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Exact Solutions of The Regularized Long-Wave Equation: The Hirota Direct Method Approach to Partially Integrable Equations

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Abstract : The Hirota direct method has been used to obtain analytic solutions of the regularized long-wave equation (nonlinear evolution and wave equations) which constructing the soliton (solitary) solution of the regularized long-wave equation (RLW) is presented. We considered a transformation of the RLW equation to the Hirota bilinear form and applied the Hirota perturbation to this equation. The obtained results are exact one-solitary wave solutions of RLW.

Keywords : Berry-Esseen theorem, central limit theorem, Stein's method, concentration inequality, uniform and non-uniform bounds.
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1 Introduction

The regularized long-wave (RLW) equation is famous nonlinear wave equation which gives the phenomena of dispersion and weak nonlinearity, including magneto hydrodynamic wave in plasma, phonon packets in nonlinear crystals, and nonlinear transverse waves in shallow water or in ion-acoustic. In 1971 [5], Ryogo Hirota developed the Hirota direct method, he applied the method to construct multi-solitons of the Korteweg-de Vries (KdV) equation for multiple collisions of solitons which is integrable nonlinear partial differential equations. These solutions are the elastic collision of soliton solutions. That is the RLW equation; it is non-integrable which different from the KdV equation. The KdV equation is

$$u_t + uu_x - u_{xxx} = 0 \tag{1.1}$$

From the RLW equation, we obtain the solitary waves which are inelastic [7]. In 2006, Pekcan [2] applied the Hirota direct method to find solutions of non-integrable equations and also described the extensions of the Kadomtsev-Petviashvili (KP) and the Boussinesq (Bo) equations. Those mentioned papers related in integrable equations, but in this research we applied the Hirota method to construct solitary waves of incompletely integrable nonlinear partial differential equation that is the RLW equation. This equation is

$$u_t + uu_x - u_{xxt} = 0 \tag{1.2}$$

The processes to evaluate this equation are the same as that has shown in the reference paper [2] (section 1.1) as follows, the first step of this method is to transform the RLW equation into a quadratic form in the independent variables. The new form of the equation is called 'bilinear form'. In the second step, we write the bilinear form of this equation as a polynomial of a special differential operator, Hirota D-operator. This polynomial of D-operator is called 'Hirota bilinear form'. The last step of the method was using the finite perturbation expansion in the Hirota bilinear form and analyzed the coefficients of the perturbation parameter and its powers separately. Here the information we gained makes us to reach the exact solution of the equation.

Let us show that the Hirota method worked and referred to the section 1.1 of Pekcan [2] as follows.

2 Preliminaries

We reviewed the Hirota direct method in four steps by following Hietarinta's article [4] . Let $F[u] = F(u, u_x, u_t, ...)$ be a nonlinear partial differential equation.

Step 1: *Bilinearization*: Transform F[u] to a quadratic form in the dependent variables by a bilinearizing transformation $u = T(f(x, t, ...), g(x, t, ...))$ and called this form the bilinear form of $F[u]$. Note that for some equations we may not be found such a transformation.

Step 2: *Transformation to the Hirota bilinear form*:

Definition 2.1. *Let $S : \mathbb{C}^n \to \mathbb{C}$ be a space of differentiable functions. Then Hirota D-operator $D : S \to S$ is defined as*

$$[D_x^{m_1} D_t^{m_2} \dots]\{f \cdot g\} = [(\partial_x - \partial_{x'})(\partial_t - \partial_{t'})] f(x, t, \dots) \cdot g(x', t', \dots)|_{x'=x, t'=t} \tag{2.1}$$

where m_i, $i = 1, 2, \dots$ are positive integers and $x, t, \dots$ are independent variables.

By using some sort of combination of Hirota D-operator, we try to write the bilinear form of $F[u]$ as a polynomial of D-operator, say $P(D)$. Let us state some propositions and corollaries on $P(D)$ [4].

Proposition 2.2. *Let $P(D)$ act on two differentiable functions $f(x, t, \dots)$ and $g(x, t, \dots)$. Then we have*

$$P(D)\{f \cdot g\} = P(-D)\{g \cdot f\}. \tag{2.2}$$

Corollary 2.3. *Let $P(D)$ act on two differentiable functions $f(x,t,\ldots)$ and $g=1$ then we have*

$$P(D)\{f\cdot 1\}=P(\partial)f,\quad P(D)\{1\cdot f\}=P(-\partial)f. \tag{2.3}$$

Proposition 2.4. *Let $P(D)$ act on two differentiable functions e^{θ_1} and e^{θ_2} where $\theta_i=k_ix+\omega_it+\ldots+\alpha_i$ with $k_i,\omega_i,\ldots,\alpha_i$ are constants for $i=1,2$. Then we have*

$$P(D)\{e^{\theta_1}\cdot e^{\theta_2}\}=P(k_1-k_2,\ldots,\alpha_1-\alpha_2)e^{\theta_1+\theta_2}. \tag{2.4}$$

For a shorter notation, we use $P(p_1-p_2)$ instead of $P(k_1-k_2,\ldots,\alpha_1-\alpha_2)$

Corollary 2.5. *If we have $P(D)\{a\cdot a\}=0$ where a is any nonzero constant, then we have $P(0,0,\ldots)=0$.*

Definition 2.6. *We say that a nonlinear partial differential equation can be written in Hirota bilinear form if it is equivalent to*

$$\sum_{\alpha,\beta=1}^{m}P^{\eta}_{\alpha\beta}(D)f^{\alpha}f^{\beta}=0,\eta=1,\ldots,r \tag{2.5}$$

for some m,r and linear operators $P^{\eta}_{\alpha\beta}(D)$. The f^i 's are new dependent variables.

Remark 2.7. *There is no systematic way to write a nonlinear partial differential equation in Hirota bilinear form.*

Remark 2.8. *For some nonlinear partial differential equations we may need more than one Hirota bilinear equation.*

Step 3: *Application of the Hirota perturbation*: We substitute the finite perturbation expansions of the differentiable functions $f(x,t,\ldots)$ and $g(x,t,\ldots)$ which are

$$f(x,t,\ldots)=f_0+\sum_{m=1}^{N}\varepsilon^m f_m(x,t,\ldots),\quad g(x,t,\ldots)=g_0+\sum_{m=1}^{N}\varepsilon^m g_m(x,t,\ldots) \tag{2.6}$$

into the Hirota bilinear form. Here to avoid the trivial solution f_0,g_0 are constants with the condition $(f_0,g_0)\neq(0,0)$. For the sake of applicability of the method we take the functions f_m and g_m, $m=1,\ldots,N$ as exponential functions. The ε is constant called the perturbation parameter. For instance for $N=2$, we take

$$f=f_0+\varepsilon f_1+\varepsilon^2 f_2, g=g_0+\varepsilon g_1+\varepsilon^2 g_2 \tag{2.7}$$

where $f_1=e^{\theta_1}+e^{\theta_2}$ for $\theta_i=k_ix+\omega_it+\ldots,i=1,2$. We can decide the other terms of the functions f and g in the process of the method.

Step 4: *Examination of the coefficients of the perturbation parameter ε*: We make the coefficients of the perturbation parameter ε and its powers appeared in the Hirota perturbation to vanish. From these coefficients we obtain the functions $f(x,t,\ldots)$ and $g(x,t,\ldots)$. Hence by using them in the bilinearizing transformation $u=T[f(x,t,\ldots),g(x,t,\ldots)]$, we find the exact solution of $F[u]$.

3 The regularized long-wave (RLW) equation

In this section we applied the Hirota direct method to solve the regularized long-wave equation in Eq.(2).

Step 1. *Bilinearized*: We used the transformation

$$v(x,t) = 4\frac{\partial^2}{\partial x^2}\ln f(x,t), \tag{3.1}$$

for the RLW equation.
Assume that $u = u_0 + v$, thus v corresponds Eq.(2) in the form

$$v_t + u_0 v_x + v v_x - v_x xt = 0. \tag{3.2}$$

Let

$$F(x,t) = 4\frac{\partial}{\partial x}\ln f(x,t), \tag{3.3}$$

and $v = F_x$. Substitute $v = F_x$ into Eq.(11), then we have

$$F_{xt} + u_0 F_{xx} + F_x F_{xx} - F_{xxxt} = 0, \tag{3.4}$$

Integrate Eq. (13) with respect to x yields

$$F_t + u_0 F_x + \frac{1}{2}(F_x)^2 - F_{xxt} + a(t) = 0, \tag{3.5}$$

where $a(t)$ is arbitrary function.
Substitute Eq.(12) into Eq.(14), we get

$$(\ln f)_{xt} + u_0(\ln f)_{xx} + 2[(\ln f)_{xx}]^2 - (\ln f)_{xxxt} + \frac{1}{2}b(t) = 0. \tag{3.6}$$

Hence from Eq.(15), becomes

$$(ff_{xt} - f_x f_t) + u_0(ff_{xx} - f_x f_x) + \tfrac{2}{f^2}[ff_{xx} - f_x f_x]^2 - [ff_{xxxt} - ff_{xxx}$$

$$-3(f_{xx}f_{xt} + f_{xx}f_x)] - \frac{6}{f^2}[f_{xx}f_x f_t + f_x^2 f_t] + \frac{6}{f^2}[f_x^3 f_t] + \frac{1}{2}b(t) = 0. \tag{3.7}$$

Step 2. *Transformation to the bilinear form*: We use the Hirota method and write the bilinear form of the RLW equation in the Hirota bilinear form.
We consider $D_x D_t$ applied on the product of $f \cdot f$,

$$D_x D_t\{f \cdot f\} = (\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial x'})(\tfrac{\partial}{\partial t} - \tfrac{\partial}{\partial t'})\{f(x,t)\cdot f(x',t')\}|_{x'=x,t'=t}$$

$$= f_{xt}f + ff_{xt} - f_tf_x - f_xf_t$$

$$= 2(ff_{xt} - f_tf_x), \tag{3.8}$$

considerD_x^2,

$$D_x^2\{f \cdot f\} = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^2\{f(x,t) \cdot f(x',t')\}|_{x'=x,t'=t}$$

$$= 2(ff_{xx} - (f_x)^2), \tag{3.9}$$

and consider $D_x^2D_t$

$$D_x^3D_t\{f \cdot f\} = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^3(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})\{f(x,t) \cdot f(x',t')\}|_{x'=x,t'=t}$$

$$= 2[ff_{xxxt} - f_{xxx}f_t) - 3f_xfxxt + 3f_{xx}f_{xt}], \tag{3.10}$$

Thus the Eq.(16) can be written in the bilinear form

$$P(D)\{f \cdot f\} = (D_xD_t + u_0D_x^2 - D_x^3D_t + b)\{f \cdot f\}$$

$$+\frac{1}{f^2}D_x^2\{f \cdot f\}(D_x^2 + 3D_xD_t)\{f \cdot f\} = 0. \tag{3.11}$$

Suppose that

$$c(x,t) = \frac{1}{f^2}D_x^2(D_x^2 + 3D_xD_t)\{f \cdot f\}, \tag{3.12}$$

and Eq.(20) becomes the bilinear equation

$$[D_xD_t + (u_0 + c)D_x^2 - D_x^3D_t + b]\{f \cdot f\} = 0. \tag{3.13}$$

Thus from Eq.(21) and (22), yield

$$(D_x^2 + 3D_xD_t - c)\{f \cdot f\} = 0. \tag{3.14}$$

Compare Eq.(22) and (23) with the bilinear form of the KdV equation (1) which corresponding the bilienear form of the RLW equation (i.e. Eq.(22) and (23)). We see that the RLW equation (1.2) transforms to the KdV equation (2). That is, let $t \to x$ in the third term into Eq.(22) and set $c = 0, u_0 = 1$ to get

$$[D_xD_t + D_x^2 - D_x^4 + b]\{f \cdot f\} = 0. \tag{3.15}$$

Step 3. *Application the finite perturbation expansion in the bilinear form*: Insert $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$ into Eq.(24), we get

$$P(D)\{f \cdot f\} = P(D)\{1 \cdot 1\} + \varepsilon P(D)\{f_1 \cdot 1 + 1 \cdot f_1\}$$

$$+\varepsilon^2 P(D)\{f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2\} + \dots = 0, \tag{3.16}$$

where ε is a constant of the perturbation parameter.

4 Solitary solution of the RLW equation

One-solitary solution

Constructing one-solition (solitary) solution, let $f = 1 + \varepsilon f_1$ where $f_1 = e^{\theta_1}$ and $\theta_1 = k_1 x + \omega_1 t + \ldots + \alpha_1$. Note that $f_j = 0$ for all $j \geq 2$ Performing the coefficients $\varepsilon^m, m = 0, 1, 2$ vanish.

Substitute f into the Eq.(24) with $b(t)$ we have the coefficient ε^0 that is

$$P(D)(0, 0, \ldots) = 0 \quad since \quad P(D)\{1 \cdot 1\} = 0. \tag{4.1}$$

For the coefficient ε^1,

$$\begin{aligned} P(D)\{f_1 \cdot 1 + 1 \cdot f_1\} &= P(\partial)e^{\theta_1} + P(-\partial)e^{\theta_1} \\ &= 2P(p_1)e^{\theta_1} = 0. \end{aligned} \tag{4.2}$$

Therefore, we get the dispersion $P(p_1) = 0$, and the coefficient ε^2 also vanish, because of

$$\begin{aligned} P(D)\{f_1 \cdot f_1\} &= P(D)\{e^{\theta_1} \cdot e^{\theta_1}\} \\ &= 2P(p_1 - p_1)e^{2\theta_1} = 0. \end{aligned} \tag{4.3}$$

Hence we can define ε (without loss of generality) so $f = 1 + \varepsilon e^{\theta_1}$ Using the results of the Hirota method, it is necessary that$b(t) = 0$ and

$$\begin{aligned} &\omega_1 k_1 + k_1^2 - k_1^4 = 0, \text{ or} \\ &\qquad \omega_1 = k_1^3 - k_1. \end{aligned} \tag{4.4}$$

From Eq.(2), we have one-solitary solution of the RLW equation

$$u(x,t) = 4\frac{\partial^2}{\partial x^2}\ln(1 + e^{\theta_1}) = k_1^2 sech^2(\frac{1}{2}\theta_1). \tag{4.5}$$

[$sech^2$:Hyperbolic secant]

5 Conclusions

One-solitary wave solutions of the regularized long-wave equation that is the non-completely integrable nonlinear partial differential equation can be constructed by using the Hirota direct method.

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