{1, 2, 3, ..., t - 1\}$.

(ii) Join $(1, 6)$, $(3, 5)$, $(2, 5)$, $(4, 6)$ of the $i$th $4 \times 6$ chessboard to $(2, 2)$, $(4, 1)$, $(1, 1)$, $(3, 2)$ of the $(i+1)$th $4 \times 6$ chessboard, respectively, for all $i \in \{1, 2, 3, ..., t - 1\}$.

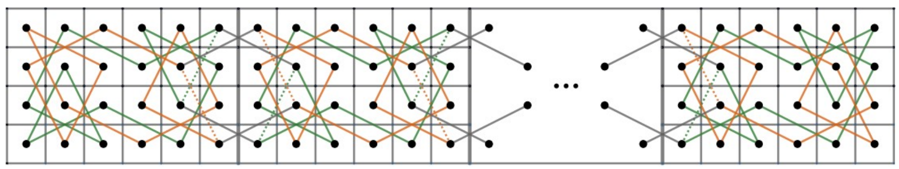


Figure 14: Two cycles $C_1'$ and $C_2'$ on the $4 \times 6t$ chessboard

Finally, for $n \geq 12$ and $n \equiv 2k \pmod 6$ where $k \in \{0, 1, 2\}$, we connect $4 \times (6 + 2k)$ chessboard with $4 \times 6t$ chessboard, $t \geq 1$, and delete the edges $(1, 6 + 2k) - (3, 5 + 2k)$ and $(2, 5 + 2k) - (4, 6 + 2k)$ of the $4 \times (6 + 2k)$ and $(2, 2) - (4, 1)$ and $(1, 1) - (3, 2)$ of the $4 \times 6t$ chessboard. Then, join $(1, 6 + 2k)$, $(3, 5 + 2k)$, $(2, 5 + 2k)$, $(4, 6 + 2k)$ of the $4 \times (6 + 2k)$ chessboard to $(2, 2)$, $(4, 1)$, $(1, 1)$, $(3, 2)$ of the $4 \times 6t$ chessboard, respectively, as shown in Figures 15, 16 and 17.

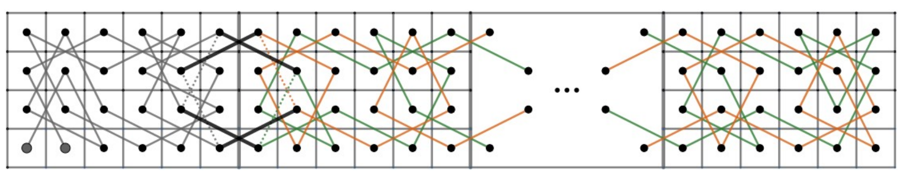


Figure 15: The required open knight's tour for $4 \times n$ chessboard where $n \equiv 0 \pmod 6$

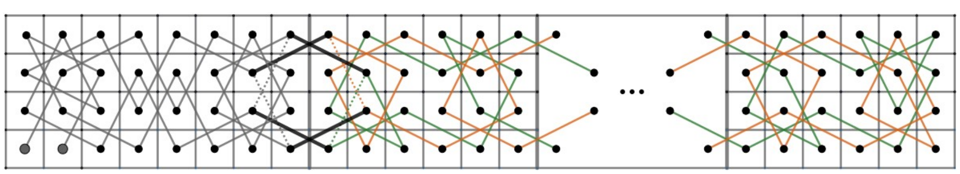


Figure 16: The required open knight's tour for $4 \times n$ chessboard where $n \equiv 2 \pmod 6$

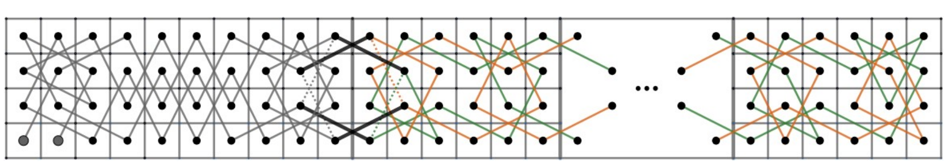


Figure 17: The required open knight's tour for $4 \times n$ chessboard where $n \equiv 4 \pmod 6$

□

**Lemma 2.4.** [5] *A* $4 \times n$ *chessboard contains an open knight's tour which begins with* $(1, 3)$ *and ends with* $(4, 1)$ *when* $n$ *is odd and* $n \geq 3$.

*Proof.* For the proof, see [5] or visit gaebler.us/share/Knight_tour.html. □

**Lemma 2.5.** *A* $5 \times n$ *chessboard contains an open knight's tour which begins with* $(5, 1)$ *and ends with* $(2, n-1)$ *when* $n$ *is odd and* $n \geq 5$.

*Proof.* The required open knight's tours for $5 \times 5$ and $5 \times 7$ chessboards are shown in Figure 18.

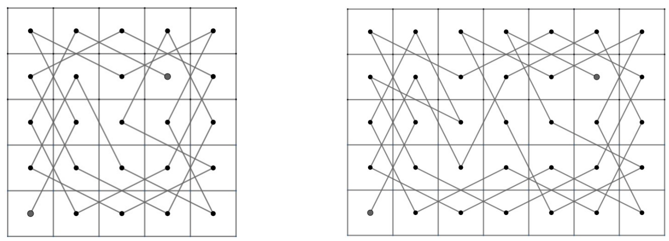


Figure 18: The required open knight's tours for $5 \times 5$ and $5 \times 7$ chessboards

Next, we construct a path $P$ (dash line) and a cycle $C$ (solid line) on the $5 \times 4$ chessboard as shown in Figure 19.

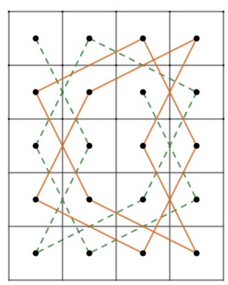


Figure 19: A path $P$ and a cycle $C$ on the $5 \times 4$ chessboard

Then, we construct a path $P'$ and a cycle $C'$ on $5 \times n$ chessboard where $n \equiv 0 \pmod 4$ and $n \geq 8$ as follow.

For $n = 4t$ where $t \geq 2$, the $5 \times n$ chessboard is obtained from $t$ copies of $5 \times 4$ chessboard. We construct a path $P'$ on $5 \times 4t$ chessboard by joining $(2, 3)$ of the $i$th $5 \times 4$ chessboard to $(1, 1)$ of the $(i+1)$th $5 \times 4$ chessboard for all $i \in \{1, 2, 3, ..., t-1\}$. Next, we can construct a cycle $C'$ on the $5 \times 4t$ chessboard by

(i) deleting the edge $(3, 3) - (5, 4)$ of the $i$th $5 \times 4$ chessboard and $(2, 1) - (4, 2)$ of the $(i+1)$th $5 \times 4$ chessboard and

(ii) joining $(3, 3)$ and $(5, 4)$ of the $i$th $5 \times 4$ chessboard to $(2, 1)$ and $(4, 2)$ of the $(i+1)$th $5 \times 4$ chessboard, respectively, for all $i \in \{1, 2, 3, ..., t-1\}$.

The path $P'$ and cycle $C'$ on the $5 \times 4t$ chessboard are shown in Figure 20.

Finally, for $n \geq 9$ and $n \equiv 2k - 1 \pmod 4$ where $k \in \{1, 2\}$, we connect the $5 \times (5 + 2(k-1))$ chessboard with the $5 \times 4t$ chessboard, $t \geq 1$, and delete the edges $(3, 4 + 2(k-1)) - (5, 5 + 2(k-1))$ of the $5 \times (5 + 2(k-1))$ chessboard and $(2, 1) - (4, 2)$ of the $5 \times 4t$ chessboard and join $(2, 4 + 2(k-1))$, $(3, 4 + 2(k-1))$ and $(5, 5 + 2(k-1))$ of the $5 \times (5 + 2(k-1))$ chessboard to $(1, 1)$, $(2, 1)$ and $(4, 2)$ of the $5 \times 4t$, respectively, as shown in Figures 21 and 22.

□

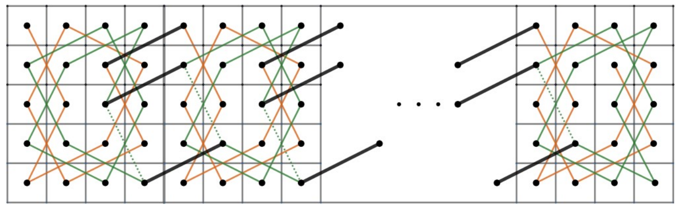


Figure 20: The path $P'$ and cycle $C'$ on the $5 \times 4t$ chessboard

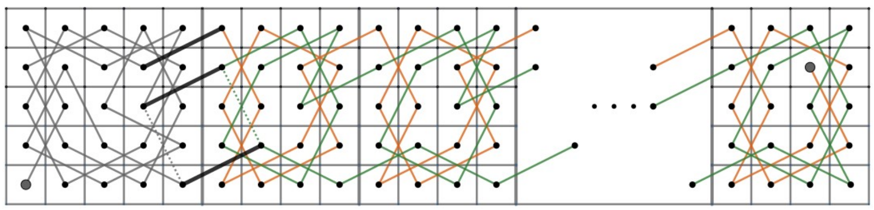


Figure 21: The required open knight's tour for $5 \times n$ chessboard where $n \equiv 1 \pmod 4$

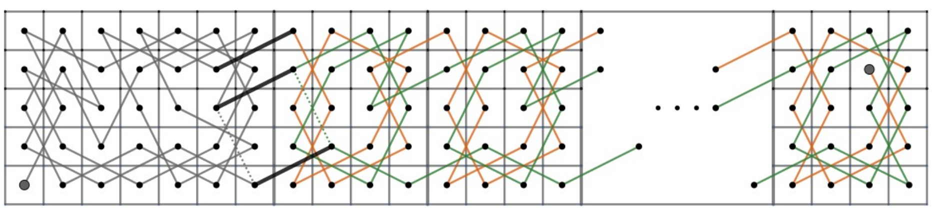


Figure 22: The required open knight's tour for $5 \times n$ chessboard where $n \equiv 3 \pmod 4$

From open knight's tours that we construct on Lemmas 2.1. 2.4 and 2.5, we can construct a larger knight's tour as follow.

**Theorem 2.6.** *An $m \times n$ chessboard contains an open knight's tour which begins with $(m, 1)$ and ends with $(2, n-1)$ provided that*

*(a) $m = 3$, $n$ is odd and $n \geq 7$, or*

*(b) $m$ and $n$ are odd and $m, n \geq 5$.*

*Proof.* Without loss of generality, let us assume that $m \leq n$. For $m = 3$, $n$ is odd and $n \geq 7$, by flipping and re-labeling a $3 \times n$ chessboard in Lemma 2.1, the $3 \times n$ chessboard has an open knight's tour which begins with $(3, 1)$ and ends with $(2, n-1)$.

For $m = 5$ and $n$ is odd, by Lemma 2.5, an $5 \times n$ chessboard has an open knight's tour which begins with $(5, 1)$ and ends with $(2, n-1)$.

For $m$ and $n$ is odd and $m \geq 7$, we consider 3 cases as follow.

**case 1** $m \equiv 3 \pmod 6$. Then, we can partition the $m \times n$ chessboard into $l$ $3 \times n$ sub-chessboards where $l$ is odd and $l \geq 3$. For $i \in \{1, 3, 5, ..., l\}$, by flipping and re-labeling a $3 \times n$ chessboard in Lemma 2.1, the $i$th $3 \times n$ sub-chessboard has an open knight's tour which begins with $(3, 1)$ and ends with $(2, n-1)$. For $i \in \{2, 4, 6, ..., l-1\}$, by Lemma 2.1, the $i$th $3 \times n$ sub-chessboard has an open knight's tour which begins with $(2, 2)$ and ends with $(3, n)$. Next, to construct an open knight's tour on the $m \times n$ chessboard, we join $(3, 1)$ of the $i$th $3 \times n$ sub-chessboard to $(2, 2)$ of the $(i+1)$th $3 \times n$ sub-chessboard, $i \in \{1, 3, 5..., l-2\}$ and join $(3, n)$ of the $i$th $3 \times n$ sub-chessboard to $(2, n-1)$ of the $(i+1)$th $3 \times n$ sub-chessboard where $i \in \{2, 4, 6, ..., l-1\}$ as shown in Figure 23

**case 2** $m \equiv 5 \pmod 6$. Then, we can partition the $m \times n$ chessboard into an $5 \times n$ sub-chessboard and $l$ $3 \times n$ sub-chessboards where $l$ is even and $l \geq 2$. By Lemma 2.1 and Case 1, we can construct an

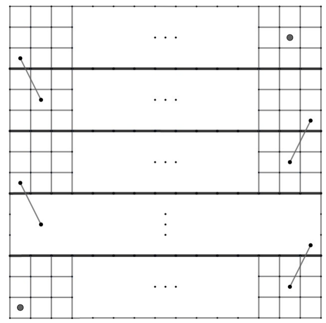


Figure 23: Joining of $l$ $3 \times n$ sub-chessboards

open knight's tour starting from $(2, n-1)$ of the first $3 \times n$ sub-chessboard and terminating at $(3, n)$ of the $l$th $3 \times n$ sub-chessboard. Finally, we join $(3, n)$ of the $l$th $3 \times n$ sub-chessboard to $(2, n-1)$ of the $5 \times n$ sub-chessboard which contains an open knight's tour by Lemma 2.4 as shown in Figure 24.

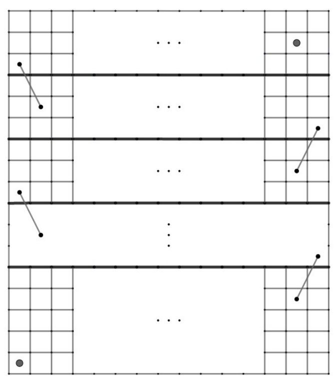


Figure 24: Joining of $l$ $3 \times n$ sub-chessboards and $5 \times n$ sub-chessboard

**case 3** $m \equiv 1 \pmod 6$. Then, we can partition the $m \times n$ chessboard into an $4 \times n$ sub-chessboard and $l$ $3 \times n$ sub-chessboards where $l$ is odd and $l \geq 1$. By Lemma 2.1 and case 1, we can construct an open knight's tour starting from $(2, n-1)$ of the first $3 \times n$ sub-chessboard and terminating at $(3, 1)$ of the $l$th $3 \times n$ sub-chessboard. Finally, we join $(3, 1)$ of the $l$ $3 \times n$ sub-chessboard to $(1, 3)$ of the $4 \times n$ sub-chessboard which contains an open knight's tour by Lemma 2.4 as shown in Figure 25.

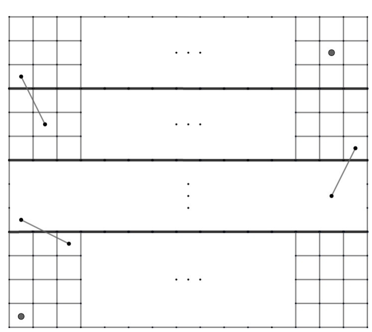


Figure 25: Joining of $l$ $3 \times n$ sub-chessboards and $4 \times n$ sub-chessboard

This completes the proof of the theorem. □

# 3 Closed knight's tour on $(m, m, r)$-ringboards

**Theorem 3.1.** *An $(m, m, r)$-ringboard with $m \geq 3$ and $m > 2r$ has a closed knight's tour if and only if*

*(a) $m = 3$ and $r = 1$, or*

*(b) $m \geq 7$ and $r \neq 1, 2$.*

*Proof.* First, for $m \geq 4$, $(m, m, 1)$-ringboard has four corner positions that the knight cannot move to any other places on that chessboard. For $m \geq 4$, by [4], $(m, m, 2)$-ringboard has no closed knight's tours.

Conversely, for $m = 3$ and $r = 1$, $(3, 3, 1)$-ringboard has a closed knight's tour as shown in Figure 26.

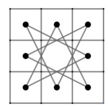


Figure 26: The closed knight's tour on $(3, 3, 1)$-ringboard

Next, we assume that $m \geq 7$, $r \geq 3$ and $m > 2r$. We separate the proof into two cases.

**Case 1** $m = 2k + 1$ is odd where $k \geq 3$.

**Case 1.1** Closed knight's tours for $(7, 7, 3)$, $(9, 9, 3)$ and $(11, 11, 3)$-ringboards are shown in Figures 27, 28 and 29.

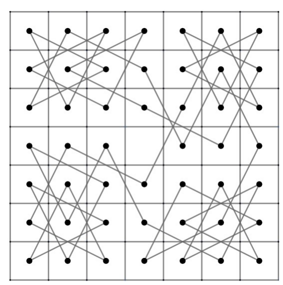


Figure 27: A closed knight's tour for $(7, 7, 3)$-ringboard

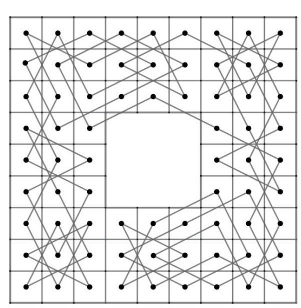


Figure 28: A closed knight's tour for $(9, 9, 3)$-ringboard

**Case 1.2** For $m \geq 13$ and $r \in \{3, 5, 6, ..., k\}$, we partition $(m, m, r)$-ringboard into four $r \times (m - r)$ subchessboards, see Figure 30(a) for $(13, 13, 5)$-ringboard. Since $r$ or $m - r$ is even, by [2], each $r \times (m - r)$ chessboard contains a closed knight's tour having edges $(1, m - r - 1) - (3, m - r)$ and $(r - 1, 4) - (r, 2)$. Thus, if we use the position on the $(m, m, r)$-ringboard, there are 4 Hamiltonian cycles having 6 edges, namely $(1, m - r - 1) - (3, m - r)$, $(2, m - r + 1) - (4, m - r + 2)$, $(m - r - 1, m) - (m - r, m - 2)$, $(m - r + 1, m - 1) - (m - r + 2, m - 3)$, $(m, r + 2) - (m - 2, r + 1)$, and $(m - 1, r) - (m - 3, r - 1)$.

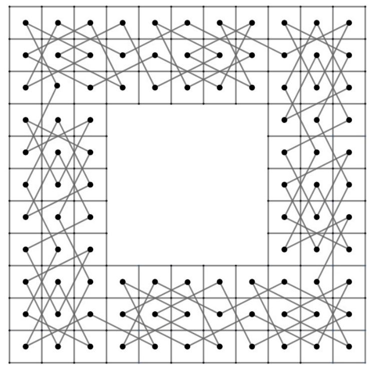
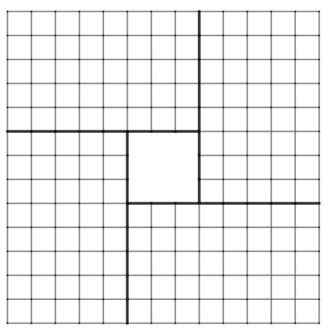
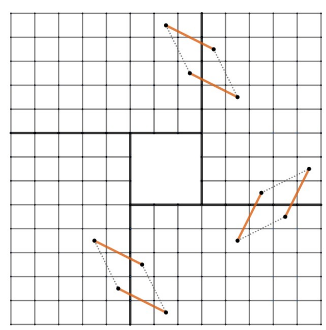


Figure 29: A closed knight's tour for $(11, 11, 3)$-ringboard

Next, to construct a closed knight's tour on the $(m, m, r)$ ringboard, we delete six edges: $(1, m-r-1)-(3, m-r)$, $(2, m-r+1)-(4, m-r+2)$, $(m-r-1, m)-(m-r, m-2)$, $(m-r+1, m-1)-(m-r+2, m-3)$, $(m, r+2)-(m-2, r+1)$, $(m-1, r)-(m-3, r-1)$ and join six edges: $(1, m-r-1)-(2, m-r+1)$, $(3, m-r)-(4, m-r+2)$, $(m-r-1, m)-(m-r+1, m-1)$, $(m-r, m-2)-(m-r+2, m-3)$, $(m, r+2)-(m-1, r)$, $(m-2, r+1)-(m-3, r-1)$ as shown in Figure 30(b) for $(13, 13, 5)$-ringboard.



(a) Four partitions of $(13, 13, 5)$-ringboard



(b) Joining four $5 \times 8$ sub-chessboards to obtain $(13, 13, 5)$-ringboard

Figure 30: $(13, 13, 5)$-ringboard

Note that the above construction still work if we let $m = 11$ and $r = 5$.

**Case 1.3** for $m \geq 9$ and $r = 4$, we partition the $(m, m, 4)$-ringboard by divided into four $4 \times (m-4)$ sub-chessboards. Since $m-4$ is odd, by Lemma 2.2, each $4 \times (m-4)$ chessboard contains an open knight's tour which begins with $(4, 2)$ and ends with $(1, m-5)$. Thus, if we use the position on $(m, m, 4)$-ringboard, there are 4 Hamiltonian paths having 8 end vertices, namely $(4, 2)$, $(1, m-5)$, $(2, m-3)$, $(m-5, m)$, $(m-3, m-1)$, $(m, 6)$, $(m-1, 4)$ and $(6, 1)$.

Next, to construct a closed knight's tour on the $(m, m, 4)$-ringboard, we join four edges: $(1, m-5)-(2, m-3)$, $(m-5, m)-(m-3, m-1)$, $(m, 6)-(m-1, 4)$, $(6, 1)-(4, 2)$ as shown in Figure 31 for the $(11, 11, 4)$-ringboard.

**Case 2** $m = 2k+2$ where $k \geq 3$.

**Case 2.1** Closed knight's tour for $(8, 8, 3)$-ringboard is shown in Figures 32.

**Case 2.2** For $m \geq 10$, $r$ is odd and $r \geq 3$, partition the $(m, m, r)$-ringboard into four $r \times (m-r)$ sub-chessboards. Since $m-r$ is odd, by Theorem 2.6, each $r \times (m-r)$ sub-chessboard contains an open knight's tour which begins with $(r, 1)$ and ends with $(2, m-r-1)$. Thus, if we use the position on

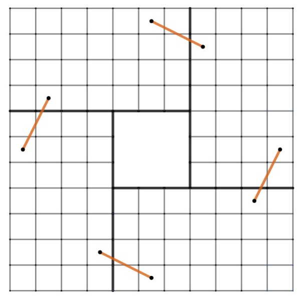


Figure 31: Joining four $4 \times 7$ sub-chessboards to obtain the $(11, 11, 4)$-ringboard

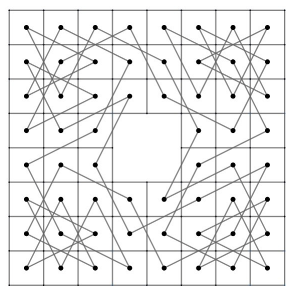


Figure 32: A closed knight's tour for the $(8, 8, 3)$-ringboard

$(m, m, r)$-ringboard, there are 4 Hamiltonian paths having 8 end vertices, namely $(r, 1)$, $(2, m-r-1)$, $(1, m-r+1)$, $(m-r-1, m-1)$, $(m-r+1, m)$, $(m-1, r+2)$, $(m, r)$ and $(r+2, 2)$.

Next, to construct a closed knight's tour on the $(m, m, r)$-ringboard, we join four edges: $(2, m-r-1)-(1, m-r+1)$, $(m-r-1, m-1)-(m-r+1, m)$, $(m-1, r+2)-(m, r)$ and $(r+2, 2)-(r, 1)$ as shown in Figure 33 for the $(16, 16, 7)$-ringboard.

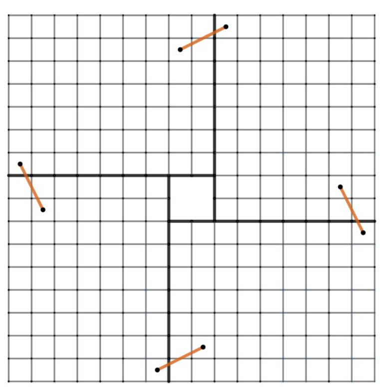


Figure 33: Joining four $7 \times 9$ sub-chessboards to obtain the $(16, 16, 7)$-ringboard

**Case 2.3** for $m \geq 14$, $r$ is even and $r \geq 6$, we partition the $(m, m, r)$-ringboard four $r \times (m-r)$ sub-chessboards. Since $r$ and $m-r$ are even, by [2] and the construction described in Case 1.2, the $(m, m, r)$-ringboard has a closed knight's tour.

**Case 2.4** For $m \geq 10$ and $r = 4$, we partition the $(m, m, 4)$-ringboard four $4 \times (m-4)$ sub-chessboards. Since $m-4$ is even and by Lemma 2.3, each $4 \times (m-4)$ sub-chessboard contains an open knight's tour which begins with $(4, 1)$ and ends with $(4, 2)$. Thus, if we use the position on the $(m, m, 4)$-ringboard, there are 4 Hamiltonian paths having end 8 vertices, namely $(4, 1)$, $(4, 2)$, $(1, m-3)$, $(2, m-3)$, $(m-3, m)$,

$(m-3, m-1)$, $(m, 4)$ and $(m-1, 4)$.

Next, to construct a closed knight's tour on the $(m, m, 4)$-ringboard, we delete six edges: $(1, m-4) - (3, m-5)$, $(2, m-5) - (4, m-4)$, $(2, m-2) - (4, m-3)$, $(m-5, m-1) - (m-4, m-3)$, $(m-3, 5) - (m-1, 6)$, $(5, 4) - (6, 2)$ and join ten edges: $(2, m-5) - (1, m-3)$, $(4, m-4) - (2, m-3)$, $(1, m-4) - (2, m-2)$, $(3, m-5) - (4, m-3)$, $(m-5, m-1) - (m-3, m)$, $(m-4, m-3) - (m-3, m-1)$, $(m-3, 5) - (m-1, 4)$, $(m-1, 6) - (m, 4)$, $(4, 1) - (6, 2)$ and $(4, 2) - (5, 4)$ as shown in Figure 34 for the $(12, 12, 4)$-ringboard.

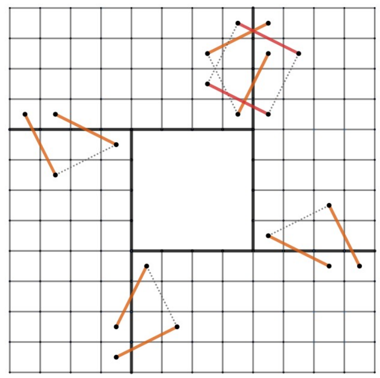


Figure 34: Joining four $4 \times 8$ sub-chessboards to obtain the $(12, 12, 4)$-ringboard

□

**Acknowledgement(s) :** The first author would like to thank the Science Achievement Scholarship of Thailand for financial support throughout my study.

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(Received 22 November 2018)
(Accepted 10 June 2019)