Thai Journal of Mathematics
Volume 17 (2019) Number 3 : 775–787
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence Theorems for G-Nonexpansive Mappings in CAT(0) Spaces Endowed with Graphs

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Abstract : In this paper, we introduce the concept of a CAT(0) space endowed with a graph. Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in an underlying space will be presented. This result extends and generalizes the result of Tiammee, Kaewkhao and Suantai (2015).

Keywords : CAT(0) space; directed graph; nonexpansive mapping; Browder's convergence theorem; Halpern iteration process.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

Let (X, d) be a metric space. A maping $T : X \to X$ is said to be a *nonexpansive mapping* if there is $k \in (0, 1]$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. If in the case of $k < 1$, we call T a *contraction.* A point $x \in X$ is called a *fixed point* of T if $Tx = x$. In this paper, we use the notation $F(T)$ stand for the set of all fixed

[0]This research was partially supported by the Rearch Fund for DPST Graduate with First Placement, Institute for the Promotion of Teaching Science and Technology (IPST).
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points of T.

A *geodesic* joining points x and y in a metric space X is a mapping c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x,\ c(l) = y$ and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0,l]$. In particular, c is an *isometry* and $d(x,y) = l$. The image γ of c is called a *geodesic (or metric) segment* joining x and y. When it is unique this geodesic is denoted by $[x,y]$. We write $\alpha x \oplus (1-\alpha)y$ stand for the point $c(\alpha 0 + (1-\alpha)l) \in X$. The space X is said to be a *(uniquely) geodesic space* if every two points of X are joined by a (unique) geodesic. A geodesic space X is said to be a *CAT(0) space* if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane, i.e.,

$$d(a,b) \leq d_{\mathbb{R}^2}(\overline{a}, \overline{b}), \tag{1.1}$$

for any $a, b \in \Delta(x,y,z)$ and $\overline{a}, \overline{b} \in \overline{\Delta}(x,y,z)$.

The first famous fixed point theorem in a metric space is established by Stefan Banach [1] in 1922. They investigated the theorem, called Banach contraction principle, which telling that a self mapping on a complete metric space X has a unique fixed point. Browder [2] used the Banach's result to prove the convergence theorem for the implicit iterative in a Hilbert space, called the Browder's convergence theorem.

In 2008, Jachymski combined the knowledge of the original fixed point theory and graph theory. First of all, they introduced a concept of a metric space endowed with a graph as the following: For any metric space (X,d) and a directed graph $G = (V(G), E(G))$, if $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta = \{(x,x) : x \in X\} \subseteq E(G)$, the triple (X,d,G) is called a *metric space endowed with a graph*. Let C be a nonempty subset of a metric space endowed with graph (X,d,G). Suppose $T : C \to C$ preserves edges of G and satisfy $d(Tx,Ty) \leq kd(x,y)$ for any $x, y \in X$ for some $k \in \mathbb{R}^+$. Then

(1) if $k < 1$, we call T a *G-contraction*, and

(2) if $k \leq 1$, we call T a *G-nonexpansive mapping*.

The following theorem, a generalization of Banach contraction principle, has been presented in [3]:

Theorem 1.1 ([3])**.** *Suppose that a metric space endowed with graph (X,d,G) have the **Property P**:*

for any $\{x_n\}_{n\in\mathbb{N}}$ if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$,

then there is a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let T be a G-contraction, and $X_T = \{x \in X : (x, T(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$.

In 2015, Tiammee et al. [4] extened the Browders convergence theorem for G-nonexpansive mappings in Hilbert spaces endowed with graphs. In the prove of their theorem, they have to replace the Property P to the stronger one, called the **Property G**: for every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Theorem 1.2 ([4]). *Let C be a bounded closed convex subset of Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose C has Property G. Let $T : C \to C$ be G-nonexpansive. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Define $T_n : C \to C$ by*

$$T_n x = (1 - \alpha_n)Tx + \alpha_n x_0$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \to 0$ as $n \to \infty$. Then the following hold:

(i) T_n has a fixed point $u_n \in C$;

(ii) $F(T) \neq \emptyset$

(iii) if $F(T) \times F(T) \subseteq E(G)$ and Px_0 is dominated by $\{u_n\}$, then the sequence $\{u_n\}$ converges strongly to Px_0 where P is the metric projection onto $F(T)$.

Motivating by the above results, in this paper, we will present the Brower convergence theorem in the framework of CAT(0) spcaes endowed with graph. The conditions on the set C in our result has been relaxed. The convergence theorems of the Halpern's iteration scheme for a family of G-nonexpansive mappings are also presented.

2 Preliminaries

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a *dominating set* if every $v \in V(G) \setminus X$ there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that x *dominates* v or v is *dominated by* x. Let $v \in V(G)$, a set $X \subseteq V(G)$ is *dominated by* v if $(v, x) \in E(G)$ for any $x \in X$ and we say that X *dominates* v if $(x, v) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops. Let G be a directed graph, and $E(G)$ the set of edges of G. We say $E(G)$ is a *convex set* if, for any $\alpha \in [0, 1]$,

$$(\alpha x + (1 - \alpha)y, \alpha u + (1 - \alpha)v) \in E(G)$$

for all $(x, y), (u, v) \in E(G)$.

Let X be a metric space. The following statements are equivalent for a uniquely geodesic space X:

(i) X is a CAT(0) space;

(ii) X satisfies the **(CN)-inequality**: If $x, y \in X$ and $\alpha \in (0, 1)$, then

$$d^2(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) - \alpha(1 - \alpha)d^2(x, y),$$

for all $z \in X$;

(iii) X satisfies the **law of cosine**: If $a = d(x, z), b = d(y, z), c = d(x, y)$ and ξ is the Alexandrov angle at z between $[x, z]$ and $[y, z]$, then

$$c^2 \geq a^2 + b^2 - 2ab\cos\xi.$$

Lemma 2.1 ([5]). *Let X be a CAT(0) space. Then for each $p, q, x, y \in X$ and $\alpha \in [0,1]$, we have*

$$d(\alpha p \oplus (1-\alpha)q, \alpha x \oplus (1-\alpha)y) \le \alpha d(p,x) + (1-\alpha)d(q,y).$$

For any nonempty subset C of X, let $\pi = \pi_C$ be the projection mapping from X to C. It is known that if C is closed and convex, the mapping π is well-defined, nonexpansive and satisfies

$$d^2(x,y) \ge d^2(x,\pi x) + d^2(\pi x, y) \text{ for all } x \in X \text{ and } y \in C. \tag{2.1}$$

In 2011, Dhompongsa et al. [6] introduced the following concepts of convex combination in CAT(0) spaces. Let $v_1, v_2, \dots, v_n \subset X$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in (0,1)$ with $\sum_{i=1}^n \lambda_i = 1$. Using the result in [7], the partial sum of $\lambda_1 v_1, \lambda_2 v_2. \dots, \lambda_n v_n$ can be written by:

$$\bigoplus_{i=1}^{n} \lambda_i v_i := (1-\lambda_n)\left(\frac{\lambda_1}{1-\lambda_n}v_1 \oplus \frac{\lambda_2}{1-\lambda_n}v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}v_{n-1}\right) \oplus \lambda_n v_n. \tag{2.2}$$

Let $\{\lambda_n\} \subset (0,1)$ be such that $\sum_{n=1}^\infty \lambda_n = 1$. Let $\{v_n\} \subset X$ be bounded and v_0 be an arbitrary point in X. Suppose $\lambda'_n = \sum_{i=n+1}^\infty \lambda_i$ and $\sum_{i=n}^\infty \lambda'_i \to 0$ as $n \to \infty$. Set

$$s_n := \left(\sum_{i=1}^n \lambda_i\right) w_n \oplus \lambda'_n v_0, \tag{2.3}$$

where $w_1 = v_1$ and for each $n \ge 2$,

$$w_n = \frac{\lambda_1}{\sum_{i=1}^n \lambda_i}v_1 \oplus \frac{\lambda_2}{\sum_{i=1}^n \lambda_i}v_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^n \lambda_i}v_n.$$

Then $s_n \to x$ as $n \to \infty$ for some $x \in X$. In [6], they use the element x for representing the infinite summation of $\lambda_1 v_1, \lambda_2 v_2 \dots$, *i.e.*,

$$x = \bigoplus_{n=1}^{\infty} \lambda_n v_n.$$

By (2.3), $d(s_n, w_n) \le \lambda'_n d(w_n, v_0)$, it follows that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} w_n$. Thus the limit x is independent of the choice of v_0.

The followings are importance properties of the convex combination in CAT(0) spaces introduced in [6].

Lemma 2.2. *If y_0 and v_n belong to X, $d(v_n, y_0) = d(x, y_0)$ for all n where $x = \bigoplus_{n=1}^\infty \lambda_n v_n$, then $v_n = x$ for all n.*

Lemma 2.3 ([6], Lemma 3.8). *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $\{T_n : n \in \mathbb{N}\}$ be a family of single-valued nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Define $T : C \to C$ by*

$$Tx = \bigoplus_{n=1}^{\infty} \lambda_n t_n x$$

for all $x \in C$ where $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and $\sum_{i=n}^{\infty} \lambda'_i \to 0$ as $n \to \infty$. Then T is nonexpansive and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

The following results are vary useful in the proof of our main results.

Lemma 2.4 ([8]). *Let $(a_1, a_2, ...) \in l^\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq 0$.*

Lemma 2.5 ([9]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} t_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + u_n \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6 ([10]). *Let C be a closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $k \in (0,1)$, the mapping $S_k : C \to C$ defined by*

$$S_k x = ku \oplus (1-k)Tx \quad \text{for } x \in C \tag{2.5}$$

has a unique fixed point $x_k \in C$, that is,

$$x_k = S_k x_k = ku \oplus (1-k)Tx_k. \tag{2.6}$$

Then $F(T) \neq \emptyset$ if and only if $\{x_k\}$ given by (2.6) is bounded as $k \to 0$. In this case, the following statements hold:

(i) $\{x_k\}$ converges to the unique fixed point z_0 of T which is nearest to u;

(ii) $d^2(u, z_0) \leq \mu_n d^2(u, x_n)$, for all Banach limits μ and all bounded sequences $\{x_n\}$ with $d(x_n, Tx_n) \to 0$.

3 Main Results

3.1 Browder's Convergence Thorem

Lemma 3.1. *Let (X, d, G) be a CAT(0) space endowed with graph. Suppose $T : X \to X$ is a G-nonexpansive mapping. If X has a Property P, then T is continuous.*

Proof. Let $\{x_n\}$ be a sequence in X converging to some $x \in X$. Let $\{Tx_{n_k}\}$ be any subsequence of $\{Tx_n\}$. Since $x_{n_k} \to x$ as $k \to 0$, by Property P, there exists a subsequence $\{x_{m_k}\}$ such that $(x_{m_k}, x) \in E(G)$ for each $k \in \mathbb{N}$. Since T is G-nonexpansive and $(x_{m_k}, x) \in E(G)$ we obtain

$$d(Tx_{m_k}, Tx) \leq d(x_{m_k}, x) \to 0 \quad as\ k \to \infty.$$

Hence $Tx_{m_k} \to Tx$. By the double extract subsequence principle, we conclude that $Tx_n \to Tx$. Therefore T is continuous. □

In what follows, we will prove the Brower's convergence theorem for a G-nonexpansive mapping on a bounded closed and star-shaped subset C of a CAT(0) space X under the hypothesis that X satisfies the property P. We first present the definition of a star-shaped set in a CAT(0) space.

Definition 3.2 ([11])**.** Let X be a CAT(0) space. A subset C is said to be *star-shaped* if there exists $p \in C$ such that $(1-t)p \oplus tx \in C$ for any $x \in C$ and $t \in [0,1]$. In this case, C is also called p-star-shaped, where p is the center of the star.

Remark 3.1. The assumption "C is p-star-shape"is weaker than the convexity of C.

Theorem 3.3. *Let (X, d, G) be a CAT(0) space endowed with graph having Property P and C be a nonempty subset of X. Suppose $T : C \to C$ is a G-nonexpansive mapping and $F(T) \times F(T) \subseteq E(G)$. If $E(G)$ is convex, then $F(T)$ is closed and convex.*

Proof. Suppose $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. By Property P, there is a subsequence $\{x_{n_k}\}$ of$\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Since T is G-nonexpansive, we obtain

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx) \\ &= d(x, x_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k}) + d(x_{n_k}, x) \to 0. \end{aligned}$$

Therefore $x = Tx$, i.e., $x \in F(T)$. This shows that $F(T)$ is closed.

Let $x, y \in F(T)$ and $\lambda \in [0, 1]$. Denote $z = \lambda x + (1 - \lambda)y$. By the convexity of $E(G)$, we obtain

$$(x, z) = (\lambda x + (1 - \lambda)x, \lambda x + (1 - \lambda)y) \in E(G).$$

Similarly, we also have $(y, z) \in E(G)$. Finally, we will show by contradiction, that $z \in F(T)$. Suppose the contrary i.e., $z \neq Tz$. Using the (CN)-inequality and the

G-nonexpansiveness of T, we have

$$\begin{aligned} d^2\left(\frac{z\oplus Tz}{2}, x\right) &\le \frac{d^2(z,x)}{2} + \frac{d^2(Tz,x)}{2} - \frac{d^2(z,Tz)}{4} \\ &= \frac{d^2(z,x)}{2} + \frac{d^2(Tz,Tx)}{2} - \frac{d^2(z,Tz)}{4} \\ &\le \frac{d^2(z,x)}{2} + \frac{d^2(z,x)}{2} - \frac{d^2(z,Tz)}{4} \\ &= d^2(z,x) - \frac{d^2(z,Tz)}{4} \\ &< d^2(z,x). \end{aligned}$$

Therefore $d\left(\frac{z\oplus Tz}{2}, x\right) < d(z,x)$ and, by the similar argument, we also get $d\left(\frac{z\oplus Tz}{2}, y\right)$ $\le d(z,y)$. Hence

$$\begin{aligned} d(x,y) \le\ & d\left(x, \frac{z\oplus Tz}{2}\right) + d\left(y, \frac{z\oplus Tz}{2}\right) \\ <\ & d(x,z) + d(y,z) \\ =\ & d(x,y). \end{aligned}$$

Which lead us a contradiction. Thus $F(T)$ is convex. □

Now, we already to prove our first main result.

Theorem 3.4. *Let (X,d,G) be a complete CAT(0) space endowed with graph. Assume that there exists $p\in C$ such that $(p,Tp)\in E(G)$. Let C be a bounded closed p-star-shaped of X which has Property P and $E(G)$ is convex. Let $T : C \to C$ be a G-nonexpansive mapping. Define $T_n : C \to C$ by*

$$T_n x = (1-\alpha_n)Tx \oplus \alpha_n p$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \to 0$. Then all of the followings hold:

(i) T_n has a fixed point $u_n \in C$;

(ii) $F(T) \neq \emptyset$; and

(iii) if $F(T)\times F(T) \subseteq E(G)$ and $(u_n,u_k)\in E(G)$ for all $n,k\in\mathbb{N}$, then the sequence $\{u_n\}$ converges strongly to $v^* \in F(T)$ which is nearest to p.

Proof. We first show that T_n is G-contraction for all $n\in\mathbb{N}$. Let $n\in\mathbb{N}$ and $x,y\in C$ such that $(x,y)\in E(G)$. Since T is G-nonexpansive, we obtain T_n is also nonexpansive. Since T is edge-preserving, $(Tx,Ty)\in E(G)$. By the convexity of $E(G)$, we have $(T_nx, T_ny) = ((1-\alpha_n)Tx\oplus\alpha_n p, (1-\alpha_n)Ty\oplus\alpha_n p)\in E(G)$. Hence T_n is G-contraction. For each sequence $\{x_n\}$ in C such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, by Property P, there is a subseqence $\{x_{n_k}\}$ such that

$(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$. Since $E(G)$ is convex and $(p,p) \in E(G)$, so $(p, T_n p) = ((1-\alpha_n)p \oplus \alpha_n p, (1-\alpha_n)Tp \oplus \alpha_n p) \in E(G)$. Then T_n has a fixed point, i.e., $u_n = T_n u_n$, because $X_{T_n} = \{x \in X : (x, T_n(x)) \in E(G)\} \neq \emptyset$.

To prove (ii) & (iii), let $\{u_m\}$ be any subsequence of $\{u_n\}$. Since $\alpha_n \to 0$ as $n \to \infty$, there exists a monotone decreasing subsequence $\{\alpha_{m_k}\}$ of $\{\alpha_m\}$. Let $\{u_{m_k}\}$ be a subsequence of $\{u_m\}$ corresponds with the coefficient $\{\alpha_{m_k}\}$. We show that $\{u_{m_k}\}$ is a Cauchy sequence. Indeed, let $l, k \in \mathbb{N}$ and suppose without loss of generality that $l < k$. So $\alpha_{m_l} > \alpha_{m_k}$. Consider $\overline{\Delta}(p, Tu_{m_k}, Tu_{m_l})$, the comparison triangle of $\Delta(p, Tu_{m_k}, Tu_{m_l})$ in $\mathbb{R}^2$. For convenience, we take $\overline{p} = (0,0)$ $d = \overline{u_{m_l}} - \overline{u_{m_k}}$, $a = 1 - \alpha_{m_l}$ and $b = 1 - \alpha_{m_k}$. We have $\overline{u_{m_k}} = b\overline{Tu_{m_k}}$ and $\overline{u_{m_l}} = a\overline{Tu_{m_l}}$. Consider

$$\begin{aligned}\left\| \frac{1}{a}(\overline{u_{m_k}} + d) - \frac{1}{b}\overline{u_{m_k}} \right\|^2 &= \left\| \left(\frac{1}{a}\overline{u_{m_k}} - \frac{1}{b}\overline{u_{m_k}} \right) + \frac{1}{a}d \right\|^2 \\ &= \left\| \frac{1}{a}\overline{u_{m_l}} - \frac{1}{b}\overline{u_{m_k}} \right\|^2 \\ &= \|\overline{Tu_{m_l}} - \overline{Tu_{m_k}}\|^2 \leq \|d\|^2.\end{aligned}$$

Thus

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 \|\overline{u_{m_k}}\|^2 + \left(\frac{1}{a}\right)^2 \|d\|^2 + 2\left\langle \left(\frac{1}{a} - \frac{1}{b}\right)\overline{u_{m_k}}, \frac{1}{a}d \right\rangle \leq \|d\|^2.$$

Therefore

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 \|\overline{u_{m_k}}\|^2 + \left(\frac{1}{a} - 1\right)^2 \|d\|^2 \leq \frac{2}{a}\left(\frac{1}{b} - \frac{1}{a}\right)\langle \overline{u_{m_k}}, d\rangle.$$

This means $\langle \overline{u_{m_k}}, d\rangle \geq 0$. Since $\overline{u_{m_l}} = \overline{u_{m_k}} + d$, we have

$$\begin{aligned}\|\overline{u_{m_l}}\|^2 &= \langle \overline{u_{m_k}} + d, \overline{u_{m_k}} + d\rangle \\ &= \|\overline{u_{m_k}}\|^2 + \|d\|^2 + 2\langle \overline{u_{m_k}}, d\rangle \\ &\geq \|\overline{u_{m_k}}\|^2 + \|\overline{u_{m_l}} - \overline{u_{m_k}}\|^2 \geq \|\overline{u_{m_k}}\|^2.\end{aligned}$$

This show that the sequence $\{\|\overline{u_{m_k}}\|^2\}$ is monotone decreasing. By the boundedness of C, we can conclude that $\|\overline{u_{m_k}}\|^2 \to M$, for some $M > 0$ as $k \to \infty$. From (1.1), we have

$$d^2(u_{m_k}, u_{m_l}) \leq \|\overline{u_{m_k}} - \overline{u_{m_l}}\|^2 \leq \|\overline{u_{m_l}}\|^2 - \|\overline{u_{m_k}}\|^2 \to 0$$

as $k, l \to \infty$. Hence $\{u_{m_k}\}$ is a Cauchy sequence. By the completeness of X, it converges to some $v^* \in C$. From the continuity of the metric d, we can say that $d(v^*, Tv^*) = \lim_{k\to\infty} \alpha_{m_k} d(p, Tu_{m_k}) = 0$. Therefore $v^* \in F(T)$ and (ii) has been proved.

Now, let x^* be an another fixed point of T. Taking $u_{m_l} = u_1 = x^*$, then

$$\begin{aligned} d(p, x^*) = \|\overline{x^*}\|^2 &\geq \|\overline{u_{m_k}}\|^2 + \|\overline{x^*} - \overline{u_{m_k}}\|^2 \\ &\geq \|\overline{u_{m_k}}\|^2 + d^2(x^*, u_{m_k}) \\ &\geq d^2(p, u_{m_k}) + d^2(x^*, u_{m_k}). \end{aligned}$$

By taking limit with $k \to \infty$, we have $d^2(p, x^*) \geq d^2(p, v^*) + d^2(v^*, x^*)$. Hence v^* is projection on $F(T)$. By double extract subsequence principle, we can conclude that the sequence $\{u_n\}$ converges to $v^* \in F(T)$. □

3.2 Halpern Iteration Process for G-Nonexpansive Mappings

In this section, we will prove the strong convergence theorem for a family of $G-$nonexpansive mappings in a complete CAT(0) space endowed with graph by using the Halpern iteration process

Theorem 3.5. *Let C be a convex subset of a complete CAT(0) space endowed with graph (X, d, G). Suppose that G is transitive and $E(G)$ is convex. Let $T : C \to C$ be edge-preserving and $\{\alpha_n\}$ be a sequence in $[0, 1]$. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)Tx_n \quad \forall n \geq 2, \tag{3.1}$$

where $u \in C$ such that $(u, Tu) \in E(G)$. If $\{x_n\}$ dominates u, then (x_n, x_{n+1}), (u, x_n) and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$.

Proof. We prove by induction. Since $E(G)$ is convex, (u, u) and (u, Tu) are in $E(G)$, we have $(u, x_1) \in E(G)$. Then $(Tu, Tx_1) \in E(G)$, since T is edge-preserving. Because G is transitive, we have $(u, Tx_1) \in E(G)$. By convexity of $E(G)$ and (u, Tx_1), $(Tu, Tx_1) \in E(G)$, we get $(x_1, Tx_1) \in E(G)$. By assumption, $(x_1, u) \in E(G)$. So, by convexity of $E(G)$, we get $(x_1, x_2) \in E(G)$.

Next, assume that (x_k, x_{k+1}), (u, Tx_k) and (x_k, Tx_k) are in $E(G)$. Then $(Tx_k, Tx_{k+1}) \in E(G)$, since T is edge-preserving. By transitivity of G, we have $(u, Tx_{k+1}) \in E(G)$. By convexity of $E(G)$ and (u, Tx_{k+1}), $(Tx_k, Tx_{k+1}) \in E(G)$, we get $(x_{k+1}, Tx_{k+1}) \in E(G)$. Since u is dominated by $\{x_n\}$, we have $(x_{k+1}, u) \in E(G)$. By convexity of $E(G)$, we get $(x_{k+1}, x_{k+2}) \in E(G)$.

Therefore, by induction, we can conclude that (x_n, x_{n+1}), (u, x_n) and (x_n, Tx_n) are in $E(G)$, for all $n \in \mathbb{N}$. □

Remark 3.2. The sequence $\{x_n\}$ generated by (3.1) is called the *Halpern iteration process.*

Theorem 3.6. *Let C be a nonempty convex subset of a complete CAT(0) space endowed with graph (X, d, G). Suppose G is transitive and $E(G)$ is convex. Let $T : C \to C$ be a G-nonexpansive mapping with nonempty fixed point set $F(T)$ and*

$F(T)\times F(T)\subseteq E(G)$. Assume that $u\in C$ such that $(u,Tu)\in E(G)$ and $x_1\in C$ is an arbitrarity chosen and $\{x_n\}$ is iteratively generated by

$$x_{n+1}=\alpha_n u\oplus(1-\alpha_n)Tx_n \quad \forall n\geq 2, \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying
(C1) $\lim_{n\to\infty}\alpha_n=0$;
(C2) $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(C3) $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$ *or* $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}}=1$.
If $\{x_n\}$ is dominated by p for some $p\in F(T)$ and $\{x_n\}$ dominates u, then $\{x_n\}$ converges strongly to $z\in F(T)$ which is nearest to u.

Proof. We first show that the sequence $\{x_n\}$ is bounded. Let p be any point in $F(T)$. Consider

$$\begin{aligned} d(x_{n+1},p)&=d(\alpha_n u\oplus(1-\alpha_n)Tx_n,p)\\ &\leq \alpha_n d(u,p)+(1-\alpha_n)d(x_n,p)\\ &\leq \max\{d(u,p),d(x_n,p)\}. \end{aligned}$$

This implies that $\{x_n\}$ is bounded. By the nonexpansiveness of T and $(x_n,z)\in E(G)$, we have $d(Tx_n,Tp)\leq d(x_n,p)\leq \max\{d(u,p),d(x_1,p)\}$. This shows that $\{Tx_n\}$ is also bounded. Consider the following calculation:

$$\begin{aligned} d(x_{n+1},x_n)&=d(\alpha_n u\oplus(1-\alpha_n)Tx_n,\alpha_{n-1}u\oplus(1-\alpha_{n-1})Tx_{n-1})\\ &\leq d(\alpha_n u\oplus(1-\alpha_n)Tx_n,\alpha_n u\oplus(1-\alpha_n)Tx_{n-1})\\ &+d(\alpha_n u\oplus(1-\alpha_n)Tx_{n-1},\alpha_{n-1}u\oplus(1-\alpha_{n-1})Tx_{n-1})\\ &\leq (1-\alpha_n)d(Tx_n,Tx_{n-1})+|\alpha_n-\alpha_{n-1}|d(u,Tx_{n-1})\\ &\leq (1-\alpha_n)d(x_n,x_{n-1})+|\alpha_n-\alpha_{n-1}|M, \end{aligned}$$

for some $M\geq 0$. By $(C2)$, $(C3)$ and Lemma 2.5, we can conclude that $d(x_n,x_{n+1})\to 0$ as $n\to\infty$. Consequently, by $(C1)$,

$$\begin{aligned} d(x_n,Tx_n)&\leq d(x_n,x_{n+1})+d(x_{n+1},Tx_n)\\ &=d(x_n,x_{n+1})+d(\alpha_n u\oplus(1-\alpha_n)Tx_n,Tx_n)\\ &\leq d(x_n,x_{n+1})+\alpha_n d(u,Tx_n)\to 0. \end{aligned}$$

From Lemma 2.6, let $z=\lim_{k\to\infty}x_k$, x_k given by (2.6), is the nearest point to u. Consider

$$\begin{aligned} d^2(x_{n+1},z)&=d^2(\alpha_n u\oplus(1-\alpha_n)Tx_n,z)\\ &\leq \alpha_n d^2(u,z)+(1-\alpha_n)d^2(Tx_n,z)-\alpha_n(1-\alpha_n)d^2(u,Tx_n)\\ &\leq (1-\alpha_n)d^2(x_n,z)+\alpha_n(d^2(u,z)-(1-\alpha_n)d^2(u,Tx_n)). \end{aligned} \tag{3.3}$$

Lemma 2.4 guarantee that

$$\limsup_{n\to\infty}(d^2(u,z)-d^2(u,x_n))\leq 0. \tag{3.4}$$

Moreover, by law of cosine, we get that

$$\begin{aligned} d^2(x_n, Tx_n) &\geq d^2(x_n, u) + d^2(Tx_n, u) - 2d(x_n, u)d(Tx_n, u) \\ &= (d(x_n, u) - d(Tx_n, u))^2 \geq 0. \end{aligned}$$

Since $d^2(x_n, Tx_n) \to 0$, this implies that $d(x_n, u) - d(Tx_n, u) \to 0$, and

$$\lim_{n\to\infty} d(u, x_n) = \lim_{n\to\infty} d(u, Tx_n) = \lim_{n\to\infty} (1 - \alpha_n)d(u, Tx_n). \tag{3.5}$$

From (3.4) and (3.5),

$$\begin{aligned} \limsup_{n\to\infty}(d^2(u, z) - (1 - \alpha_n)d^2(u, Tx_n)) &= d^2(u, z) - \limsup_{n\to\infty}(1 - \alpha_n)d^2(u, Tx_n) \\ &= d^2(u, z) - \limsup_{n\to\infty} d^2(u, x_n) \\ &= \limsup_{n\to\infty}(d^2(u, z) - d^2(u, x_n)) \leq 0 \end{aligned}$$

Hence, from (3.3) and Lemma 2.5, we get $\lim_{n\to\infty} d^2(x_n, z) = 0$. The proof has been completed. □

This following lemma, the extension of Lemma 2.3, will be used for proving the last main theorem.

Lemma 3.7. *Let C be a nonempty closed convex subset of a complete CAT(0) space endowed with graph (X, d, G), and let $\{T_n : n \in \mathbb{N}\}$ be a family of single-valued G-nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Define $T : C \to C$ by*

$$Tx = \bigoplus_{n=1}^{\infty} \lambda_n T_n x$$

for all $x \in C$, where $\{\lambda_n\} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and $\sum_{i=n}^{\infty} \lambda_i' \to 0$ as $n \to \infty$. Then T is G-nonexpansive and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Let $y_0 \in \bigcap_{n=1}^{\infty} F(T_n)$ be arbitrary given. Since $d(T_n(x), y_0) \leq d(x, y_0)$, for all $n \in \mathbb{N}$, $\{T_n(x)\}$ is bounded. For each $n \in \mathbb{N}$, let $w_n : C \to C$ given by

$$w_n x = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} T_1 x \oplus \frac{\lambda_2}{\sum_{i=1}^{n} \lambda_i} T_2 x \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} T_n x. \tag{3.6}$$

Since $w_1 = T_1$, so w_1 is G-nonexpansive. Suppose that w_k is G-nonexpansive and let $x, y \in X$ be such that $(x, y) \in E(G)$. Consider

$$\begin{aligned} d(w_{k+1}x, w_{k+1}y) &= d\left(\frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{k+1} \lambda_i} w_k x \oplus \frac{\lambda_{k+1}}{\sum_{i=1}^{k+1} \lambda_i} T_{k+1}x, \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{k+1} \lambda_i} w_k y \oplus \frac{\lambda_{k+1}}{\sum_{i=1}^{k+1} \lambda_i} T_{k+1}y\right) \\ &\leq \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{k+1} \lambda_i} d(w_k x, w_k y) + \frac{\lambda_{k+1}}{\sum_{i=1}^{k+1} \lambda_i} d(T_{k+1}x, T_{k+1}y) \\ &\leq d(x, y). \end{aligned}$$

So w_{k+1} is nonexpansive. This show that w_n is nonexpansive. By the convexity of $E(G)$, we have

$$(w_nx, w_ny) = \left(\frac{\sum_{i=1}^{n-1}\lambda_i}{\sum_{i=1}^{n}\lambda_i}w_{n-1}x \oplus \frac{\lambda_n}{\sum_{i=1}^{n}\lambda_i}T_nx, \frac{\sum_{i=1}^{n-1}\lambda_i}{\sum_{i=1}^{n}\lambda_i}w_{n-1}y \oplus \frac{\lambda_n}{\sum_{i=1}^{n}\lambda_i}T_ny\right) \in E(G).$$

Thus w_n is G-nonexpansive and T is nonexpansive. If $(x, y) \in E(G)$ by convexity of $E(G)$, we obtain that

$$(Tx, Ty) = \left(\bigoplus_{n=1}^{\infty}\lambda_nT_nx, \bigoplus_{n=1}^{\infty}\lambda_nT_ny\right) \in E(G).$$

Therefore T is also G-nonexpansive. It is easy to see that $\bigcap_{n=1}^{\infty}F(T_n) \subset F(T)$. Now, let $x_0 \in F(T)$. Since $y_0 = w_ny_0$, we have

$$\begin{aligned} d(x_0, y_0) =& d(Tx_0, y_0) = \lim_{n\to\infty} d(s_nx_0, y_0) \\ &\le \lim_{n\to\infty}[\lambda_1 d(T_1x_0, y_0) + \cdots + \lambda_n d(T_nx_0, y_0) + \lambda'_n d(y_0, y_0)] \\ &= \sum_{n=1}^{\infty}\lambda_n d(T_nx_0, y_0) \le d(x_0, y_0). \end{aligned}$$

Thus $d(T_nx_0, y_0) = d(x_0, y_0)$ for all $n \in \mathbb{N}$, by Lemma 2.2, $T_nx_0 = x_0$ for all $n \in \mathbb{N}$. □

Using the result in the above Lemma, we can extent Theorem 3.6 to the family of G-nonexpansive mappings.

Theorem 3.8. *Let C be a nonempty convex subset of a complete CAT(0) space endowed with graph (X, d, G). Suppose G is transitive and $E(G)$ is convex. Let $\{T_n : C \to C\}$ be a countable family of G-nonexpansive mappings with $\bigcap_{n=1}^{\infty}F(T_n)$. Let $\{\lambda_n\} \subset (0, 1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$ and $\sum_{i=n}^{\infty}\lambda'_i \to 0$ as $n \to 0$. Suppose that $u, x_1 \in C$ are arbitrary chosen and x_n is defined by*

$$x_{n+1} = \alpha_nu \oplus (1 - \alpha_n)w_nx_n \quad \forall n \ge 2, \tag{3.7}$$

where w_n defined by (3.6) and $\{\alpha_n\} \in (0, 1)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$.

If $\{x_n\}$ is dominated by p for some $p \in \bigcap_{n=1}^{\infty}F(T_n)$ and $\{x_n\}$ dominates u. Then $\{x_n\}$ converges to $z \in \bigcap_{n=1}^{\infty}F(T_n)$ which is nearest to u.

Proof. Let $\{w_n\}$ and T be as in the proof of lemma 3.7, so w_n is G-nonexpansive and $\bigcap_{n=1}^{\infty}F(w_n) = F(T) = \bigcap_{n=1}^{\infty}F(T_n)$ and $w_n(p) = p$ for all $p \in F(T)$. Then we follow the proof from Theorem 3.6 by replace w_n by T_n. Then the proof is complete. □

Acknowledgement : The authors would like to thank Associate Prof. Dr. Kasamsuk Ungchittrakul and the reviewers for their valuable comments.

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(Received 30 July 2018)
(Accepted 18 June 2019)