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Remarks on Brouwer Fixed Point Theorem for Some Surfaces in $\mathbb{R}^3$

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Abstract : Let X be a surface in $\mathbb{R}^3$. A subset E of X is said to be convex if, for each $p, q \in E$, it contains each shortest geodesic joining p and q. A surface in $\mathbb{R}^3$ is said to have the fixed point property if each continuous mapping $T : E \to E$ from a compact convex subset E of X has a fixed point. In this paper, we give some examples of surfaces in $\mathbb{R}^3$ that do not have the fixed point property. Moreover, we show that the surface $z = y^2$ and the upper hemisphere of the sphere of radius r centered at $(0, 0, 0)$ with north pole and equator removed have the fixed point property.

Keywords : Brouwer fixed point theorem; fixed point property; surfaces in $\mathbb{R}^3$.
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1 Introduction

In 1911, Brouwer [1] proved the well known theorem, Brouwer fixed point theorem, that each continuous function from a closed ball of $\mathbb{R}^n$ into itself has a fixed point. In 1930, Schuader [2] proved a generalization of the theorem, Schuader fixed point theorem, that each continuous function from a compact convex subset

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of a Banach space into itself has a fixed point. In 1941, Kakutani [3] proved, Kakutani fixed point theorem, that each multi-valued mapping $T : E \to 2^X \setminus \emptyset$ with a closed graph and such that Tx is convex, for each $x \in E$, has a fixed point. In 2001, Cauty [4] proved that each continuous function from a compact convex subset of a topological vector space into itself has a fixed point. In 2010, Butsan, Dhompongsa, and Fupinwong [5] proved the following theorem. As a consequence, each continuous function from a compact convex subset of a CAT(0) space into itself has a fixed point.

Theorem 1.1. *Let E be a compact convex subset of a convex metric space X. Then E has the fixed point property for continuous mappings, that is, each continuous function from E into itself has a fixed point.*

Brouwer fixed point theorem has been generalized to lots of spaces by lots of authors. See, e.g., Tychonoff [6], Fan [7], Day [8], Brower [9], Henderson and Livesay [10], Himmellberg [11], Riech [12], [13], Park [14], Lau and Yao [15], Dhompongsa and Nantadilok [16], Chuensupantharat et al. [17], Kumam and Dhompongsa [18].

In this paper, a subset E of a surface in $\mathbb{R}^3$ is said to be convex if, for each p and q in E, it contains each shortest geodesic joining p and q. A surface X in $\mathbb{R}^3$ is said to have the fixed point property if each continuous function from a compact convex subset of X into itself has a fixed point.

It is known that the distance of points in a surface is the length of a shortest geodesic joining them. Some surfaces are CAT(0) spaces, so, by using Theorem 1.1, they have the fixed point property. Unfortunately, geodesics in lots of surfaces are mystery. Finding their closed forms is always difficult. So it is not obvious to conclude that the surfaces have the fixed point property.

We introduce a method for proving the fixed point property of surfaces in $\mathbb{R}^3$ in this paper. By using this method, we show that the cylinder $z = y^2$ and the upper hemisphere of the sphere of radius r centered at $(0, 0, 0)$ with north pole and equator removed have the fixed point property. Consequently, we extend Brouwer fixed point theorem to some surfaces in $\mathbb{R}^3$. Moreover, some examples of surfaces in $\mathbb{R}^3$ that do not have the fixed point property are given.

2 Preliminaries

A real-valued function $f : I \to \mathbb{R}$ on an open interval I is called smooth if the derivative and all the higher-order derivatives of f exist and are continuous. A real-valued function $f : G \to \mathbb{R}$ on an open set G in $\mathbb{R}^n$, $n = 2, 3$, is called smooth if all partial derivatives of f, of all orders, exist and are continuous.

Let $m, n \in \{1, 2, 3\}$. A function $F = (f_1, f_2, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is smooth if $f_1, f_2, ..., f_n$ are all smooth. A smooth function $F = (f_1, f_2, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be regular if the Jacobian matrix of F at p has rank n for each $p \in \mathbb{R}^n$. A smooth function $\xi : U \to \mathbb{R}^3$ on an open subset U of $\mathbb{R}^2$ is called a coordinate patch if it is one-to-one and regular.

Define the natural coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ of $\mathbb{R}^3$ by

$$x(p) = p_1,\ y(p) = p_2,\ z(p) = p_3,$$

for each $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. Obviously, x, y, and z are all smooth.

Let $v, p \in \mathbb{R}^3$. We denote by v_p a tangent vector to $\mathbb{R}^3$ at p, where v is its vector part and p is its point of application. We picture v_p as the arrow from p to $p + v$. For example, if $v = (1, 2, 0)$ and $p = (-1, -1, -1)$, then v_p is the arrow from $(-1, -1, -1)$ to $(0, 1, -1)$. The set $T_p(\mathbb{R}^3)$ of all tangent vectors to $\mathbb{R}^3$ at p is called the tangent space of $\mathbb{R}^3$ at p.

For each $p \in \mathbb{R}^3$, the dot product $v_p \cdot w_p$ of tangent vectors $v_p = (v_1, v_2, v_3)_p, w_p = (w_1, w_2, w_3)_p$ to $\mathbb{R}^3$ at p is defined by

$$v_p \cdot w_p = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Tangent vectors v_p and w_p are said to be normal if $v_p \cdot w_p = 0$.

For each $p \in \mathbb{R}^3$, the norm $||v_p||$ of a tangent vector $v_p = (v_1, v_2, v_3)_p$ to $\mathbb{R}^3$ at p is defined by

$$||v_p|| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

From now on, for convenience, , we shall omit the point of application p from the notation of a tangent vector v_p.

A vector field V on $\mathbb{R}^3$ is a function of $\mathbb{R}^3$ into $\bigcup_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3)$ satisfying $V(p) \in T_p(\mathbb{R}^3)$ for each $p \in \mathbb{R}^3$.

Let V_1, V_2, V_3 be vector fields on $\mathbb{R}^3$, and let $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ be all smooth. Define the vector field $f_1 V_1 + f_2 V_2 + f_3 V_3$ by

$$(f_1 V_1 + f_2 V_2 + f_3 V_3)(p) = f_1(p) V_1(p) + f_2(p) V_2(p) + f_3(p) V_3(p),$$

for each $p \in \mathbb{R}^3$.

Define the vector fields U_1, U_2, U_3 on $\mathbb{R}^3$ by

$$U_1(p) = (1, 0, 0),\ U_2(p) = (0, 1, 0),\ U_3(p) = (0, 0, 1),$$

for each $p \in \mathbb{R}^3$. A vector field V on $\mathbb{R}^3$ is called smooth if there are smooth functions $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ such that $V = f_1 U_1 + f_2 U_2 + f_3 U_3$.

Let f be a smooth real-valued function on $\mathbb{R}^3$. Define the differential $df : \bigcup_{p \in \mathbb{R}^3} T_p(\mathbb{R}^3) \to \mathbb{R}$ of f by

$$df(v) = v_1 \frac{\partial f}{\partial x}(p) + v_2 \frac{\partial f}{\partial y}(p) + v_3 \frac{\partial f}{\partial z}(p),$$

for each tangent vector $v = (v_1, v_2, v_3)$ to $\mathbb{R}^3$ at p. We denote by $X : f(x, y, z) = c$, $c \in \mathbb{R}$, the set $X = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : f(p_1, p_2, p_3) = c\}$ satisfying $df \neq 0$ on $T_p(X)$, for each $p \in X$. In this case, it is known that $\nabla f = \frac{\partial f}{\partial x} U_1 + \frac{\partial f}{\partial y} U_2 + \frac{\partial f}{\partial z} U_3$ and $-\nabla f$ are nonvanishing normal vector fields on X, that is, for some $p_0 \in X$, $\nabla f(p_0) \neq (0, 0, 0)$ but $\nabla f(p) \cdot v = 0$, for each tangent vector v to $\mathbb{R}^3$ at p (see, e.g., [19] and [20]). In particular, $2yU_2 - U_3$ and $-2yU_2 + U_3$ are nonvanishing normal vector fields on the cylinder $X : y^2 - z = 0$.

Let I be an open interval and let $\alpha_1, \alpha_2, \alpha_3 : I \to \mathbb{R}$. The function $\alpha = (\alpha_1, \alpha_2, \alpha_3) : I \to \mathbb{R}^3$ is called a curve in $\mathbb{R}^3$ if $\alpha_1, \alpha_2, \alpha_3$ are all smooth.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$. The velocity vector $\alpha'(t)$ of α at $t \in I$ is the tangent vector

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$$

to $\mathbb{R}^3$ at $\alpha(t)$. The acceleration vector $\alpha''(t)$ of α at $t \in I$ is the tangent vector

$$\alpha''(t) = (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t))$$

to $\mathbb{R}^3$ at $\alpha(t)$. A curve $\alpha : I \to \mathbb{R}^3$ is in a subset X of $\mathbb{R}^3$ if $\alpha(I) \subset X$.

Let $h : J \to I$ be a smooth function from an open interval J into an open interval I, and let $\alpha : I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$. Then the curve $\alpha \circ h : J \to \mathbb{R}^3$ is said to be a reparametrization of α by h.

A curve $\alpha : I \to \mathbb{R}^3$ is regular if $\alpha'(t) \neq 0$ for each $t \in I$. A curve $\alpha : I \to \mathbb{R}^3$ is unit-speed if $\alpha'(t) = 1$ for each $t \in I$.

The following lemma shows the existence of a unit-speed reparametrization of a regular curve in $\mathbb{R}^3$. The detail proof can be found in [19] and [20].

Lemma 2.1. *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$. Define a smooth function $s : I \to \mathbb{R}$ by*

$$s(t) = \int_a^t ||\alpha'(u)||du,$$

where a is a number in I. Then s is one-to-one and the inverse function $t : J \to I$ of s is a smooth function from an open interval J into I. Moreover, the curve $\alpha \circ t : J \to \mathbb{R}^3$ is a unit-speed reparametrization of α.

Let X be a subset of $\mathbb{R}^3$. X is a surface in $\mathbb{R}^3$ if for each $p \in X$ there exists a coordinate patch $\xi : U \to X$ on an open set U in $\mathbb{R}^2$ with $p \in \xi(U)$. In particular, if $\xi(u, v) = (u, v, v^2)$ for each $(u, v) \in \mathbb{R}^2$ then ξ is a coordinate patch with $X : y^2 - z = 0 \subset \xi(U)$. So $X : y^2 - z = 0$ is a surface in $\mathbb{R}^3$.

A tangent vector $v \in T_p(\mathbb{R}^3)$ is tangent to a surface X at $p \in X$ if v is a velocity of a curve in X. The set $T_p(X)$ of all tangent vectors to X at p is called the tangent plane of X at p. It is known that $T_p(X)$ is a 2-dimensional vector subspace of $\mathbb{R}^3$ (see, e.g., [19] and [20]).

A curve $\alpha : I \to \mathbb{R}^3$ in a surface X is a geodesic of X if its acceleration α'' is normal to M, that is, for each $t \in I$,

$$\alpha''(t) \cdot v = 0$$

for each $v \in T_{\alpha(t)}(X)$.

The following lemma is well-known (see, e.g., [19] and [20]).

Lemma 2.2. *Let α be a unit-speed curve in a surface X in $\mathbb{R}^3$. If α lies in a plane which is orthogonal to X along α then α is a geodesic of X.*

Let $\xi(u,v) = (u,v,v^2)$ for each $(u,v) \in \mathbb{R}^2$. And let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a unit-speed reparametrization of the curve $t \mapsto \xi(0,t)$. Since α lies in the yz-plane which is orthogonal to the cylinder $X : y^2 - z = 0$, so it follows from Lemma 2.2 that α is a geodesic of $y^2 - z = 0$. Similarly, the unit-speed reparametrizations of the curve $t \mapsto \xi(u,t)$ and $t \mapsto \xi(t,v)$ are also geodesics of $y^2 - z = 0$, for each $(u,v) \in \mathbb{R}^2$.

3 Main Results

First, we prove the following proposition.

Proposition 3.1. *Let X be a surface in $\mathbb{R}^3$, and let $\alpha : [a,b] \to X$ be a closed geodesic with $\alpha(a) = \alpha(b)$. If α is one-to-one on (a,b) and $\alpha([a,b])$ is convex, then X does not have the fixed point property.*

Proof. Define $T : \alpha([a,b]) \to \alpha([a,b])$ by

$$T(\alpha(t)) = \begin{cases} \alpha(t + \frac{b-a}{2}), & \text{if } t \leq \frac{a+b}{2}, \\ \alpha(t - \frac{b-a}{2}), & \text{if } t > \frac{a+b}{2}. \end{cases}$$

It can be seen that T is continuous but does not have any fixed points. □

From the above proposition, the following surfaces do not have the fixed point property.

1) The cylinder

$$X : x^2 + y^2 = r^2$$

does not have the fixed point property since it contains the closed geodesic $\alpha(t) = (r\cos t, r\sin t, 0)$.

2) The usual parametrization of a torus is

$$\xi = ((R + r\cos x)\cos y, (R + r\cos x)\sin y, rsinx).$$

The torus $\xi(\mathbb{R}^2)$ does not have the fixed point property since it contains the closed geodesic $\alpha(t) = ((R + r)\cos t, (R + r)\sin t, 0)$.

Let $X = \{p \in \mathbb{R}^3 : \|p\| = 1\} \backslash (0,0,1)$. It can be seen that the curve $\alpha(t) = (\cos t, \sin t, 0)$ is in X. However, we are not able to conclude that X does not have the fixed point property by using Proposition 3.1. Indeed, the image of α in this case is not convex.

Consider the surface $z = y^2$. It is not obvious to see that it has the fixed point property, although it is homeomorphic to $\mathbb{R}^2$. In face, from Hadamard theorem, it is a CAT(0) space since it is simply connected and the Gaussian curvature at each its point is zero. It follows from Theorem 1.1 that it has the fixed point property.

We can show that $z = y^2$ has the fixed point property by using another method. Geodesics in $z = y^2$ have an interesting property that is very useful for showing the fixed point property. The interesting property is proved in the following lemma.

Lemma 3.2. *Let $X : z - y^2 = 0$ and $\xi = \left(x, y, y^2\right)$. And let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a shortest geodesic in X joining $\xi(u_1, v_1)$ and $\xi(u_2, v_2)$ with $u_1 < u_2$. Assume that $\alpha(t_1) = \xi(u_1, v_1)$ and $\alpha(t_2) = \xi(u_2, v_2)$. Then:*

1) If $t_1 < t_2$, then $\alpha_1'(t_1) > 0$.

2) If $t_1 > t_2$, then $\alpha_1'(t_2) < 0$.

Proof. Note that $V = -2yU_2 + U_3$ is the normal vector field on X. Since $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a geodesic in X, so α'' is normal to X. Thus there exists a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $\alpha'' = f(V \circ \alpha)$. Then

$$(\alpha_1'', \alpha_2'', \alpha_3'') = \alpha'' = f(V \circ \alpha) = (0, -2f\alpha_2, f).$$

Therefore, $\alpha_1'' = 0$.

1) Let $t_1 < t_2$. Assume to the contrary that $\alpha_1'(t_1) \leq 0$. It follows from $\alpha_1'' = 0$ and $\alpha_1'(t_1) \leq 0$ that α_1 is decreasing. Then

$$\alpha_1(t_1) \geq \alpha_1(t_2).$$

From

$$(u_1, v_1, v_1^2) = \xi(u_1, v_1) = \alpha(t_1) = (\alpha_1(t_1), \alpha_2(t_1), \alpha_3(t_1))$$

and

$$(u_2, v_2, v_2^2) = \xi(u_2, v_2) = \alpha(t_2) = (\alpha_1(t_2), \alpha_2(t_2), \alpha_3(t_2)),$$

Then $u_1 \geq u_2$. This leads to the contradiction. So we conclude that $\alpha_1'(t_1) > 0$.

2) Let $t_1 > t_2$. Assume to the contrary that $\alpha_1'(t_2) \geq 0$. It follows from $\alpha_1'' = 0$ and $\alpha_1'(t_2) \geq 0$ that α_1 is increasing. Then

$$\alpha_1(t_2) \leq \alpha_1(t_1).$$

From $u_1 = \alpha_1(t_1)$ and $u_2 = \alpha_1(t_2)$, then $u_2 \leq u_1$. This leads to the contradiction. So we conclude that $\alpha_1'(t_2)$ is less than zero. □

Define, for each $p_1, p_2 \in \mathbb{R}^2$,

$$[p_1, p_2] = \{(1 - \alpha)p_1 + \alpha p_2 : \alpha \in [0, 1]\}.$$

The following theorem shows that the cylinder $z = y^2$ has the fixed point property by using the property proved in Lemma 3.2.

Theorem 3.3. *Let $X : z - y^2 = 0$, and let $\xi = \left(x, y, y^2\right)$. If E is a compact convex subset of X. Then there is a retraction r from a 2-dimensional interval I^2 into $\xi^{-1}(E)$. Consequently, X has the fixed point property.*

Proof. If E is a curve segment, obviously, E has the fixed point property. So we assume that E is not a curve segment. Let

$$a = \inf\{\alpha \in \mathbb{R} : (\alpha, \beta) \in \xi^{-1}(E), \exists \beta \in \mathbb{R}\}$$

and

$$b = \sup\{\alpha \in \mathbb{R} : (\alpha, \beta) \in \xi^{-1}(E), \exists \beta \in \mathbb{R}\}.$$

Note that $(a, \beta_1), (b, \beta_2) \in \xi^{-1}(E)$, for some $\beta_1, \beta_2 \in \mathbb{R}$, since $\xi^{-1}(E)$ is compact. For each $u \in [a, b]$, let

$$u^* = \inf\{\beta \in \mathbb{R} : (u, \beta) \in \xi^{-1}(E)\}$$

and

$$u^{**} = \sup\{\beta \in \mathbb{R} : (u, \beta) \in \xi^{-1}(E)\}.$$

Since E is compact convex,

$$\xi^{-1}(E) = \bigcup_{u \in [a,b]} [u^*, u^{**}].$$

Let

$$c = \inf_{u \in [a,b]} u^*, \;\; d = \sup_{u \in [a,b]} u^{**},$$

and

$$I^2 = [a, b] \times [c, d].$$

Define $r : I^2 \to \xi^{-1}(E)$ by

$$r(u, v) = \begin{cases} (u, v), & \text{if } (u, v) \in \xi^{-1}(E), \\ (u, u^{**}), & \text{if } v > u^{**}, \\ (u, u^*), & \text{if } v < u^*. \end{cases}$$

To show that r is continuous. Let $\{p_n\}$ be a sequence in I^2 with

$$\lim_{n \to \infty} p_n = p.$$

Write $p = (u, v), p_n = (u_n, v_n), r(p) = (u, r(v))$, and $r(p_n) = (u_n, r(v_n))$, for each $n \in \mathbb{N}$. It follows that $\lim_{n \to \infty} u_n = u$ and $\lim_{n \to \infty} v_n = v$. If p is in the interior of $\xi^{-1}(E)$, it can be seen that

$$\lim_{n \to \infty} r(p_n) = \lim_{n \to \infty} p_n = p = r(p).$$

Assume that p is not in the interior. It follows from the convexity of E that $v > u^{**}$ or $v < u^*$. We may assume by passing to a subsequence that $\{u_n\}$ is strictly increasing and $\lim_{n \to \infty} r(v_n) = w$, for some $w \in [c, d]$. Then

$$\lim_{n \to \infty} r(p_n) = \lim_{n \to \infty} (u_n, r(v_n)) = (u, w).$$

We have $(u, w) \in \xi^{-1}(E)$ since $\xi^{-1}(E)$ is compact. Note that the proof is similar to the following one if $\{u_n\}$ is strictly decreasing. There are two cases to be considered:

1) If $v > u^{**}$, then $r(v) = u^{**}$. From $v > u^{**}$ and $\lim v_n = v$, we may assume without loss of generality that there exists $n_0 \in \mathbb{N}$ such that $r(v_n) = u_n^{**}$, for each $n \geq n_0$. Let α be a geodesic joining $\xi(u, u^{**})$ and $\xi(u_{n_0}, u_{n_0}^{**})$ with $\alpha(t_1) = \xi(u_{n_0}, u_{n_0}^{**})$, $\alpha(t_2) = \xi(u, u^{**})$, and $t_1 > t_2$. Note that $\alpha(t)$ is in E, for each $t \in [t_2, t_1]$ since E is convex. It follows from Lemma 3.2 that $\alpha_1'(t_2) < 0$. Then there exists $\delta > 0$ such that α_1 is strictly decreasing on $(t_2, t_2+\delta)$. For each $n \geq n_0$ with $u_n \geq \alpha_1(t_2 + \delta/2)$, from the continuity of α_1, there exists s_n in $(t_2, t_2 + \delta)$ with

$$\alpha_1(s_n) = u_n.$$

Since $\alpha(s_n)$ is in E, for each $n \geq n_0$ with $u_n \geq \alpha_1(t_2 + \delta/2)$, so

$$\alpha_2(s_n) \leq u_n^{**}.$$

Note that $\lim_{n\to\infty} s_n = t_2$ since $\{\alpha_1(s_n)\} = \{u_n\}$ is a strictly increasing sequence with

$$\lim_{n\to\infty} \alpha_1(s_n) = \lim_{n\to\infty} u_n = u = \alpha_1(t_2)$$

and α_1 is strictly decreasing on $(t_2, t_2 + \delta)$. Then

$$w = \lim_{n\to\infty} r(v_n) = \lim_{n\to\infty} u_n^{**} \geq \lim_{n\to\infty} \alpha_2(s_n) = \alpha_2(t_2) = u^{**}.$$

From

$$w \leq \sup\{\beta \in \mathbb{R} : (u, \beta) \in \xi^{-1}(E)\} = u^{**},$$

so

$$w = u^{**}.$$

Therefore,

$$\lim_{n\to\infty} r(p_n) = \lim_{n\to\infty} (u_n, u_n^{**}) = (u, w) = (u, u^{**}) = (u, r(v)) = r(p).$$

2) If $v < u^*$, then $r(v) = u^*$. We may assume that there exists $n_0 \in \mathbb{N}$ such that $r(v_n) = u_n^*$, for each $n \geq n_0$. If α is a geodesic joining $\xi(u, u^*)$ and $\xi(u_{n_0}, u_{n_0}^*)$ with $\alpha(t_1) = \xi(u_{n_0}, u_{n_0}^*)$, $\alpha(t_2) = \xi(u, u^*)$, and $t_1 > t_2$, it follows from Lemma 3.2 that $\alpha_1'(t_2) < 0$. So α_1 is strictly decreasing on $(t_2, t_2 + \delta)$, for some $\delta > 0$. For each $n \geq n_0$ with $u_n \geq \alpha_1(t_2 + \delta/2)$, there exists s_n in $(t_2, t_2 + \delta)$ with

$$\alpha_1(s_n) = u_n.$$

Since $\alpha(s_n)$ is in E, for each $n \geq n_0$ with $u_n \geq \alpha_1(t_2 + \delta/2)$, so

$$\alpha_2(s_n) \geq u_n^*.$$

Note that $\lim_{n\to\infty} s_n = t_2$. Therefore,

$$w = \lim_{n\to\infty} r(v_n) = \lim_{n\to\infty} u_n^* \leq \lim_{n\to\infty} \alpha_2(s_n) = \alpha_2(t_2) = u^*.$$

From

$$w \geq \inf\{\beta \in \mathbb{R} : (u, \beta) \in \xi^{-1}(E)\} = u^*,$$

thus

$$w = u^*.$$

Then

$$\lim_{n\to\infty} r(p_n) = \lim_{n\to\infty} (u_n, u_n^*) = (u, w) = (u, u^*) = (u, r(v)) = r(p).$$

Therefore, $r : I^2 \to \xi^{-1}(E)$ is continuous.

If $T : E \to E$ is a continuous function, then $\xi^{-1}T\xi : \xi^{-1}(E) \to \xi^{-1}(E)$ is continuous. It follows that $\xi^{-1}T\xi r : I^2 \to \xi^{-1}(E)$ is continuous. From Schuader fixed point theorem, $\xi^{-1}T\xi r$ has a fixed point, say q. Then

$$q = (\xi^{-1}T\xi r)q.$$

Note that $q = (\xi^{-1}T\xi r)q$ is in $\xi^{-1}(E)$ since the image of $\xi^{-1}T\xi$ is $\xi^{-1}(E)$. Then $rq = q$. Therefore,

$$\xi q = (T\xi r)q = (T\xi)rq = (T\xi)q = T(\xi q).$$

Thus T has a fixed point. □

From the last paragraph of the proof of the above theorem, we have the following lemma.

Lemma 3.4. *Let E be a topological space, $\xi : B \to E$ be a homeomorphism from a Banach space B into E, and $T : E \to E$ be a continuous function. If there exists a retraction $r : C \to \xi^{-1}(E)$ of a compact convex subset C of B into $\xi^{-1}(E)$, then T has a fixed point.*

Let $\xi : \mathbb{R}^2 \to \mathbb{R}^3$ be the geographical parametrization

$$\xi = (r \cos x \cos y, r \sin x \cos y, r \sin y).$$

And let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a geodesic in the surface $\xi((\pi/4, 3\pi/4) \times (0, \pi/2))$ joining $\xi(u_1, v_1)$ and $\xi(u_2, v_2)$ with $u_1 < u_2$. Assume that $\alpha(t_1) = \xi(u_1, v_1)$ and $\alpha(t_2) = \xi(u_2, v_2)$. It follows that the image of α is a part of a great circle in the sphere of radius r centered at $(0, 0, 0)$. Then $\xi(u_1, v_1)$ and $\xi(u_2, v_2)$ divide the image into two parts, the short one and the long one. It can be seen that the short one is in $\xi((\pi/4, 3\pi/4) \times (0, \pi/2))$ but the long one is not. Let $S = \alpha([a, b])$ denote the short part of the great circle. Then $t_1, t_2 \in [a, b]$. If $u_1 = \pi/2$ and $t_1 < t_2$, since $u_1 < u_2$, then $(\xi^{-1} \circ \alpha)_1'(t_1) > 0$. Indeed, S will be the long part of the image if $(\xi^{-1} \circ \alpha)_1'(t_1) < 0$, which leads to the contradiction. Similarly, if $u_2 = \pi/2$ and $t_1 > t_2$, then $(\xi^{-1} \circ \alpha)_1'(t_2) < 0$.

From the above face, we obtain the following lemma.

Lemma 3.5. *Let ξ be the geographical parametrization*

$$\xi = (r\cos x \cos y, r\sin x \cos y, r\sin y).$$

And let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a shortest geodesic in the surface $\xi((\pi/4, 3\pi/4) \times (0, \pi/2))$ joining $\xi(u_1, v_1)$ and $\xi(u_2, v_2)$ with $u_1 < u_2$. Assume that $\alpha(t_1) = \xi(u_1, v_1)$ and $\alpha(t_2) = \xi(u_2, v_2)$. Then:

1) If $u_1 = \pi/2$ and $t_1 < t_2$, then $(\xi^{-1} \circ \alpha)_1'(t_1) > 0$.

2) If $u_2 = \pi/2$ and $t_1 > t_2$, then $(\xi^{-1} \circ \alpha)_1'(t_2) < 0$.

The following theorem shows that the upper hemisphere of the sphere of radius r centered at $(0, 0, 0)$ with north pole and equator removed has the fixed point property. Note that its Gaussian curvature is $1/r^2$, greater than zero. Moreover, $(0, r/\sqrt{2}, r/\sqrt{2})$ and $(0, -r/\sqrt{2}, r/\sqrt{2})$ are in it but can not be joined by any geodesics of it. This implies that it is not a convex metric space. So Theorem 1.1 is useless in this case.

Theorem 3.6. *Let ξ be the geographical parametrization*

$$\xi = (r\cos x \cos y, r\sin x \cos y, r\sin y).$$

If E is a compact convex subset of the surface

$$X = \xi([0, 2\pi] \times (0, \pi/2)),$$

then there is a retraction r from a 2-dimensional interval I^2 onto $\xi^{-1}(E)$. Consequently, X has the fixed point property.

Proof. Let E be a compact convex subset of X. We may assume that E is not a curve segment. Observe that $x(\xi^{-1}(E))$ must be a proper subset of $[0, 2\pi]$ since X does not contain $(0, 0, r)$. From the symmetry of X, we may assume that $x(\xi^{-1}(E)) \subset (0, 2\pi)$. Note that ξ is one-to-one on $(0, 2\pi) \times (0, \pi/2)$, so $\xi : (0, 2\pi) \times (0, \pi/2) \to X$ is a patch.

Let

$$a = \inf\{\alpha \in \mathbb{R} : (\alpha, \beta) \in \xi^{-1}(E), \exists \beta \in \mathbb{R}\}$$

and

$$b = \sup\{\alpha \in \mathbb{R} : (\alpha, \beta) \in \xi^{-1}(E), \exists \beta \in \mathbb{R}\}.$$

For each $u \in [a, b]$, let

$$u^* = \inf\{\beta \in \mathbb{R} : (u, \beta) \in \xi^{-1}(E)\}$$

and

$$u^{**} = \sup\{\beta \in \mathbb{R} : (u, \beta) \in \xi^{-1}(E)\}.$$

Since E is compact convex, we have

$$\xi^{-1}(E) = \bigcup_{u \in [a,b]} [u^*, u^{**}].$$

Let

$$c = \inf_{u\in[a,b]} u^*, \;\; d = \sup_{u\in[a,b]} u^{**},$$

and

$$I^2 = [a, b] \times [c, d].$$

Note that

$$\xi^{-1}(E) \subset I^2 \subset (0, 2\pi) \times (0, \pi/2).$$

Define $r : I^2 \to \xi^{-1}(E)$ by

$$r(u,v) = \begin{cases} (u,v), & \text{if } (u,v) \in \xi^{-1}(E), \\ (u,u^{**}), & \text{if } v > u^{**}, \\ (u,u^*), & \text{if } v < u^*. \end{cases}$$

To show that r is continuous. Let $\{p_n\}$ be a sequence in I^2 with

$$\lim_{n\to\infty} p_n = p.$$

If p is in the interior of $\xi^{-1}(E)$, then

$$\lim_{n\to\infty} r(p_n) = \lim_{n\to\infty} p_n = p = r(p).$$

Assume that p is not in the interior. From the convexity of E, we have $v > u^{**}$ or $v < u^*$. Write $p = (u, v), p_n = (u_n, v_n), r(p) = (u, r(v))$, and $r(p_n) = (u_n, r(v_n))$, for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Without loss of generality, we assume that $\{u_n\}$ is strictly increasing and $\lim_{n\to\infty} r(v_n) = w$, for some $w \in [c, d]$. Therefore,

$$\lim_{n\to\infty} r(p_n) = \lim_{n\to\infty} (u_n, r(v_n)) = (u, w).$$

Note that $(u, w) \in \xi^{-1}(E)$. From the symmetry of X, we may assume that $u = \pi/2$. There are two cases to be considered:

case 1. If $v > u^{**}$, then $r(v) = u^{**}$. From $v > u^{**}$ and $\lim_{n\to\infty}(u_n, v_n) = (u, v)$, we may assume that there exists $n_0 \in \mathbb{N}$ such that $r(v_n) = u_n^{**}$, and $|u_n - u| < \pi/4$, for each $n \ge n_0$. Let α be a geodesic joining $\xi(u, u^{**})$ and $\xi(u_{n_0}, u_{n_0}^{**})$ with $\alpha(t_1) = \xi(u_{n_0}, u_{n_0}^{**})$, $\alpha(t_2) = \xi(u, u^{**})$, and $t_1 > t_2$. From Lemma 3.5, $(\xi^{-1} \circ \alpha)_1'(t_2) < 0$. Then there exists $\delta > 0$ such that $(\xi^{-1} \circ \alpha)_1$ is strictly decreasing on $(t_2, t_2 + \delta)$. For each $n \ge n_0$ with $u_n \ge (\xi^{-1} \circ \alpha)_1(t_2 + \delta/2)$, from the continuity of $(\xi^{-1} \circ \alpha)_1$, there exists s_n in $(t_2, t_2 + \delta)$ with

$$(\xi^{-1} \circ \alpha)_1(s_n) = u_n.$$

Since $(\xi^{-1} \circ \alpha)(s_n)$ is in E, for each $n \ge n_0$ with $u_n \ge (\xi^{-1} \circ \alpha)_1(t_2 + \delta/2)$, so

$$(\xi^{-1} \circ \alpha)_2(s_n) \le u_n^{**}.$$

Note that $\lim_{n\to\infty} s_n = t_2$. Then

$$w = \lim_{n\to\infty} r(v_n) = \lim_{n\to\infty} u_n^{**} \geq \lim_{n\to\infty} (\xi^{-1}\circ\alpha)_2(s_n) = (\xi^{-1}\circ\alpha)_2(t_2) = u^{**}.$$

Since

$$w \leq \sup\{\beta \in \mathbb{R} : (u,\beta) \in \xi^{-1}(E)\} = u^{**},$$

so

$$w = u^{**}.$$

Then

$$\lim_{n\to\infty} r(p_n) = \lim_{n\to\infty} (u_n, u_n^{**}) = (u,w) = (u,u^{**}) = (u, r(v)) = r(p).$$

case 2. If $v < u^*$, then $r(v) = u^*$. We may assume that there exists $n_0 \in \mathbb{N}$ such that $r(v_n) = u_n^*$ and $|u_n - u| < \pi/4$, for each $n \geq n_0$. If α is a geodesic joining $\xi(u,u^*)$ and $\xi(u_{n_0}, u_{n_0}^*)$ with $\alpha(t_1) = \xi(u_{n_0}, u_{n_0}^*)$, $\alpha(t_2) = \xi(u,u^*)$, and $t_1 > t_2$, from Lemma 3.5, then $(\xi^{-1}\circ\alpha)_1'(t_2) < 0$. Thus $(\xi^{-1}\circ\alpha)_1$ is strictly decreasing on $(t_2, t_2+\delta)$, for some $\delta > 0$. For each $n \geq n_0$ with $u_n \geq (\xi^{-1}\circ\alpha)_1(t_2+\delta/2)$, there exists s_n in $(t_2, t_2+\delta)$ with $(\xi^{-1}\circ\alpha)_1(s_n) = u_n$. Since $(\xi^{-1}\circ\alpha)(s_n)$ is in E, for each $n \geq n_0$ with $u_n \geq (\xi^{-1}\circ\alpha)_1(t_2+\delta/2)$, so

$$(\xi^{-1}\circ\alpha)_2(s_n) \geq u_n^*.$$

Therefore,

$$w = \lim_{n\to\infty} r(v_n) = \lim_{n\to\infty} u_n^* \leq \lim_{n\to\infty} (\xi^{-1}\circ\alpha)_2(s_n) = (\xi^{-1}\circ\alpha)_2(t_2) = u^*.$$

It follows from

$$w \geq \inf\{\beta \in \mathbb{R} : (u,\beta) \in \xi^{-1}(E)\} = u^*$$

that

$$w = u^*.$$

Then

$$\lim_{n\to\infty} r(p_n) = \lim_{n\to\infty} (u_n, u_n^*) = (u,w) = (u,u^*) = (u, r(v)) = r(p).$$

Therefore, $r : I^2 \to \xi^{-1}(E)$ is a retraction. It follows from Lemma 3.4 that X has the fixed point property. □

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