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Convergence Theorems of the Modified SP-Iteration for G-Asymptotically Nonexpansive Mappings with Directed Grahps

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Abstract : In this paper, weak and strong convergence theorems of the modified SP-iteration are established for three G-asymptotically nonexpansive mappings in Banach spaces. We give some numerical example for supporting our main theorem and compare convergence rate between the modified SP-iteration and the modified Noor iteration.

Keywords : G-asymptotically nonexpansive mapping; directed graph; common fixed point; Banach space.
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1 Introduction

Let C be a nonempty subset of a Banach space X. Let $F(T)$ be the set of all fixed points of a mapping T. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$, $\{k_n\} \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

whenever $x, y \in C$ and all $n \geq 1$. The set of common fixed points of three mappings T_1, T_2 and T_3 denoted by $F = \cap_{i=1}^{n} F(T_i)$.

Some mathematicians have studied the convergence theorem of common fixed points for asymptotically nonexpansive mappings in Banach spaces. The details of the research of such mathematician is shown in the references [1] to [4].

In 2008, the mathematician namely Jachymski [5] has generalized Banach's contraction principle to mappings on a metric space endowed with a graph by using the combination concepts between the fixed point theory and the graph theory. After the successful research of Jachymski, there are next generation of mathematicians widely brought the concept of graph theory as a tool to study on the convergence theorem of common fixed point in Hilbert spaces and Banach spaces. Here are a couple of examples of such mathematicians.

In 2015, Tiammee et al. [6] prove the theory of the strong convergence of the Halpern iteration for a G-nonexpansive mapping in Hilbert spaces with a directed graph. In 2018, Suparatulatorn et al. [7] prove the weak and strong convergence theorems of the modified S-iteration for finding a common fixed point of two G-nonexpansive mappings in Banach spaces with directed graphs.

In this paper, we study the following iteration

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_3^n x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T_2^n z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1^n y_n, \end{cases} \tag{1.1}$$

for all $n \geq 1$, where $x_1 \in C$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ and is called the modified SP-iteration. We establish some strong and weak convergence theorems for three G-asymptotically nonexpansive mappings in a uniformly convex Banach space endowed with a directed graph. Last but not least, we also present the numerical example for the modified SP-iteration scheme to compare with the modified Noor iteration which has been shown in the reference [8].

2 Preliminaries

In this section, we provide and recall some definitions and lemmas which will be used in the next sections.

Let C be a nonempty subset of a real Banach space X. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, i.e., $\triangle = \{(x,x) : x \in C\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edges. Thus we can identify the graph G with the pair $(V(G), E(G))$. A mapping $T : C \to C$ is said to be *G-asymptotically nonexpansive* if T satisfies the following conditions:

(i) T preserves edges of G (or T is edge-preserving), i.e.,

$$(x,y) \in E(G) \implies (Tx, Ty) \in E(G).$$

(ii) if there exists a sequence $\{k_n\}$, $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||,$$

whenever $(x, y) \in E(G)$ and each $n \geq 1$.

Definition 2.1. The *conversion* of a graph G is the graph obtained from G by reversing the direction of edges denoted by G^{-1} and

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Definition 2.2. Let x and y be vertices of a graph G. A path in G from x to y of length $N(N \in \mathbb{N} \cup \{0\})$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices for which $x_0 = x$, $x_N = y$, and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \ldots, N-1$.

Definition 2.3. A graph G is said to be *connected* if there is a path between any two vertices of the graph G.

Definition 2.4. Let $x_0 \in V(G)$ and $A \subseteq V(G)$. We say that

(i) *A is dominated by x_0* if $(x_0, x) \in E(G)$ for all $x \in A$.

(ii) *A dominates x_0* if for each $x \in A$, $(x, x_0) \in E(G)$.

Definition 2.5. A directed graph $G = (V(G), E(G))$ is said to be *transitive* if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

Definition 2.6. ([5])A mapping $T : X \to X$ is called *G-continuous* if given $u \in X$ and a sequence $\{u_n\}$ for $n \in \mathbb{N}$, $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$ imply $Tu_n \to Tu$.

Definition 2.7. A mapping $T : C \to C$ is called *G-semicompact* if for a sequence $\{x_n\}$ in C with $(x_n, x_{n+1}) \in E(G)$ and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$.

Definition 2.8. Let C be a nonempty subset of a Banach space X and let $T : C \to X$ be a mapping. Then, T is said to be *G-demiclosed* at $y \in X$ if, for any sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to $x \in C$, $\{Tx_n\}$ converges strongly to y and $(x_n, x_{n+1}) \in E(G)$ imply $Tx = y$.

Definition 2.9. ([9])A Banach space X is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for all $y \in X$ with $x \neq y$.

Property G [6] Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a directed graph with $V(G) = C$. We said that C has the *Property G* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ with $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Lemma 2.10 ([10])**.** *Suppose X is a Banach space satisfying Opial's condition and C is a nonempty weakly compact convex subset of X and $T : C \to C$ is an asymptotically nonexpansive mapping. Also, suppose $\{x_n\}$ is a sequence in C converging weakly to x and for which the sequence $\{x_n - Tx_n\}$ converges strongly to 0. Then $\{T^n x\}$ converges weakly to x.*

Lemma 2.11 ([11])**.** *Let $\{a_n\}, \{b_n\}$and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n,$$

for all $n = 1, 2, \dots$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim\limits_{n\to\infty} a_n$ *exists;*

(ii) $\lim\limits_{n\to\infty} a_n = 0$ *whenever* $\liminf\limits_{n\to\infty} a_n = 0$.

Lemma 2.12 ([12])**.** *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$||\lambda x + (1 - \lambda)y||^p \leq \lambda||x||^p + (1 - \lambda)||y||^p - w_p(\lambda)g(||x - y||),$$

for all x, y in $B_r = \{x \in X : ||x|| \leq r\}, \lambda \in [0, 1]$, where

$$w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda).$$

Lemma 2.13 ([13])**.** *Suppose C has Property G : $\{x_n\} \rightharpoonup x$ and $(x_n, x_{n+1}) \in E(G)$, there exists a subsequence $\{x_{n_k}\}$ such that for each k, $(x_{n_k}, x) \in E(G)$. Let T be a G-asymptotically nonexpansive mapping on C with asymptotic coefficient $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Then $I - T$ is G-demiclosed at 0.*

Lemma 2.14 ([14])**.** *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim\limits_{n\to\infty} ||x_n - u||$ and $\lim\limits_{n\to\infty} ||x_n - v||$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converges weakly to u and v, respectively, then $u = v$.*

3 Weak and Strong Convergence Theorems

In this section, we prove weak and strong convergence theorems of the modified SP-iteration scheme (1.1) for three G-asymptotically nonexpansive mappings in a Banach space endowed with a directed graph. Thoughtout of this section, let C be a nonempty closed, bounded and convex subset of a Banach space X with a directed graph $G = (V(G), E(G))$ such that $V(G) = C$ and $E(G)$ is convex. We also suppose that the graph G is transitive. Suppose $T_1, T_2, T_3 : C \to C$ are three G-asymptotically nonexpansive mappings with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$.

For prove our main theorems, we need the following propositons and lemmas.

Proposition 3.1. *Let $z_0 \in F$ be such that (x_0, z_0), (z_0, x_0) are in $E(G)$. Then (x_n, z_0), (y_n, z_0), (z_n, z_0), (z_0, x_n), (z_0, y_n), (z_0, z_n), (x_n, y_n), (x_n, z_n) and (x_n, x_{n+1}) are in $E(G)$.*

Proof. We proceed by induction. Since T_1, T_2 and T_3 are edge-preserving, it can be easily seen that T_1^n, T_2^n and T_3^n are also edge-preserving for all $n \in \mathbb{N}$. From $(x_0, z_0) \in E(G)$ and since T_3^n is edge-preserving, we get $(T_3^n x_0, z_0) \in E(G)$ and so $(z_0, z_0) \in E(G)$ because $E(G)$ is convex. Since T_2^n is edge-preserving and $(z_0, z_0) \in E(G)$, we have $(T_2^n z_0, z_0) \in E(G)$ and then $(y_0, z_0) \in E(G)$. Thus, since T_1^n is edge-preserving and $(y_0, z_0) \in E(G)$, we get $(T_1^n y_0, z_0) \in E(G)$. By the convexity of $E(G)$ and (y_0, z_0), $(T_1^n y_0, z_0) \in E(G)$, we get $(x_1, z_0) \in E(G)$. Thus, by edge-preserving of T_3^n, $(T_3^n x_1, z_0) \in E(G)$. Again, by the convexity of $E(G)$ and (x_1, z_0), $(T_3^n x_1, z_0) \in E(G)$, we have $(z_1, z_0) \in E(G)$. Since T_2^n is edge-preserving and $(z_1, z_0) \in E(G)$, we get $(T_2^n z_1, z_0) \in E(G)$ and so $(y_1, z_0) \in E(G)$ and hence $(T_1^n y_1, z_0) \in E(G)$. Next, we assume that $(x_k, z_0) \in E(G)$. Since T_3^n is edge-preserving, we obtain $(T_3^n x_k, z_0) \in E(G)$ and so $(z_k, z_0) \in E(G)$, by $E(G)$ is convex. Since T_2^n is edge-preserving and $(z_k, z_0) \in E(G)$, we get $(T_2^n z_k, z_0) \in E(G)$. By the convexity of $E(G)$ and (z_k, z_0), $(T_2^n z_k, z_0) \in E(G)$, we get $(y_k, z_0) \in E(G)$. Since T_1^n is edge-preserving, we have $(T_1^n y_k, z_0) \in E(G)$. By the convexity of $E(G)$, we get $(x_{k+1}, z_0) \in E(G)$. Then, since T_3^n is edge-preserving and $(x_{k+1}, z_0) \in E(G)$, we have $(T_3^n x_{k+1}, z_0) \in E(G)$ and so $(z_{k+1}, z_0) \in E(G)$ because $E(G)$ is convex. Hence, by edge-preserving of T_2^n, we obtain $(T_2^n z_{k+1}, z_0) \in E(G)$ and so $(y_{k+1}, z_0) \in E(G)$. Therefore (x_n, z_0), (y_n, z_0), $(z_n, z_0) \in E(G)$ for all $n \geq 1$. Using a similar argument, we can show that (z_0, x_n), (z_0, y_n) and $(z_0, z_n) \in E(G)$ under the assumption that $(z_0, x_0) \in E(G)$. By the transitivity of G, we obtain (x_n, y_n), (x_n, z_n) and $(x_n, x_{n+1}) \in E(G)$. This completes the proof. □

Proposition 3.2. *Let X be a Banach space with a directed graph G and let $T : C \to C$ be G-asymptotically nonexpansive mapping. If X has the Property G, then T is G-continuous.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. We show that $Tx_n \to Tx$. To show this, let $\{Tx_{n_k}\}$ be a subsequence of $\{Tx_n\}$. Since $(x_n, x_{n+1}) \in E(G)$ and G is transitive, we obtain $(x_{n_k}, x_{n_{k+1}}) \in E(G)$. Since $x_{n_k} \to x$ and $(x_{n_k}, x_{n_{k+1}}) \in E(G)$, by the Property G, there is a subsequence $\{x_{n'_k}\}$ of $\{x_{n_k}\}$ such that $(x_{n'_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Since T is G-asymptotically nonexpansive mapping and $(x_{n'_k}, x) \in E(G)$, we obtain

$$||Tx_{n'_k} - Tx|| \leq k_1 ||x_{n'_k} - x|| \to 0$$

as $k \to \infty$. Thus $Tx_{n'_k} \to Tx$. By the double extract subsequence principle, we include that $Tx_n \to Tx$. Then T is G-continuous. □

Lemma 3.3. *Let X be a uniformly convex Banach space and $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_1 \in C$ and $z_0 \in F$. Then* *(i)* $\lim_{n\to\infty} ||x_n - z_0||$ *exists.*

(ii) *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1,$ *then* $\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0.$

(iii) *If* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$ *then* $\lim_{n\to\infty} ||T_2^n z_n - z_n|| = 0.$

(iv) *If* $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1,$ *then* $\lim_{n\to\infty} ||T_3^n x_n - x_n|| = 0.$

Proof. Let $z_0 \in F$. By Proposition 3.1, (x_n, z_0), (y_n, z_0) and $(z_n, z_0) \in E(G)$. Choose a number $r > 0$ such that $C \subseteq B_r$ and $C - C \subseteq B_r$. By Lemma 2.12, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$||\lambda x + (1-\lambda)y||^2 \le \lambda||x||^2 + (1-\lambda)||y||^2 - w_2(\lambda)g(||x-y||) \tag{3.1}$$

for all $x, y \in B_r, \lambda \in [0,1]$, where $w_2(\lambda) = \lambda(1-\lambda)^2 + \lambda^2(1-\lambda)$. It follows from (3.1) and G-asymptotically nonexpansiveness of T_3 that

$$\begin{aligned} ||z_n - z_0||^2 &= ||(1-\gamma_n)(x_n - z_0) + \gamma_n(T_3^n x_n - z_0)||^2 \\ &\le (1-\gamma_n)||x_n - z_0||^2 + \gamma_n||T_3^n x_n - z_0||^2 - w_2(\gamma_n)g(||T_3^n x_n - x_n||) \\ &\le (1-\gamma_n)||x_n - z_0||^2 + \gamma_n k_n^2||x_n - z_0||^2 - w_2(\gamma_n)g(||T_3^n x_n - x_n||) \\ &= (1-\gamma_n + \gamma_n k_n^2)||x_n - z_0||^2 - w_2(\gamma_n)g(||T_3^n x_n - x_n||). \end{aligned} \tag{3.2}$$

Again, it follows from (3.1) and G-asymptotically nonexpansiveness of T_2 that

$$\begin{aligned} ||y_n - z_0||^2 &= ||(1-\beta_n)(z_n - z_0) + \beta_n(T_2^n z_n - z_0)||^2 \\ &\le (1-\beta_n)||z_n - z_0||^2 + \beta_n||T_2^n z_n - z_0||^2 - w_2(\beta_n)g(||T_2^n z_n - z_n||) \\ &\le (1-\beta_n)||z_n - z_0||^2 + \beta_n k_n^2||z_n - z_0||^2 - w_2(\beta_n)g(||T_2^n z_n - z_n||) \\ &= (1-\beta_n + \beta_n k_n^2)||z_n - z_0||^2 - w_2(\beta_n)g(||T_2^n z_n - z_n||) \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned} ||x_{n+1} - z_0||^2 &= ||(1-\alpha_n)(y_n - z_0) + \alpha_n(T_1^n y_n - z_0)||^2 \\ &\le (1-\alpha_n)||y_n - z_0||^2 + \alpha_n||T_1^n y_n - z_0||^2 - w_2(\alpha_n)g(||T_1^n y_n - y_n||) \\ &\le (1-\alpha_n)||y_n - z_0||^2 + \alpha_n k_n^2||y_n - z_0||^2 - w_2(\alpha_n)g(||T_1^n y_n - y_n||) \\ &= (1-\alpha_n + \alpha_n k_n^2)||y_n - z_0||^2 - w_2(\alpha_n)g(||T_1^n y_n - y_n||) \\ &\le (1-\alpha_n + \alpha_n k_n^2)((1-\beta_n + \beta_n k_n^2)||z_n - z_0||^2 \\ &\quad - w_2(\beta_n)g(||T_2^n z_n - z_n||)) - w_2(\alpha_n)g(||T_1^n y_n - y_n||) \\ &\le (1-\alpha_n + \alpha_n k_n^2)((1-\beta_n + \beta_n k_n^2)((1-\gamma_n + \gamma_n k_n^2)||x_n - z_0||^2 \\ &\quad - w_2(\gamma_n)g(||T_3^n x_n - x_n||)) - w_2(\beta_n)g(||T_2^n z_n - z_n||)) \\ &\quad - w_2(\alpha_n)g(||T_1^n y_n - y_n||) \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n+\alpha_n k_n^2)((1-\beta_n+\beta_n k_n^2)(1-\gamma_n+\gamma_n k_n^2)||x_n-z_0||^2\\
&\quad -(1-\beta_n+\beta_n k_n^2)w_2(\gamma_n)g(||T_3^n x_n-x_n||)\\
&\quad -w_2(\beta_n)g(||T_2^n z_n-z_n||))-w_2(\alpha_n)g(||T_1^n y_n-y_n||)\\
&= (1-\alpha_n+\alpha_n k_n^2)(1-\beta_n+\beta_n k_n^2)(1-\gamma_n+\gamma_n k_n^2)||x_n-z_0||^2\\
&\quad -(1-\alpha_n+\alpha_n k_n^2)(1-\beta_n+\beta_n k_n^2)w_2(\gamma_n)g(||T_3^n x_n-x_n||)\\
&\quad -(1-\alpha_n+\alpha_n k_n^2)w_2(\beta_n)g(||T_2^n z_n-z_n||)\\
&\quad -w_2(\alpha_n)g(||T_1^n y_n-y_n||)\\
&\le ||x_n-z_0||^2+(k_n^2-1)(\beta_n+\alpha_n+\alpha_n\beta_n k_n^2+\gamma_n+\gamma_n\beta_n k_n^2\\
&\quad +\gamma_n\alpha_n k_n^2+\alpha_n\beta_n\gamma_n k_n^4)||x_n-z_0||^2\\
&\quad -(1-\beta_n+\beta_n k_n^2)w_2(\gamma_n)g(||T_3^n x_n-x_n||)\\
&\quad -(1-\alpha_n+\alpha_n k_n^2)w_2(\beta_n)g(||T_2^n z_n-z_n||)\\
&\quad -w_2(\alpha_n)g(||T_1^n y_n-y_n||)
\end{aligned}
$$

Since $\{k_n\}$ and C are bounded, there exists a constant $M>0$ such that

$$(\beta_n+\alpha_n+\alpha_n\beta_n k_n^2+\gamma_n+\gamma_n\beta_n k_n^2+\gamma_n\alpha_n k_n^2+\alpha_n\beta_n\gamma_n k_n^4)||x_n-z_0||^2\le M$$

for all $n\ge 1$. It follows that

$$
\begin{aligned}
(1-\beta_n+\beta_n k_n^2)w_2(\gamma_n)g(||T_3^n x_n-x_n||) \le{}& ||x_n-z_0||^2-||x_{n+1}-z_0||^2\\
&+M(k_n^2-1) \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
(1-\alpha_n+\alpha_n k_n^2)w_2(\beta_n)g(||T_2^n z_n-z_n||) \le{}& ||x_n-z_0||^2-||x_{n+1}-z_0||^2\\
&+M(k_n^2-1) \qquad (3.5)
\end{aligned}
$$

and

$$w_2(\alpha_n)g(||T_1^n y_n-y_n||)\le ||x_n-z_0||^2-||x_{n+1}-z_0||^2+M(k_n^2-1). \qquad (3.6)$$

(i) From (3.4), we obtain $||x_{n+1}-z_0||^2\le ||x_n-z_0||^2+M(k_n^2-1)$. Since $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$, it follows from Lemma 2.11 that $\lim_{n\to\infty}||x_n-z_0||$ exists.

(ii) If $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$, there exist some real number $\delta>0$ and a positive integer n_0 such that

$$w_2(\alpha_n)=\alpha_n(1-\alpha_n)^2+\alpha_n^2(1-\alpha_n)\ge\delta>0,$$

for all $n \geq n_0$. It follows from (3.6) that for any natural number $m \geq n_0$,

$$\begin{aligned}\sum_{n=n_0}^{m} g(||T_1^n y_n - y_n||) &\leq \sum_{n=n_0}^{m} w_2(\alpha_n) g(||T_1^n y_n - y_n||) \\ &\leq ||x_{n_0} - z_0||^2 - ||x_{m+1} - z_0||^2 + M \sum_{n=n_0}^{m} (k_n^2 - 1) \\ &\leq ||x_{n_0} - z_0||^2 - M \sum_{n=n_0}^{m} (k_n^2 - 1). \qquad (3.7)\end{aligned}$$

Since $0 \leq t^2 - 1 \leq 2t(t-1)$ for all $t \geq 1$, the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $m \to \infty$ in inequality (3.7). Thus

$$\sum_{n=n_0}^{\infty} g(||T_1^n y_n - y_n||) < \infty,$$

and therefore $\lim_{n\to\infty} g(||T_1^n y_n - y_n||) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0$.

(iii) If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$, then by using a similar method, together with inequality (3.5), it can be shown that $\lim_{n\to\infty} ||T_2^n z_n - z_n|| = 0$.

(iv) If $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $\liminf_{n\to\infty} \beta_n > 0$, then by using a similar method, together with inequality (3.4), it can be shown that $\lim_{n\to\infty} ||T_3^n x_n - x_n|| = 0$. □

Lemma 3.4. *Let X be a uniformly convex Banach space and $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$. If*

(*i*) $\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0$,

(*ii*) $\lim_{n\to\infty} ||T_2^n z_n - z_n|| = 0$,

(*iii*) $\lim_{n\to\infty} ||T_3^n x_n - x_n|| = 0$,

then $\lim_{n\to\infty} ||T_i x_n - x_n|| = 0$ *for all* $i = 1, 2, 3$.

Proof. Let $z_0 \in F$ be such that (x_0, z_0), (z_0, x_0) are in $E(G)$. By Proposition 3.1, we get (x_n, z_n), (x_n, y_n) and $(x_n, x_{n+1}) \in E(G)$. Note that $||z_n - x_n|| \leq \gamma_n ||T_3^n x_n - x_n||$. By (iii), we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \qquad (3.8)$$

Again note that $||y_n - x_n|| \leq ||z_n - x_n|| + \beta_n||T_2^n z_n - z_n||$. Using (3.8) and (ii), we obtain

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{3.9}$$

Further, note that $||x_{n+1} - x_n|| \leq ||y_n - x_n|| + \alpha_n||T_1^n y_n - y_n||$. Using (3.9) and (i), we get

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.10}$$

Observe that

$$\begin{aligned} ||x_{n+1} - T_1^n x_{n+1}|| &\leq ||x_{n+1} - x_n|| + ||T_1^n x_n - T_1^n x_{n+1}|| + ||T_1^n x_n - x_n|| \\ &\leq ||x_{n+1} - x_n|| + k_n||x_n - x_{n+1}|| + k_n||x_n - y_n|| \\ &\quad + ||T_1^n y_n - y_n|| + ||y_n - x_n|| \\ &= (1 + k_n)||x_{n+1} - x_n|| + (1 + k_n)||x_n - y_n|| + ||T_1^n y_n - y_n||. \end{aligned}$$

Using (3.9), (3.10) and (i), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - T_1^n x_{n+1}|| = 0.$$

Thus

$$\begin{aligned} ||x_{n+1} - T_1 x_{n+1}|| &\leq ||x_{n+1} - T_1^{n+1} x_{n+1}|| + ||T_1 x_{n+1} - T_1^{n+1} x_{n+1}|| \\ &\leq ||x_{n+1} - T_1^{n+1} x_{n+1}|| + k_1||x_{n+1} - T_1^n x_{n+1}|| \to 0 \end{aligned}$$

as $n \to \infty$, which implies

$$\lim_{n\to\infty} ||T_1 x_n - x_n|| = 0.$$

Note that

$$\begin{aligned} ||x_{n+1} - T_2^n x_{n+1}|| &\leq ||x_{n+1} - x_n|| + ||T_2^n x_n - T_2^n x_{n+1}|| + ||T_2^n x_n - x_n|| \\ &\leq ||x_{n+1} - x_n|| + k_n||x_n - x_{n+1}|| + k_n||x_n - z_n|| \\ &\quad + ||T_2^n z_n - z_n|| + ||z_n - x_n|| \\ &= (1 + k_n)||x_{n+1} - x_n|| + (1 + k_n)||x_n - z_n|| + ||T_2^n z_n - z_n||. \end{aligned}$$

Using (3.8), (3.10) and (ii), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - T_2^n x_{n+1}|| = 0.$$

Thus

$$\begin{aligned} ||x_{n+1} - T_2 x_{n+1}|| &\leq ||x_{n+1} - T_2^{n+1} x_{n+1}|| + ||T_2 x_{n+1} - T_2^{n+1} x_{n+1}|| \\ &\leq ||x_{n+1} - T_2^{n+1} x_{n+1}|| + k_2||x_{n+1} - T_2^n x_{n+1}|| \to 0 \end{aligned}$$

as $n \to \infty$, which implies

$$\lim_{n\to\infty} ||T_2x_n - x_n|| = 0.$$

Again note that

$$\begin{aligned}||x_{n+1} - T_3^n x_{n+1}|| &\le ||x_{n+1} - x_n|| + ||T_3^n x_n - T_3^n x_{n+1}|| + ||T_3^n x_n - x_n|| \\ &\le ||x_{n+1} - x_n|| + k_n||x_n - x_{n+1}|| + ||T_3^n x_n - x_n||.\end{aligned}$$

Using (3.10) and (iii), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - T_3^n x_{n+1}|| = 0.$$

Thus

$$\begin{aligned}||x_{n+1} - T_3x_{n+1}|| &\le ||x_{n+1} - T_3^{n+1}x_{n+1}|| + ||T_3x_{n+1} - T_3^{n+1}x_{n+1}|| \\ &\le ||x_{n+1} - T_3^{n+1}x_{n+1}|| + k_3||x_{n+1} - T_3^n x_{n+1}|| \to 0\end{aligned}$$

as $n \to \infty$, which implies

$$\lim_{n\to\infty} ||T_3x_n - x_n|| = 0.$$

□

Theorem 3.5. *Let X be a uniformly convex Banach space satisfying the Opial's condition and let C be a nonempty closed and convex subset of X. Let T_1, T_2 and T_3 be three G-asymptotically nonexpansive mappings on C with the nonempty common fixed point set $F = \cap_{i=1}^3 F(T_i)$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in* $[0,1]$ *satisfying*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, *and*

(*iii*) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Assume that C has the Property G. Let $x_0 \in C$ be fixed so that (x_0, z_0) and (z_0, x_0) are in $E(G)$ for some $z_0 \in F$. If $\{x_n\}$ is a sequence defined by recursion (1.1), then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. Let $z_0 \in F$ be such that $(x_0, z_0), (z_0, x_0) \in E(G)$. It follows from Lemma 3.3 (i) that $\lim_{n\to\infty} ||x_n - z_0||$ exists. So $\{x_n\}$ is bounded, hence it has a weakly convergent subsequence. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. For, let u and v be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, respectively. By Lemma 3.4, we have $\lim_{n\to\infty} ||T_1x_n - x_n|| = 0$ and $I - T_1$ is G-demiclosed with respect to zero by Lemma 2.13, therefore we obtain $T_1u = u$. Similarly, $T_2u = u$ and $T_3u = u$. Again in the same fashion, we can prove that $v \in F$. By Lemma 2.14, we have $u = v$. Thus $\{x_n\}$ converges weakly to a common fixed point in F. □

Theorem 3.6. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T_1, T_2 and T_3 be three G-asymptotically nonexpansive mappings on C with the nonempty common fixed point set $F = \cap_{i=1}^{3} F(T_i)$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, *and*

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Assume that C has the Property G and one of T_1, T_2 and T_3 is G-semicompact. Let $x_0 \in C$ be fixed so that (x_0, z_0) and (z_0, x_0) are in $E(G)$ for some $z_0 \in F$. If $\{x_n\}$ is a sequence defined by recursion (1.1), then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Proof. We may assume that T_1 is G-semicompact. By Lemma 3.3, we obtain $\{x_n\}$ is bounded. From Lemma 3.4, we get

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$$

for all $i = 1, 2, 3$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z_0$ as $k \to \infty$.Thus

$$\lim_{k\to\infty} ||x_{n_k} - T_i x_{n_k}|| = 0$$

for all $i = 1, 2, 3$. By Proposition 3.2, we obtain T_1, T_2 and T_3 are G-continuous. It follows that

$$||z_0 - T_i z_0|| = \lim_{k\to\infty} ||x_{n_k} - T_i x_{n_k}|| = 0$$

for all $i = 1, 2, 3$. This yield $z_0 \in F$ so that $\{x_{n_k}\}$ converges strongly to $z_0 \in F$. But again by Lemma 3.4, $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F$ therefore $\{x_n\}$ must itself converge to $z_0 \in F$. This completes the proof. □

4 Numerical Example

In this section, we give an example of the numerical experiments for supporting our main theorem. The next definitions give the idea of the comparison of the rate of convergence between the two iterative methods.

Definition 4.1 ([15]). *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{m_n\}$ are two iterations which converge to a fixed point q of T. Then $\{x_n\}$ is said to converge faster than $\{m_n\}$ if*

$$||x_n - q|| \leq ||m_n - q||$$

for all $n \geq 1$.

Definition 4.2 ([16]). *Suppose $\{a_n\}$ is a sequence that converges to a, with $a_n \neq a$ for all n. If positive constants λ and α exist with*

$$\lim_{n\to\infty} \frac{|a_{n+1} - a|}{|a_n - a|^{\alpha}} = \lambda,$$

then $\{a_n\}$ converges to a of order α, with asymptotic error constant λ. If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent, and if $\alpha = 2$, the sequence is quadratically convergent.

Definition 4.3 ([17]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, b, respectively. Assume there exists*

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(i) If $l = 0$, then it is said that the sequence $\{a_n\}$ converges to a faster than the sequence $\{b_n\}$ to b.

(ii) If $0 < l < \infty$, then we say that the sequences $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Definition 4.4 ([18]). *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{m_n\}$ are two iterations which converge to a fixed point q of T. We say that $\{x_n\}$ converges faster than $\{m_n\}$ to q if*

$$\lim_{n\to\infty} \frac{||x_n - q||}{||m_n - q||} = 0.$$

We now give an example which shows numerical experiment for supporting our main results and comparing the rate of convergence of the modified SP-iteration and the modified Noor iteration.

Example 4.5. *Let $X = \mathbb{R}$ and $C = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.50 < x$, $y \leq 1.70$. Define a mapping $T_1, T_2, T_3 : C \to C$ by*

$$\begin{aligned} T_1 x &= \frac{5}{8}\arcsin(x-1) + 1 \\ T_2 x &= \frac{1}{3}\tan(x-1) + 1 \\ T_3 x &= x^{\ln x} \end{aligned}$$

for all $x \in C$. Let $\alpha_n = \frac{n+1}{5n+3}$, $\beta_n = \frac{n+4}{10n+7}$ and $\gamma_n = \frac{n+2}{8n+5}$. Choose $m_n = x_0 = 1.4$. Let $\{x_n\}$ be a sequence generated by (1.1) and $\{m_n\}$ be a sequence generated by the modified Noor iteration. We obtain the following numerical experiments for common fixed point of T_1, T_2 and T_3 and rate of convergence of $\{x_n\}$ and $\{m_n\}$ as shown in Table 1 and 2.

For the first one, we would like to illustrate the rate of convergence of the modified Noor iteration and the modified SP-iteration.

Table 1

n	modified Noor	modified SP	Rate of convergence		
	m_n	x_n	$\|m_n - 1\|$	$\|x_n - 1\|$	$\frac{\|x_n-1\|}{\|m_n-1\|}$
1	1.3292	1.1951	0.3292	0.1951	0.5926
2	1.2773	1.1091	0.2773	0.1091	0.3935
3	1.2359	1.0650	0.2359	0.0650	0.2755
4	1.2017	1.0401	0.2017	0.0401	0.1987
5	1.1731	1.0253	0.1731	0.0253	0.1462
...	...	...	...	...	...
20	1.0196	1.0001	0.0196	0.0001	0.0030

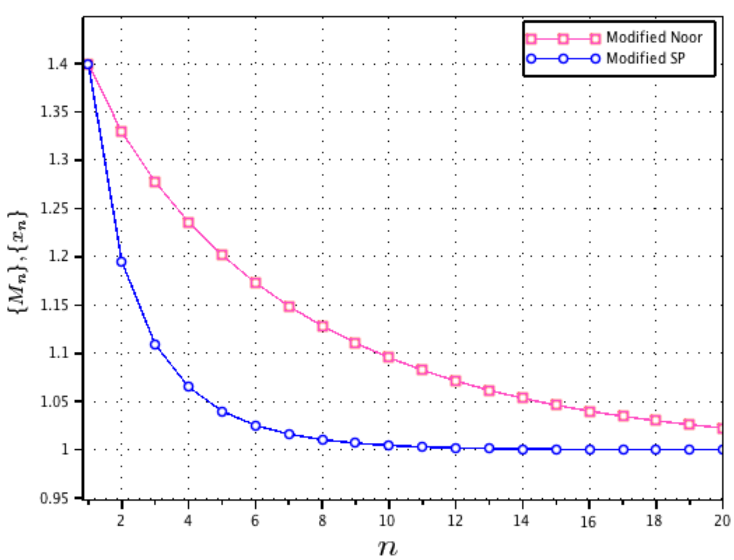

Figure 1: Comparison between the modified Noor iteration and the modified SP-iteration.

From Table 1 and Figure 1, we observe that $|x_n - 1| \leq |m_n - 1|$ and

$$\lim_{n\to\infty} \frac{|x_n - 1|}{|m_n - 1|} = 0,$$

so the sequence $\{x_n\}$ converges faster than the sequence $\{m_n\}$ generated by the modified Noor iteration.

Regarding second one, we would like to demonstrate the numerical errors of the modified Noor iteration and the modified SP-iteration.

Table 2

n	modified Noor		modified SP	
	m_n	$\|m_n - m_{n-1}\|$	x_n	$\|x_n - x_{n-1}\|$
1	1.3292	0.0519	1.1951	0.0860
2	1.2773	0.0414	1.1091	0.0441
3	1.2359	0.0342	1.0650	0.0249
4	1.2017	0.0287	1.0401	0.0148
5	1.1731	0.0242	1.0253	0.0091
...	...	...	...	...
20	1.0196	0.0026	1.0001	0.0000

We note that $x = 1$ is a common fixed point of T_1, T_2 and T_3. Also, we see that both $\{m_n\}$ and $\{x_n\}$ converge to $1 \in \cap_{i=1}^{3} F(T_i)$. For Figure 2, we would like to show the comparison of the errors between the modified Noor and the modified SP-iterations.

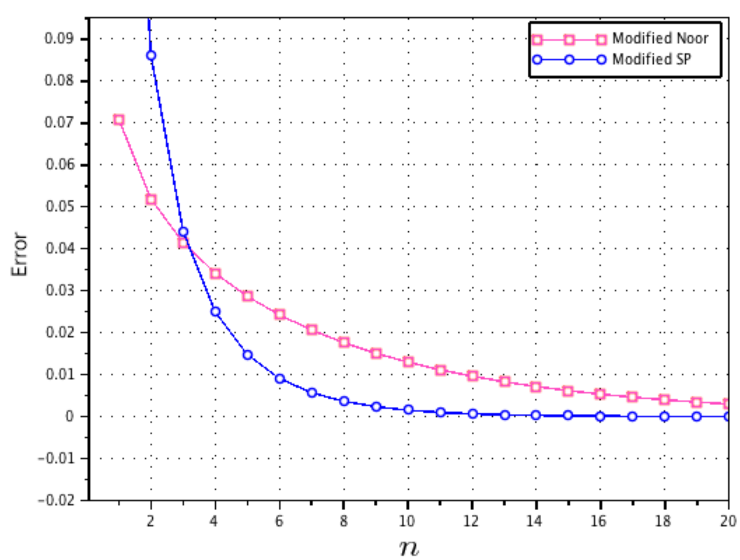

Figure 2: Comparison of errors of the modified Noor iteration and the modified SP-iteration

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