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Fourth-Order Conservative Algorithm for Nonlinear Wave Propagation: the Rosenau-KdV Equation

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Abstract : In this paper, we introduce a conservative difference method for solving the Rosenau-KdV equation. The existence of the approximate solution from the difference scheme is shown. We also prove the stability and convergence of this scheme. The presented method gives second- and fourth-order accurate in time and space, respectively. Numerical examples demonstrate the theoretical results.

Keywords : Rosenau-KdV equation; wave propagation; convergence.

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1 Introduction

A nonlinear wave phenomenon is one of the important areas of scientific research. A category of wave phenomena can be commonly expressed by using

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nonlinear partial differential equations. However, while nonlinear terms are implicated, analytical solutions of these equations are barely feasible. Therefore, numerical solution of these nonlinear partial differential equations is significantly necessary because only limited types of the equations are solvable by analytical methods. For example, the mathematical model of water wave attracts attention for a long time. These models aim to explain from smaller-scale waves such as ripples on the water surface to larger-scale waves such as a tsunami wave. There are mathematical models which describe the dynamics of wave such as the KdV equation, the Rosenau equation, the RLW equation and many others [1–11]. The KdV equation has been used in very wide applications which can be used to study wave propagation [1–4]. But the case of wave-wave and wave-wall interactions cannot be described by the KdV equation. To overcome this shortcoming of the KdV equation, Rosenau [5, 6] proposed an equation for describing the dynamics of dense discrete systems. The existence and uniqueness of the solution for the Rosenau equation were proved by Park [7]. For the further consideration of the nonlinear wave, the viscous term u_{xxx} needs to be included [10]

$$u_t + u_x + u_{xxx} + u_{xxxxt} + uu_x = 0. \tag{1.1}$$

This equation is usually called the Rosenau-KdV equation.

The behavior of the solution to the Rosenau-KdV equation with the Cauchy problem has been well studied for the past years [10, 12–14]. In this paper, we consider the following problem of the Rosenau-KdV equation with an initial condition

$$u(x,0) = u_0(x), \ \ (x_l \le x \le x_r), \tag{1.2}$$

and the boundary conditions

$$\begin{aligned} &u(x_l,t) = u(x_r,t) = 0, \ u_x(x_l,t) = u_x(x_r,t) = 0, \\ &u_{xx}(x_l,t) = u_{xx}(x_r,t) = 0, \ \ (0 \le t \le T). \end{aligned} \tag{1.3}$$

The initial-boundary value problem possesses the following conservative properties [15]:

$$Q(t) = \int_{x_l}^{x_r} u(x,t)dx = \int_{x_l}^{x_r} u_0(x,t)dx = Q(0),$$

and

$$E(t) = \|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0).$$

It is know that, the solitary wave solution for (1.1) is [10, 15]

$$u(x,t) = \left(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\right) \operatorname{sech}^4 \left[\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}\left(x - \left(\frac{1}{2} + \frac{1}{26}\sqrt{313}\right)t\right)\right].$$

Up to date, there are many numerical studies on numerical method to the initial-boundary value problem of the Rosenau-KdV equation [11, 16–18]. Finite difference methods are wildly used in most numerical works to solve the Rosenau-KdV.

Hu et al. [11] has proposed the second-order conservative finite difference scheme for the approximate solution. Zheng and Zhou [16] extended to study numerically on the generalized Rosenau-KdV. A conservative high-order accuracy scheme for the Rosenau-KdV was developed by mean of the Richardson extrapolation and Crank-Nicolson method, which was proposed by He et al. [17]. Obviously, the scheme is an two-level nonlinear implicit scheme which requires heavy algorithm to solve. To overcome this, we introduce an high-order accuracy linear three-level difference scheme for the Rosenau KdV by applying the average three-level technique and the Richardson extrapolation method. For the relevant works on the Rosenau-KdV, see [19–22] and references therein.

The content of this paper is organized as follows. In the next section, we describe a conservative implicit finite difference method. In Section 3, we discuss about solvability. The existence and uniqueness are proven in this section. In Section 4, we give complete proofs on convergence and stability of the finite difference scheme. The numerical results are given in Section 5 to confirm and illustrate our theoretical analysis. Then, we finish our paper by concluding remarks.

2 Finite Difference Scheme

In this section, we introduce a modified finite difference scheme for the formulation of (1.1)–(1.3). The solution domain $Q = \{(x,t)|\ x_l \le x \le x_r,\ 0 \le t \le T\}$, is covered by a uniform grid

$$Q_h = \{(x_i,t_n)|\ x_i = x_l + ih,\ t_n = n\tau,\ 0 \le i \le M,\ 0 \le n \le N\},$$

with spacings $h = (x_r - x_l)/M$ and $\tau = T/N$. Denote $u_i^n \approx u(x_i,t_n)$,

$$\bar{Q}_h = \{(x_i,t_n)|\ x_i = x_l + ih,\ t_n = n\tau,\ -2 \le i \le M+2,\ 0 \le n \le N\},$$

and

$$Z_h^0 = \{u^n = (u_i^n)|\ u_{-2} = u_{-1} = u_0 = u_M = u_{M+1} = u_{M+2} = 0,\ -2 \le i \le M+2\}.$$

We use the following notations for simplicity:

$$\bar{u}_i^n = \frac{u_i^{n+1} + u_i^{n-1}}{2}, \qquad (u_i^n)_{\hat{t}} = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau},$$
$$(u_i^n)_x = \frac{u_{i+1}^n - u_i^n}{h}, \qquad (u_i^n)_{\bar{x}} = \frac{u_i^n - u_{i-1}^n}{h},$$
$$(u_i^n)_{\hat{x}} = \frac{u_{i+1}^n - u_{i-1}^n}{2h}, \qquad (u_i^n)_{\ddot{x}} = \frac{u_{i+2}^n - u_{i-2}^n}{4h},$$

$$(u^n,v^n) = h\sum_{i=1}^{M-1} u_i^n v_i^n, \quad \|u^n\|^2 = (u^n,u^n), \quad \|u^n\|_\infty = \max_{1\le i\le M-1} |u_i^n|.$$

The following method is a proposed finite difference scheme to solve problem (1.1)–(1.3):

$$(u_i^n)_{\hat{t}} + \frac{4}{3}(\bar{u}_i^n)_{\hat{x}} - \frac{1}{3}(\bar{u}_i^n)_{\ddot{x}} + \frac{5}{3}(u_i^n)_{xx\bar{x}\bar{x}t} - \frac{2}{3}(u_i^n)_{x\bar{x}\hat{x}\hat{x}t} + \frac{3}{2}(\bar{u}_i^n)_{x\bar{x}\hat{x}} - \frac{1}{2}(\bar{u}_i^n)_{x\bar{x}\ddot{x}} + \varphi(u_i^n, \bar{u}_i^n) = 0, \quad 1 \le i \le M-1, \ \ 1 \le n \le N-1, \quad (2.1)$$

where

$$\varphi(u_i^n, \bar{u}_i^n) = \frac{4}{9}\left(u_i^n(\bar{u}_i^n)_{\hat{x}} + (u_i^n\bar{u}_i^n)_{\hat{x}}\right) - \frac{1}{9}\left(u_i^n(\bar{u}_i^n)_{\ddot{x}} + (u_i^n\bar{u}_i^n)_{\ddot{x}}\right),$$

$$u_i^0 = u_0(x_i), \qquad 0 \le i \le M, \quad (2.2)$$

$$u^n \in Z_h^0, \ u_0^n = u_J^n = 0, \ (u_0^n)_{\hat{x}} = (u_J^n)_{\hat{x}} = 0, \quad (u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0, \ \ 1 \le n \le N. \quad (2.3)$$

To prove that the proposed scheme is conservative, we need following lemma.

Lemma 2.1. *For any two mesh functions: $v, u \in Z_h^0$, we have*

$$(u_{\hat{x}}^n, v^n) = -(u^n, v_{\hat{x}}^n), \qquad (u_{\ddot{x}}^n, v^n) = -(u^n, v_{\ddot{x}}^n),$$
$$(u_{x\bar{x}\hat{x}\hat{x}}^n, u^n) = \|u_{x\hat{x}}\|^2, \qquad (u_{xx\bar{x}\bar{x}}^n, u^n) = \|u_{xx}^n\|^2.$$

Theorem 2.2. *Suppose that $u_0 \in H_0^2$, $u \in C^{8,3}$ then the scheme (2.1)–(2.3) is conservative*

$$Q^n = \frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1} + u_i^n\right) + \tau h \sum_{i=1}^{M-1}\left[\frac{2}{9}u_i^n\left(u_i^{n+1}\right)_{\hat{x}} - \frac{1}{18}u_i^n\left(u_i^{n+1}\right)_{\ddot{x}}\right] = Q^{n-1} = \cdots = Q^0, \quad (2.4)$$

and

$$E^n = \left(\left\|u^{n+1}\right\|^2 + \|u^n\|^2\right) + \frac{5}{3}\left(\left\|u_{xx}^{n+1}\right\|^2 + \|u_{xx}^n\|^2\right) - \frac{2}{3}\left(\left\|u_{x\hat{x}}^{n+1}\right\|^2 + \|u_{x\hat{x}}^n\|^2\right) = E^{n-1} = \cdots = E^0. \quad (2.5)$$

Proof. By multiplying (2.1) by h , summing up for i form 1 to $M-1$ and considering the boundary condition (2.3) together with Lemma 2.1, we get

$$\frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1} - u_i^{n-1}\right) + \tau h \sum_{i=1}^{M-1}\left[\frac{2}{9}\left(u_i^n\left(u_i^{n+1}\right)_{\hat{x}} - u_i^{n-1}(u_i^n)_{\hat{x}}\right) - \frac{1}{18}\left(u_i^n\left(u_i^{n+1}\right)_{\ddot{x}} - u_i^{n-1}(u_i^n)_{\ddot{x}}\right)\right] = 0.$$

Then, this gives (2.4).
We then take an inner product between (2.1) and $2\bar{u}^n = u^{n+1} + u^n$. We obtain

$$\begin{aligned}\frac{1}{2\tau}\left(\|u^{n+1}\|^2 - \|u^{n-1}\|^2\right) + \frac{5}{6\tau}\left(\|u_{xx}^{n+1}\|^2 - \|u_{xx}^{n-1}\|^2\right)\\ - \frac{1}{3\tau}\left(\|u_{x\hat{x}}^{n+1}\|^2 - \|u_{x\hat{x}}^{n-1}\|^2\right) + \frac{4}{3}\left(\bar{u}_{\hat{x}}^n, \bar{u}^n\right) - \frac{1}{3}\left(\bar{u}_{\ddot{x}}^n, \bar{u}^n\right)\\ + \frac{3}{2}\left(\bar{u}_{x\bar{x}\hat{x}}^n, \bar{u}^n\right) - \frac{1}{2}\left(\bar{u}_{x\bar{x}\ddot{x}}^n, \bar{u}^n\right) + \left(\varphi\left(u^n, \bar{u}^n\right), \bar{u}^n\right) = 0, \quad (2.6)\end{aligned}$$

by considering the boundary condition (2.3) and Lemma 2.1. According to

$$(\bar{u}_{\hat{x}}^n, \bar{u}^n) = 0, \ \ (\bar{u}_{\ddot{x}}^n, \bar{u}^n) = 0, \ \ (\bar{u}_{x\bar{x}\hat{x}}^n, \bar{u}^n) = 0, \ \ (\bar{u}_{x\bar{x}\ddot{x}}^n, \bar{u}^n) = 0, \quad (2.7)$$

and

$$\begin{aligned}&(\varphi(u^n, \bar{u}^n), \bar{u}^n) \quad (2.8)\\ &= \frac{4h}{9}\sum_{i=1}^{M-1}\left[(u_i^n(\bar{u}_i^n)_{\hat{x}} + (u_i^n\bar{u}_i^n)_{\hat{x}})\right]\bar{u}_i^n - \frac{h}{9}\sum_{i=1}^{M-1}\left[u_i^n(\bar{u}_i^n)_{\ddot{x}} + (u_i^n\bar{u}_i^n)_{\ddot{x}}\right]\bar{u}_i^n\\ &= \frac{2}{9}\sum_{i=1}^{M-1}\left(u_i^n u_i^{n+1}u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1}u_i^{n+1} + u_{i+1}^n u_i^{n+1}u_{i+1}^{n+1} - u_{i-1}^n u_{i-1}^{n+1}u_i^{n+1}\right)\\ &\quad - \frac{1}{72}\sum_{i=1}^{M-1}\left(u_i^n u_i^{n+1}u_{i+2}^{n+1} - u_i^n u_{i-2}^{n+1}u_i^{n+1} + u_{i+2}^n u_i^{n+1}u_{i+2}^{n+1} - u_{i-2}^n u_{i-2}^{n+1}u_i^{n+1}\right)\\ &= \frac{2}{9}\sum_{i=1}^{M-1}\left[\left(u_i^n u_i^{n+1}u_{i+1}^{n+1} - u_{i-1}^n u_{i-1}^{n+1}u_i^{n+1}\right) + \left(u_{i+1}^n u_i^{n+1}u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1}u_i^{n+1}\right)\right]\\ &\quad - \frac{1}{72}\sum_{i=1}^{M-1}\left[\left(u_i^n u_i^{n+1}u_{i+2}^{n+1} - u_{i-2}^n u_{i-2}^{n+1}u_i^{n+1}\right) + \left(u_{i+2}^n u_i^{n+1}u_{i+2}^{n+1} - u_i^n u_{i-2}^{n+1}u_i^{n+1}\right)\right]\\ &= 0,\end{aligned}$$

from (2.6)–(2.8) , we have

$$\left(\|u^{n+1}\|^2 - \|u^{n-1}\|^2\right) + \frac{5}{3}\left(\|u_{xx}^{n+1}\|^2 - \|u_{xx}^{n-1}\|^2\right) - \frac{2}{3}\left(\|u_{x\hat{x}}^{n+1}\|^2 - \|u_{x\hat{x}}^{n-1}\|^2\right) = 0.$$

Finally, this gives (2.5). This completes the proof. □

3 Solvability

In this section, we prove the existence and uniqueness of our proposed scheme. This implies uniquely solvable.

Theorem 3.1. *The finite difference scheme* (2.1)– (2.3) *is uniquely solvable.*

Proof. By using the mathematical induction, we can determine u^0 uniquely by initial condition and then choose a fourth-order method to compute u^1. Now, suppose $u^0, u^1, u^2, ..., u^n$ be solved uniquely. By considering the equation (2.1) for u^{n+1}, we have

$$\frac{1}{2\tau}u_i^{n+1} + \frac{2}{3}\left(u_i^{n+1}\right)_{\hat{x}} - \frac{1}{6}\left(u_i^{n+1}\right)_{\ddot{x}} + \frac{5}{6\tau}\left(u_i^{n+1}\right)_{xx\bar{x}\bar{x}} - \frac{1}{3\tau}\left(u_i^{n+1}\right)_{x\bar{x}\hat{x}\hat{x}} + \frac{3}{4}\left(u_i^{n+1}\right)_{x\bar{x}\hat{x}} - \frac{1}{4}\left(u_i^{n+1}\right)_{x\bar{x}\ddot{x}} + \frac{1}{2}\varphi(u_i^n, u_i^{n+1}) = 0, \quad (3.1)$$

where

$$\varphi(u_i^n, u_i^{n+1}) = \frac{4}{9}\left(u_i^n\left(u_i^{n+1}\right)_{\hat{x}} + \left(u_i^n u_i^{n+1}\right)_{\hat{x}}\right) - \frac{1}{9}\left(u_i^n\left(u_i^{n+1}\right)_{\ddot{x}} + \left(u_i^n u_i^{n+1}\right)_{\ddot{x}}\right).$$

By taking inner product of (3.1) with u^{n+1}, we obtain

$$\frac{1}{2\tau}\left\|u^{n+1}\right\|^2 + \frac{5}{6\tau}\left\|u_{xx}^{n+1}\right\|^2 - \frac{1}{3\tau}\left\|u_{x\hat{x}}^{n+1}\right\|^2 + \frac{1}{2}\left(\varphi(u^n, u^{n+1}), u^{n+1}\right) = 0. \quad (3.2)$$

Indeed,

$$\begin{aligned}
&\left(\varphi(u^n, u^{n+1}), u^{n+1}\right) \quad (3.3)\\
&= \frac{4h}{9}\sum_{i=1}^{M-1}\left[u_i^n\left(u_i^{n+1}\right)_{\hat{x}} + \left(u_i^n u_i^{n+1}\right)_{\hat{x}}\right]u_i^{n+1} - \frac{h}{9}\sum_{i=1}^{M-1}\left[u_i^n\left(u_i^{n+1}\right)_{\ddot{x}} + \left(u_i^n u_i^{n+1}\right)_{\ddot{x}}\right]u_i^{n+1}\\
&= \frac{2}{9}\sum_{i=1}^{M-1}\left[\left(u_i^n u_i^{n+1}u_{i+1}^{n+1} - u_{i-1}^n u_{i-1}^{n+1}u_i^{n+1}\right) + \left(u_{i+1}^n u_i^{n+1}u_{i+1}^{n+1} - u_i^n u_{i-1}^{n+1}u_i^{n+1}\right)\right]\\
&\quad - \frac{1}{36}\sum_{i=1}^{M-1}\left[\left(u_i^n u_i^{n+1}u_{i+2}^{n+1} - u_{i-2}^n u_{i-2}^{n+1}u_i^{n+1}\right) + \left(u_{i+2}^n u_i^{n+1}u_{i+2}^{n+1} - u_i^n u_{i-2}^{n+1}u_i^{n+1}\right)\right]\\
&= 0.
\end{aligned}$$

Then it follows that

$$\left\|u_{x\hat{x}}^{n+1}\right\|^2 = h\sum_{i=1}^{M-1}\left(u_i^{n+1}\right)_{x\hat{x}}^2 = \frac{h}{4}\sum_{i=1}^{M-1}\left[\left(u_i^{n+1}\right)_{xx} - \left(u_i^{n+1}\right)_{x\bar{x}}\right]^2 \le \left\|u_{xx}^{n+1}\right\|^2. \quad (3.4)$$

From (3.2) –(3.4) , hence

$$\left\|u^{n+1}\right\|^2 = \left\|u_{xx}^{n+1}\right\|^2 = 0.$$

Therefore, (3.1) has the only one solution and (2.1) u^{n+1} is uniquely solvable. This completes the proof of Theorem 3.1. □

4 Convergence and Stability

In this section, we prove the convergence and stability of the scheme (2.1)–(2.3). Let $e_i^n = v_i^n - u_i^n$, where v_i^n and u_i^n are the solutions of (1.1) and (2.1), respectively. We then obtain the following error equations:

$$r_i^n = (e_i^n)_{\hat{t}} + \frac{4}{3}(\bar{e}_i^n)_{\hat{x}} - \frac{1}{3}(\bar{e}_i^n)_{\ddot{x}} + \frac{5}{3}(e_i^n)_{xx\bar{x}\bar{x}\hat{t}} - \frac{2}{3}(e_i^n)_{x\bar{x}\hat{x}\hat{x}\hat{t}} + \frac{3}{2}(\bar{e}_i^n)_{x\bar{x}\hat{x}} - \frac{1}{2}(\bar{e}_i^n)_{x\bar{x}\ddot{x}} + M_1 - M_2 \quad (4.1)$$

where

$$M_1 = \frac{4}{9}\left(v_i^n(\bar{v}_i^n)_{\hat{x}} + (v_i^n\bar{v}_i^n)_{\hat{x}}\right) - \frac{4}{9}\left(u_i^n(\bar{u}_i^n)_{\hat{x}} + (u_i^n\bar{u}_i^n)_{\hat{x}}\right),$$

$$M_2 = \frac{1}{9}\left(v_i^n(\bar{v}_i^n)_{\ddot{x}} + (v_i^n\bar{v}_i^n)_{\ddot{x}}\right) - \frac{1}{9}\left(u_i^n(\bar{u}_i^n)_{\ddot{x}} + (u_i^n\bar{u}_i^n)_{\ddot{x}}\right),$$

and r_i^n denotes the truncation error. By using Taylor expansion, it is easy to see that $r_i^n = O(\tau^2 + h^4)$ holds as $\tau, h \to 0$. The following lemmas are essential for the proof of convergence and stability of our scheme.

Lemma 4.1. (discrete Sobolev's inequality [23]). *There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \le C_1\|u^n\| + C_2\|u_x^n\|.$$

Lemma 4.2. (discrete Gronwall's inequality [23]). *Suppose that $\omega(k)$ and $\rho(k)$ are nonnegative function and $\rho(k)$ is nondecreasing. If $C > 0$ and*

$$\omega(k) \le \rho(k) + C\tau\sum_{l=0}^{k-1}\omega(l), \quad \forall k,$$

then

$$\omega(k) \le \rho(k)e^{C\tau k}, \quad \forall k.$$

Lemma 4.3. ([11]). *Suppose that $u_0 \in H_0^2[x_l, x_r]$, then the solution u^n of* (2.1)–(2.3) *satisfies*

$$\|u\|_{L_2} \le C, \qquad \|u_x\|_{L_2} \le C,$$
$$\|u\|_{L_\infty} \le C, \qquad \|u_x\|_{L_\infty} \le C.$$

Lemma 4.4. *Suppose $u_0 \in H_0^2[x_l, x_r]$, then solution u^n satisfies $\|u^n\| \le C$ and $\|u_{xx}^n\| \le C$, which yield $\|u^n\|_\infty \le C$.*

The following theorem shows that our scheme converges to the solution with convergence rate $O(\tau^2 + h^4)$.

Theorem 4.5. *Suppose $u_0 \in H_0^2[x_l, x_r]$, then solution u^n converges to the solution of problem in the sense of $\|\cdot\|_\infty$ and the rate of convergence is $O(\tau^2 + h^4)$.*

Proof. By taking inner product on both sides of (4.1) with $2\bar{e}_i^n = e_i^{n+1} + e_i^{n-1}$, we get

$$\frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \frac{5}{6\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) - \frac{1}{3\tau}\left(\left\|e_{x\hat{x}}^{n+1}\right\|^2 - \left\|e_{x\hat{x}}^{n-1}\right\|^2\right) = (r^n, 2\bar{e}^n) - (M_1, 2\bar{e}^n) + (M_2, 2\bar{e}^n). \quad (4.2)$$

According to the Schwarz inequality, Lemma 2.1, 4.3, and 4.4, we obtain

$$\begin{aligned}(M_1, 2\bar{e}^n) &= \frac{8h}{9}\sum_{i=1}^{M-1}\left[(v_i^n(\bar{v}_i^n)_{\hat{x}} - u_i^n(\bar{u}_i^n)_{\hat{x}}) + ((v_i^n\bar{v}_i^n)_{\hat{x}} - (u_i^n\bar{u}_i^n)_{\hat{x}})\right]\bar{e}_i^n \quad (4.3)\\ &= \frac{8h}{9}\sum_{i=1}^{M-1}\left[(v_i^n(\bar{e}_i^n)_{\hat{x}} + e_i^n(\bar{u}_i^n)_{\hat{x}})\right]\bar{e}_i^n + \frac{8h}{9}\sum_{i=1}^{M-1}\left(e_i^n\bar{v} + u_i^n\bar{e}_i^n\right)(\bar{e}_i^n)_{\hat{x}}\\ &\le C\left(\|\bar{e}_{\hat{x}}^n\|^2 + \|\bar{e}^n\|^2 + \|e^n\|^2\right)\\ &\le C\left(\left\|e_{\hat{x}}^{n+1}\right\|^2 + \left\|e_{\hat{x}}^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e^{n-1}\right\|^2\right).\end{aligned}$$

Similarly, it can be easily shown that

$$(M_2, 2\bar{e}^n) \le C\left(\left\|e_{\ddot{x}}^{n+1}\right\|^2 + \left\|e_{\ddot{x}}^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e^{n-1}\right\|^2\right). \quad (4.4)$$

By the Cauchy–Schwarz inequality and a direct calculation, we obtain

$$\|e_{\ddot{x}}^n\| \le \|e_{\hat{x}}^n\| \le \|e_x^n\|, \quad (4.5)$$

$$\|e_x^n\|^2 \le \frac{1}{2}\left(\|e^n\|^2 + \|e_{xx}^n\|^2\right), \quad (4.6)$$

$$(r^n, 2\bar{e}^n) \le \|r^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right), \quad (4.7)$$

from (4.2)-(4.7), which yields

$$\begin{aligned}&\frac{1}{2}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \frac{5}{6}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) - \frac{1}{3}\left(\left\|e_{x\hat{x}}^{n+1}\right\|^2 - \left\|e_{x\hat{x}}^{n-1}\right\|^2\right)\\ &\le \tau\|r^n\|^2 + \tau C\left(\left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e_{xx}^{n+1}\right\|^2 + \|e_{xx}^n\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right).\end{aligned} \quad (4.8)$$

Set

$$B^n = \frac{1}{2}\left(\|e^n\|^2 + \left\|e^{n-1}\right\|^2\right) + \frac{5}{6}\left(\|e_{xx}^n\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right) - \frac{1}{3}\left(\|e_{x\hat{x}}^n\|^2 + \left\|e_{x\hat{x}}^{n-1}\right\|^2\right)$$

and

$$D^n = \frac{1}{2}\left(\|e^n\|^2 + \left\|e^{n-1}\right\|^2\right) + \frac{1}{2}\left(\|e_{xx}^n\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right).$$

From (3.4), we get

$$D^n \leq B^n \leq 2D^n. \tag{4.9}$$

From (3.4) and (4.8), a direct computation gives

$$B^{n+1} - B^n \leq \tau \|r^n\|^2 + 2\tau C \left(B^{n+1} + B^n\right)$$

and

$$(1 - 2\tau C)\left(B^{n+1} - B^n\right) \leq \tau \|r^n\|^2 + 4\tau C B^n.$$

If τ is sufficiently small which satisfies $1 - 2C\tau > 0$, then

$$B^{n+1} - B^n \leq \tau C \|r^n\|^2 + \tau C B^n. \tag{4.10}$$

Summing up (4.10) from 1 to n, we have

$$B^n \leq B^1 + C\tau \sum_{k=1}^{n} \left\|r^k\right\|^2 + C\tau \sum_{k=1}^{n} B^k. \tag{4.11}$$

From (4.9) and (4.11), then

$$D^n \leq 2D^1 + C\tau \sum_{k=1}^{n} \left\|r^k\right\|^2 + 2C\tau \sum_{k=1}^{n} D^k.$$

Thus we can use a fourth-order method to compute u^1, such that

$$D^1 \leq O(\tau^2 + h^4)^2,$$

and

$$\tau \sum_{k=1}^{n} \left\|r^k\right\|^2 \leq n\tau \max_{0 \leq l \leq n-1} \left\|r^l\right\|^2 \leq T \cdot O(\tau^2 + h^4).$$

By Lemma 4.2, we obtain $D^n \leq O(\tau^2 + h^4)^2$, that is

$$\|e^n\| \leq O(\tau^2 + h^4), \quad \|e^n_{xx}\| \leq O(\tau^2 + h^4).$$

From (4.6),

$$\|e^n_x\| \leq O(\tau^2 + h^4).$$

By Lemma 4.1,

$$\|e^n\|_\infty \leq O(\tau^2 + h^4).$$

This completes the proof. □

Theorem 4.6. *Under the conditions of Theorem* 4.5*, the solution* u^n *of* (2.1)-(2.3) *is stable in norm* $\|\cdot\|_\infty$.

5 Numerical Experiments

In this section, we present numerical experiments on a test problem to confirm and illustrate the accuracy of our proposed method. We then measure the accuracy of the method using $\|\cdot\|$ and $\|\cdot\|_\infty$ norm. Let $x_l = -60, x_r = 80$, and $T = 40$,

$$u_0(x) = \left(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\right) \operatorname{sech}^4 \left[\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}\,(x)\right].$$

We make a comparison between the scheme (2.1) and the scheme proposed in [11]. The results on this experiment in term of errors at the time $t = 10$ and $t = 40$ are reported in Tables 1 and 2. It is clear that the result obtained by the scheme (2.1) are more accurate then the result obtained by the scheme of [11]. As shown in Tables 3 and 4, on one particular choice of the parameters, the estimated rate is close to the theoretically predicted fourth-order rate of convergence. We can also say that when we use smaller time and space steps, numerical values are almost the same as the exact values. In Table 5, it results from the present method and the values of Q^n and E^n at any time $t \in [0, 40]$ coincide with the theory.

Absolute error distributions for the two methods with $\tau = h = 0.5$ are drawn at $t = 10$ and $t = 40$ in Figs. 1 and 2, respectively. It is can be easily observed that maximum error is taken place around the peak amplitude of the solitary wave. Figure 3 illustrates the numerical solutions of solitary waves computed by scheme (2.1) with $\tau = 0.25$ and $h = 0.5$ at $t = 0, 20, 40$, which also demonstrates the accuracy of the scheme.

Table 1: Comparison of errors $\|e\|$ between proposed scheme and Hu et al. [11].

		$\tau = 0.5,\ h = 0.5$		
		τ, h	$\frac{\tau}{4}, \frac{h}{2}$	$\frac{\tau}{16}, \frac{h}{4}$
$t = 10$	Present	$2.8557068E-02$	$1.8476735E-03$	$1.1589057E-04$
	Hu et al. [11]	$3.9304493E-02$	$4.7247751E-03$	$8.4099091E-04$
$t = 40$	Present	$9.1929440E-02$	$6.0190039E-03$	$3.7762026E-04$
	Hu et al. [11]	$1.2443028E-01$	$1.5032040E-02$	$2.6543550E-03$

Table 2: Comparison of errors $\|e\|_\infty$ between proposed scheme and Hu et al. [11].

		$\tau = 0.5,\ h = 0.5$		
		τ, h	$\frac{\tau}{4}, \frac{h}{2}$	$\frac{\tau}{16}, \frac{h}{4}$
$t = 10$	Present	$1.0886426E-02$	$7.0772788E-04$	$4.4416201E-05$
	Scheme [11]	$1.5015535E-02$	$1.8245955E-03$	$3.2598499E-04$
$t = 40$	Present	$3.2393080E-02$	$2.1309079E-03$	$1.3369728E-04$
	Scheme [11]	$4.3734420E-02$	$5.3364748E-03$	$9.4467612E-04$

Table 3: Error and convergence rate at $t = 10$.

Mesh size	$\tau = 0.5,\ h = 0.5$ τ, h	$\frac{\tau}{4}, \frac{h}{2}$	$\frac{\tau}{16}, \frac{h}{4}$	$\frac{\tau}{64}, \frac{h}{8}$
$\|e\|$	$2.8557068E-2$	$1.8476735E-3$	$1.1589057E-4$	$7.2477155E-6$
Rate		3.95007	3.99487	3.99909
$\|e\|_\infty$	$1.0886426E-2$	$7.0772788E-4$	$4.4416201E-5$	$2.7785746E-6$
Rate		3.94319	3.99404	3.99867

Table 4: Error and convergence rate at $t = 40$.

Mesh size	$\tau = 0.5,\ h = 0.5$ τ, h	$\frac{\tau}{4}, \frac{h}{2}$	$\frac{\tau}{16}, \frac{h}{4}$	$\frac{\tau}{64}, \frac{h}{8}$
$\|e\|$	$3.239308E-2$	$2.130908E-3$	$1.336973E-4$	$8.359881E-6$
Rate		3.92615	3.99443	3.99934
$\|e\|_\infty$	$9.192944E-2$	$6.019004E-4$	$3.776203E-5$	$2.361714E-5$
Rate		3.93293	3.99452	3.99903

Table 5: Two conservative invariants Q^n and E^n at various time t.

	$\tau = 0.5,\ h = 0.5$		$\tau = 0.125,\ h = 0.25$	
	Q^n	E^n	Q^n	E^n
$t = 0$	5.49891446878910	3.96961562713998	5.49818488235170	3.96890044391951
$t = 10$	5.49892677460539	3.96990909427802	5.49818496400364	3.96892306450770
$t = 20$	5.49895690079020	3.97031625894044	5.49818525479978	3.96895220660664
$t = 30$	5.49900172956839	3.97058828753461	5.49818749045364	3.96897182591078
$t = 40$	5.49879989238951	3.97076212531140	5.49817225144428	3.96898443340805

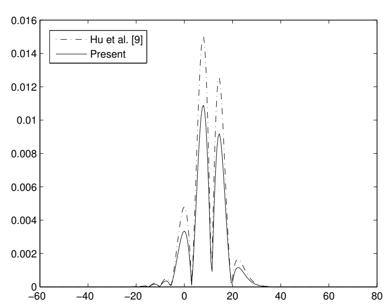

Figure 1: Absolute error distribution at $t = 10$.

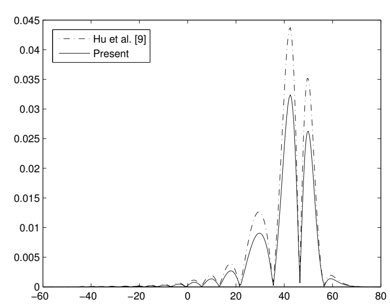

Figure 2: Absolute error distribution at $t = 40$.

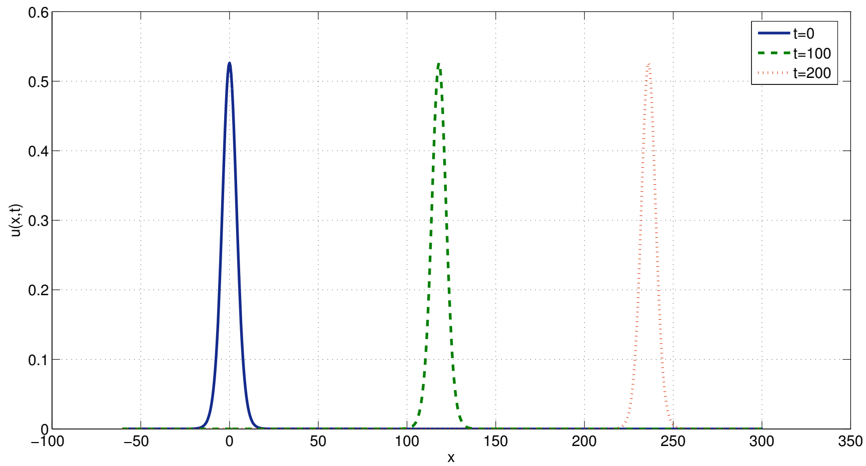

Figure 3: Numerical solutions at different times.

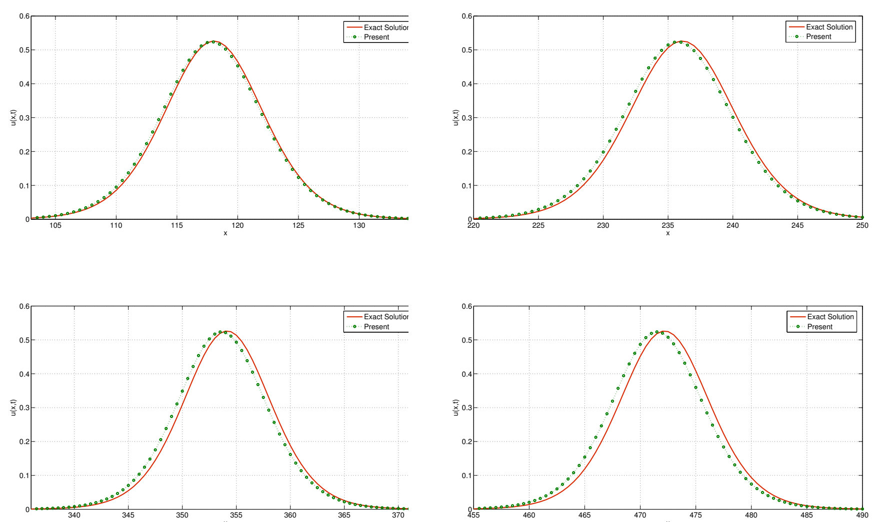

Figure 4: Numerical solutions at different times.

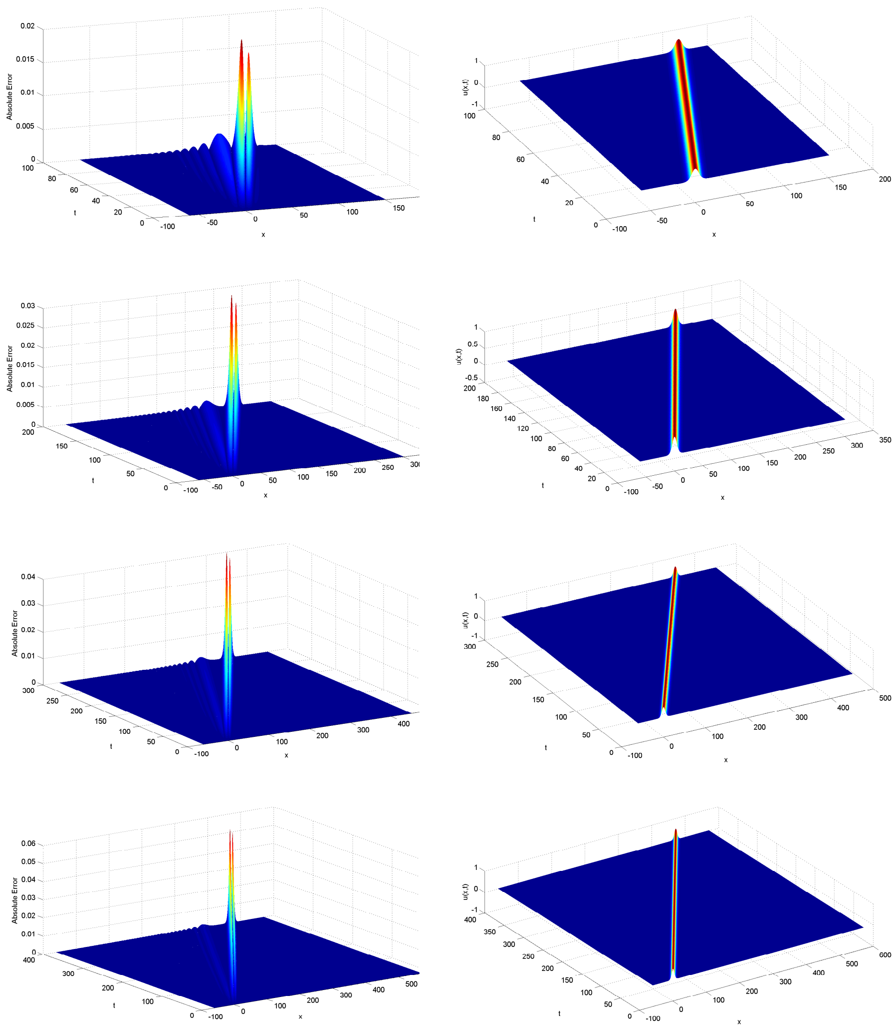

Figure 5: Numerical solutions at different times.

6 Conclusion

A conservative finite difference scheme for the Rosenau-KdV equation is introduced and analyzed. The present method gives implicit linear system, which can be easily implemented. This method is shown second- and fourth-order accurate in time and space, respectively. The numerical experiments show that the present method support the analysis of convergence rate.

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