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# Meromorphic Function Sharing Small Functions with Its Derivative

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**Abstract :** In the paper, we deal with the uniqueness problem of meromorphic function that share a set of small functions with its derivative and obtain some results which generalize the recent results due to Xu, Yi and Wang [Revista De Matematica, Teoria Y Aplicaciones, 23(2016), 291-308].

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## 1 Introduction, Definitions and Results

In this paper, by meromorphic function we shall always mean meromorphic function in the complex plane. We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna value distribution theory (see [1–3]). Let $f$ be a nonconstant meromorphic function in the complex plane. By $S(r, f)$, we mean any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite logarithmic measure. We say that the meromorphic function $\alpha$ is a small function of $f$, if $T(r, \alpha) = S(r, f)$. We denote $S(f)$ by the set

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of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to $f$.

Let $f$ be a nonconstant meromorphic function and $\alpha \in \tilde{S}(f) = S(f) \cup \{\infty\}$ and $S$ be a subset of $\tilde{S}(f)$. We define

$$E(S,f) = \bigcup_{\alpha \in S} \{z : f(z) - \alpha = 0, \quad \textit{counting multiplicity}\},$$

$$\overline{E}(S,f) = \bigcup_{\alpha \in S} \{z : f(z) - \alpha = 0, \quad \textit{ignoring multiplicity}\}.$$

If $E(S,f) = E(S,g)$, then we say that $f$ and $g$ share the set $S$ CM; if $\overline{E}(S,f) = \overline{E}(S,g)$, then we say that $f$ and $g$ share the set $S$ IM. Especially, if $S = \{\alpha\}$ and $E(S,f) = E(S,g)$, then we say that $f$ and $g$ share $\alpha$ CM; and we say that $f$ and $g$ share $\alpha$ IM if $\overline{E}(S,f) = \overline{E}(S,g)$.

During the last four decades the uniqueness theory of entire and meromorphic functions has become a prominent branch of the value distribution theory. A widely studied subtopic to the uniqueness theory has been to considering shared value problems relative to a meromorphic function $f$ and its derivative $f^{(k)}$. Many research works on entire and meromorphic function $f$ and its derivative $f^{(k)}$ have been done by many mathematicians (see [1, 4–10]). A much investigated problem in this direction is the following conjecture proposed by Brück [11].

**Conjecture 1.1.** *Let $f$ be a nonconstant entire function. Suppose that*

$$\rho_1(f) = \lim_{r\to\infty} \sup \frac{\log\log T(r,f)}{\log r}$$

*is not a positive integer or infinity. If $f$ and $f'$ share one finite value $a$ CM, then*

$$\frac{f'-a}{f-a} = c,$$

*for some nonzero constant $c$.*

In 1996, Brück [11] proved that the conjecture is true if $a = 0$ or $N(r,0;f') = S(r,f)$. In 1998, Gundersen and Yang [6] proved that the conjecture is true if $f$ is of finite order and fails, in general, for meromorphic functions. In 2005, Al-Khaladi [12] proved that the conjecture is true for meromorphic function $f$ when $N(r,0;f') = S(r,f)$. Also, in 2004, Chen and Shon [13] proved the conjecture for entire function $f$ provided $\rho_1(f) < \frac{1}{2}$.

In 2008, Yang and Zhang [14] obtained the following results.

**Theorem 1.2.** *Let $f$ be a nonconstant meromorphic function and $n$ $(\geq 12)$ be an integer. Let $F$ and $F'$ share 1 CM, where $F = f^n$. Then $F = F'$ and $f$ assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c$ is a nonzero constant.*

**Theorem 1.3.** *Let $f$ be a nonconstant entire function and $n$ $(\geq 7)$ be an integer. Let $F$ and $F'$ share 1 CM, where $F = f^n$. Then $F = F'$ and $f$ assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c$ is a nonzero constant.*

So it will be quite interesting to investigate the situation if the sharing value 1 is replaced by a small meromorphic function. In 2009, Zhang and Yang [9] obtained the following two theorems in this direction which improve Theorem 1.2 and Theorem 1.3 respectively.

**Theorem 1.4.** *Let $f$ be a nonconstant meromorphic function, $n$, $k$ be two positive integers and $\alpha$ $(\not\equiv 0,\infty)$ be a small meromorphic function of $f$. If $f^n - \alpha$ and $(f^n)^{(k)} - \alpha$ share the value 0 CM and $n > k+1+\sqrt{k+1}$, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c$ is a nonzero constant and $\lambda^k = 1$.*

**Theorem 1.5.** *Let $f$ be a nonconstant entire function, $n$, $k$ be two positive integers and $\alpha$ $(\not\equiv 0,\infty)$ be a small meromorphic function of $f$. If $f^n - \alpha$ and $(f^n)^{(k)} - \alpha$ share the value 0 CM and $n > k+1$, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c$ is a nonzero constant and $\lambda^k = 1$.*

Regarding Theorems 1.4 and 1.5, the following questions are inevitable.

**Question 1.6.** Is it possible in any way to relax the nature of sharing the small function ?

**Question 1.7.** What happen if one replace the small function $\alpha(z)$ by a set $S_m = \{\alpha(z),\ \alpha(z)\omega,\ \ldots,\ \alpha(z)\omega^{m-1}\}$ where $\omega = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$ and $m$ is a positive integer ?

Recently Xu, Yi and Wang [15] answered the above questions and obtained following two theorems. To state the results we need the following definition of weighted sharing introduced by Lahiri [7, 16] which measure how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

**Definition 1.8.** Let $l$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_l(a; f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $p$ is counted $p$ times if $p \leq l$ and l+1 times if $p > l$. If $E_l(a; f) = E_l(a; g)$, we say that $f$, $g$ share the value $a$ with weight $l$.

The definition implies that if $f$, $g$ share a value $a$ with weight $l$, then $z_0$ is an $a$-point of $f$ with multiplicity $p(\leq l)$ if and only if it is an $a$-point of $g$ with multiplicity $p(\leq l)$ and $z_0$ is an a-point of $f$ with multiplicity $p(> l)$ if and only if it is an a-point of $g$ with multiplicity $q(> l)$, where $p$ is not necessarily equal to $q$.

We write $f$, $g$ *share* $(a,l)$ to mean that $f$, $g$ share the value $a$ with weight $l$. Clearly if $f$, $g$ share $(a,l)$ then $f$, $g$ share $(a,l_1)$ for any integer $l_1$, $0 \le l_1 < l$. Also we note that $f$, $g$ share the value $a$ IM or CM if and only if $f$, $g$ share $(a,0)$ or $(a,\infty)$ respectively.

**Remark 1.9.** Let $S$ be a subset of $\tilde{S}(f)$. Then we can obtain the definitions of $E_l(S,f)$ and $E_l(S,f) = E_l(S,g)$, similarly.

The following are the results of Xu, Yi and Wang.

**Theorem 1.10.** *Let $f$ be a nonconstant meromorphic function, $n$, $k$, $l$, $m$ be positive integers and $\alpha$ ($\not\equiv 0,\infty$) be a small meromorphic function of $f$. If $E_l(S_m, f^n) = E_l(S_m,(f^n)^{(k)})$ and*

$$n > \max\left\{k+1,\ \frac{l(m+1)k+2\gamma}{2lm} + \frac{\sqrt{4\gamma(\gamma+kl)+k^2l^2(m-1)^2}}{2lm}\right\},$$

*where $\gamma = k+l+2$, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form $f(z) = ce^{\frac{\mu}{n}z}$, where $c$ is a nonzero constant and $\mu^{km} = 1$.*

**Theorem 1.11.** *Let $f$ be a nonconstant entire function, $n$, $k$, $l$, $m$ be positive integers and $\alpha$ ($\not\equiv 0,\infty$) be a small meromorphic function of $f$. If $E_l(S_m, f^n) = E_l(S_m,(f^n)^{(k)})$ and*

$$n > \max\left\{k+1,\ k+\frac{\gamma}{lm}\right\},$$

*where $\gamma = k+l+2$, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form $f(z) = ce^{\frac{\mu}{n}z}$, where $c$ is a nonzero constant and $\mu^{km} = 1$.*

Regarding Theorems 1.10 and 1.11, it is natural to ask the following question which is the motive of the authors.

**Question 1.12.** What happen if one replace $f^n$ by $f^n(f-a)^s$ in Theorems 1.10 and 1.11 where $s$ is a positive integer and $a$ is a nonzero complex constant ?

In the paper, our aim is to find out the possible answer of the above question. We prove following results which extend Theorems 1.10 and 1.11. The following theorems are the main results of the paper.

**Theorem 1.13.** *Let $f$ be a nonconstant meromorphic function, $n$, $k$, $l$, $m$, $s$ be positive integers, and $\alpha(z)$ ($\not\equiv 0,\ \infty$) be a small meromorphic function of $f$. If $E_l(S_m, f^n(f-a)^s) = E_l(S_m,(f^n(f-a)^s)^{(k)})$ and*

$$\begin{aligned} s \ \geq\ & n > \max\Bigg\{k+1, \\ & \frac{l(m+1)k+2\gamma-lms}{2lm} + \frac{\sqrt{4\gamma(\gamma+lk)+l^2\{(m-1)k+ms\}^2}}{2lm}\Bigg\}, \end{aligned} \tag{1.1}$$

*where* $\gamma = k+l+2$, *then*

$$f^{n+i} = t(f^{n+i})^{(k)},$$

*for some* $i \in \{0,1,2,\ldots,s\}$, $t^m = 1$, *and* $f$ *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n+i}z},$$

*where* $a$, $c$ *are two nonzero constant and* $\lambda^{km} = 1$.

**Theorem 1.14.** *Let* $f$ *be a nonconstant entire function,* $n$, $k$, $l$, $m$, $s$ *be positive integers, and* $\alpha(z)$ $(\not\equiv 0, \infty)$ *be a small meromorphic function of* $f$. *If* $E_l(S_m, f^n(f-a)^s) = E_l(S_m, (f^n(f-a)^s)^{(k)})$ *and*

$$s \geq n > \max\left\{k+1,\ k+\frac{\gamma}{lm}\right\}, \tag{1.2}$$

*where* $\gamma = k+l+2$, *then*

$$f^{n+i} = t(f^{n+i})^{(k)},$$

*for some* $i \in \{0,1,2,\ldots,s\}$, $t^m = 1$, *and* $f$ *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n+i}z},$$

*where* $a$, $c$ *are two nonzero constant and* $\lambda^{km} = 1$.

**Note 1.15.** *For higher derivative* $k \geq 2$, *one may observed that* $f$ *does not always assume the form* $f(z) = ce^{\frac{\lambda}{n+i}z}$, *though* $E_l(S_m, f^n(f-a)^s) = E_l(S_m, (f^n(f-a)^s)^{(k)})$.

We now give the following example for supporting the above observation.

**Example 1.16.** We assume a nonconstant meromorphic function $f$ in such a way that $f^n(f-a)^s = c_1e^z + c_2e^{wz} + c_3e^{w^2z}$, where $w$ is a non-real cube root of unity and $c_i$ are nonzero constants. Let $n=8$, $s=8$, $l=1$, $m=2$, $k=3$, then it is obvious that $E_l(S_m, f^n(f-a)^s) = E_l(S_m, (f^n(f-a)^s)^{(k)})$ and

$$\begin{aligned} s \quad \geq \quad & n > \max\Bigg\{k+1, \\ & \frac{l(m+1)k+2\gamma-lms}{2lm} + \frac{\sqrt{4\gamma(\gamma+lk)+l^2\{(m-1)k+ms\}^2}}{2lm}\Bigg\}, \end{aligned}$$

where $\gamma = k+l+2$, but $f(z) \neq ce^{\frac{\lambda}{n+i}z}$, for some $i \in \{0,1,2,\ldots,s\}$, where $a$, $c$ are two nonzero constants and $\lambda^{km} = 1$.

We now explain the following definitions and notations which are used in the paper.

**Definition 1.17.** [7] For a positive integer $p$ and $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f \mid\leq p)$ the counting function of those $a$-points of $f$ (counted with proper multiplicities) whose multiplicities are not greater than $p$. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function. Analogously we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

**Definition 1.18.** [7] Let $p$ be a positive integer or infinity. We denote by $N_p(r, a; f)$ the counting function of $a$-points of $f$, where an $a$-point of multiplicity $q$ is counted $q$ times if $q \leq p$ and $p$ times if $q > p$. Then

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq p).$$

**Definition 1.19.** [4,17] Let $F$ and $G$ be two nonconstant meromorphic functions such that $F$ and $G$ share the value 1 IM and $z_0$ be a zero of $F-1$ with multiplicity $p$ and a zero of $G-1$ with multiplicity $q$. We denote by $N_L(r, 1; F)$ the counting function of those zeros of $F-1$ for which $p > q$. In the same way, we can define $N_L(r, 1; G)$.

## 2 Lemmas

In this section, we state some lemmas which will be needed in the sequel. We will use the following notations.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.1}$$

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right), \tag{2.2}$$

$$U = \frac{F'}{F-1} - \frac{G'}{G-1}, \tag{2.3}$$

where $F$ and $G$ are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

**Lemma 2.1.** [18] *Let $f$ be a nonconstant meromorphic function, and $p$, $k$ be two positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.4}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.5}$$

**Lemma 2.2.** [2] *Let $f$ be a nonconstant meromorphic function and let $a_n(z)$ ($\not\equiv 0$), $a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0) = nT(r, f) + S(r, f).$$

**Lemma 2.3.** *Let $f$ be a nonconstant meromorphic function, and $n,\ k,\ m,\ s$ be positive integers, and $\alpha(z)$ ($\not\equiv 0,\ \infty$) be a small meromorphic function of $f$. Suppose that $F_1 = \frac{f^n(f-a)^s}{\alpha}$, $G_1 = \frac{(f^n(f-a)^s)^{(k)}}{\alpha}$, where $a$ is a nonzero constant. If $f^n(f-a)^s$ and $(f^n(f-a)^s)^{(k)}$ share $S_m$ IM and $n > k+1$, and if $H \neq 0$, then*

$$T(r,f) = O(\overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f)),$$

*where $H$ is given by* (2.1), *and* $F = (F_1)^m,\ G = (G_1)^m$.

*Proof.* From the definitions of $F,\ G,\ F_1,\ G_1$ and noting that $f^n(f-a)^s$ and $(f^n(f-a)^s)^{(k)}$ share $S_m$ IM, we see that $F,\ G$ share the value 1 IM with the possible exception of the zeros and poles of $\alpha(z)$. By the definition of $H$ we have

$$\begin{aligned} N(r,H) \quad \leq \quad & \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + \overline{N}(r,\infty;F) + N_L(r,1;F) \\ & + N_L(r,1;G) + N_0(r,0;F') + N_0(r,0;G') + S(r,f), \end{aligned}$$

where $N_0(r,0;F')$ ($N_0(r,0;G')$) denotes the counting function of those zeros of $F'$ ($G'$) which are not the zeros of $F(F-1)$ ($G(G-1)$). Now using Lemma 2.2 and arguing similarly as in Lemma 2.3 [9] we obtain

$$\begin{aligned} T(r,G) \quad \leq \quad & N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + N_L(r,1;F) \\ & + 2N_L(r,1;G) + S(r,f) \\ \leq \quad & N_2(r,0;(F_1)^m) + N_2(r,0;(G_1)^m) + 3\overline{N}(r,\infty;(F_1)^m) \\ & + N_L(r,1;F) + 2N_L(r,1;G) + S(r,f) \\ \leq \quad & N_2(r,0;f^n(f-a)^s) + mN_2(r,0;(f^n(f-a)^s)^{(k)}) + 3\overline{N}(r,\infty;f) \\ & + N_L(r,1;F) + 2N_L(r,1;G) + S(r,f) \\ \leq \quad & N_2(r,0;f) + N_2(r,a;f) + mT(r,(f^n(f-a)^s)^{(k)}) \\ & - mT(r,f^n(f-a)^s) + mN_{k+2}(r,0;f^n(f-a)^s) \\ & + 3\overline{N}(r,\infty;f) + N_L(r,1;F) + 2N_L(r,1;G) + S(r,f). \qquad (2.6) \end{aligned}$$

Since $mT(r,(f^n(f-a)^s)^{(k)}) \leq T(r,(G_1)^m) + S(r,f) \leq T(r,G) + S(r,f)$, from (2.6) we obtain

$$\begin{aligned} mT(r,f^n(f-a)^s) \quad \leq \quad & 2\overline{N}(r,0;f) + 2\overline{N}(r,a;f) + mN_{k+2}(r,0;f^n(f-a)^s) \\ & + 3\overline{N}(r,\infty;f) + N_L(r,1;F) + 2N_L(r,1;G) + S(r,f) \\ \leq \quad & \{m(k+2)+2\}[\overline{N}(r,0;f) + \overline{N}(r,a;f)] + 3\overline{N}(r,\infty;f) \\ & + N_L(r,1;F) + 2N_L(r,1;G) + S(r,f). \qquad (2.7) \end{aligned}$$

Since $n > k+1$, by Lemma 2.1 we obtain

$$\begin{aligned} N_L(r,1;F) \quad \leq \quad & N\left(r,\frac{F}{F'}\right) \leq N\left(r,\frac{F'}{F}\right) + S(r,f) \\ \leq \quad & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + S(r,f) \\ \leq \quad & \overline{N}(r,0;F_1) + \overline{N}(r,\infty;F_1) + S(r,f) \\ \leq \quad & \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f) + S(r,f). \qquad (2.8) \end{aligned}$$

Also

$$\begin{aligned} N_L(r,1;G) &\leq N\left(r,\frac{G}{G'}\right) \leq N\left(r,\frac{G'}{G}\right)+S(r,f) \\ &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,f) \\ &\leq \overline{N}(r,0;G_1)+\overline{N}(r,\infty;G_1)+S(r,f) \\ &\leq N_{k+1}(r,0;f^n(f-a)^s)+(k+1)\overline{N}(r,\infty;f)+S(r,f) \\ &\leq (k+1)[\overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}(r,\infty;f)]+S(r,f). \quad (2.9) \end{aligned}$$

Using (2.8), (2.9) and Lemma 2.2 we obtain from (2.7)

$$\begin{aligned} m(n+s)T(r,f) &\leq \{m(k+2)+2k+5\}[\overline{N}(r,0;f)+\overline{N}(r,a;f)] \\ &\quad +(2k+6)\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

This shows that

$$T(r,f) = O(\overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}(r,\infty;f)).$$

This proves the lemma. □

The following lemma can be proved in the line of the proof of Lemma 2.2 [19].

**Lemma 2.4.** *Let $f$ be a nonconstant meromorphic function and $n \geq k+1$. Let $P(z) = a_s z^s + a_{s-1}z^{s-1} + \ldots + a_1 z + a_0$ is a nonzero polynomial of degree $s$. If $f^nP(f) \equiv t(f^nP(f))^{(k)}$ for some $t$ satisfying $t^m = 1$ where $m$ is a positive integer, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0,1,2,\ldots,s\}$; and $f^{n+i} = (f^{n+i})^{(k)}$, where $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, $c$ is a nonzero constant and $\lambda^{km} = 1$.*

**Lemma 2.5.** *Let $V$ be given by (2.2), and $F$, $G$, $F_1$, $G_1$ be given as in Lemma 2.3. If $n$, $m$, $k$, $s$ are positive integers such that $n > k+1$, and $V = 0$, then*

$$f^n(f-a)^s = t(f^n(f-a)^s)^{(k)},$$

*where $t^m = 1$, and $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, for some $i \in \{0,1,2,\ldots,s\}$; $a$, $c$ are two nonzero constants and $\lambda^{km} = 1$.*

*Proof.* Since $V = 0$, integrating (2.2) we obtain

$$1-\frac{1}{F} = A\left(1-\frac{1}{G}\right), \qquad (2.10)$$

where $A\ (\neq 0)$ is a constant. Now we discuss the following two cases.

**Case (i).** Let $N(r,\infty;f) = S(r,f)$. If $A \neq 1$, from (2.10) we get

$$\overline{N}\left(r,\frac{1}{1-A};F\right) = \overline{N}(r,\infty;G) = \overline{N}(r,\infty;f) = S(r,f).$$

Using Nevanlinna's second theorem and Lemma 2.2 we obtain

$$\begin{aligned} m(n+s)T(r,f) &\leq T(r,F)+S(r,f) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F) \\ &\quad +\overline{N}\left(r,\frac{1}{1-A};F\right)+S(r,f) \\ &\leq \overline{N}(r,0;F_1)+S(r,f) \\ &\leq \overline{N}(r,0;f)+\overline{N}(r,a;f)+S(r,f) \\ &\leq 2T(r,f)+S(r,f), \end{aligned}$$

a contradiction. This, however, means that $A=1$.

**Case (ii).** Let $N(r,\infty;f)\neq S(r,f)$. Then there exists a $z_0$ ($\alpha(z_0)\neq 0,\infty$) such that $\frac{1}{f(z_0)}=0$, and therefore $\frac{1}{F(z_0)}=\frac{1}{G(z_0)}=0$. So from (2.10) we obtain $A=1$ and $F=G$, that is $f^n(f-a)^s=t(f^n(f-a)^s)^{(k)}$ for some $t$ satisfying $t^m=1$. Then the result follows from Lemma 2.4. □

**Lemma 2.6.** *Let $U$ be given by* (2.3)*, and $F$, $G$, $F_1$, $G_1$ be given as in Lemma* 2.3*. If $n$, $m$, $k$, $s$ are positive integers such that $n>k+1$, and $U=0$, then*

$$f^n(f-a)^s=t(f^n(f-a)^s)^{(k)},$$

*where $t^m=1$, and $f$ assumes the form $f(z)=ce^{\frac{\lambda}{n+i}z}$, for some $i\in\{0,1,2,\ldots,s\}$; $a$, $c$ are two nonzero constants and $\lambda^{km}=1$.*

*Proof.* Since $U=0$, integrating (2.3) we obtain

$$F=AG+1-A, \tag{2.11}$$

where $A$ ($\neq 0$) is a constant. From (2.11) and the definitions of $F$, $G$, $F_1$, $G_1$ we see that $N(r,\infty;f)=S(r,f)$. We now consider the following two cases.

**Case (i).** Let $A=1$. Then $F=G$ and hence $f^n(f-a)^s=t(f^n(f-a)^s)^{(k)}$ for some $t$ satisfying $t^m=1$. Then the conclusion follows from Lemma 2.4.

**Case (ii).** Let $A\neq 1$. We assume that $N(r,0;f)\neq S(r,f)$. Then there exists a $z_0$ ($\alpha(z_0)\neq 0$) such that $f(z_0)=0$. Since $n>k+1$, we have $F(z_0)=G(z_0)=0$. Therefore from (2.11) we get $A=1$, a contradiction.

Next we assume that $N(r,0;f)=S(r,f)$. Then from (2.11) we have

$$\begin{aligned} \overline{N}(r,1-A;F) &= \overline{N}(r,0;G) \\ &\leq N_{k+1}(r,0;f^n(f-a)^s)+k\overline{N}(r,\infty;f)+S(r,f) \\ &\leq (k+1)[\overline{N}(r,0;f)+\overline{N}(r,a;f)]+S(r,f) \\ &\leq (k+1)\overline{N}(r,a;f)+S(r,f). \end{aligned}$$

Using Nevanlinna's second theorem and Lemma 2.2 we obtain

$$\begin{aligned} m(n+s)T(r,f) &\leq T(r,F)+S(r,f) \\ &\leq \overline{N}(r,0;F)+\overline{N}(r,1-A;F)+\overline{N}(r,\infty;F)+S(r,f) \\ &\leq \overline{N}(r,0;f)+(k+2)\overline{N}(r,a;f)+\overline{N}(r,\infty;f)+S(r,f) \\ &\leq (k+2)T(r,f)+S(r,f), \end{aligned}$$

a contradiction since $n > k+1$ and $s$ is a positive integer. This proves the lemma. □

**Lemma 2.7.** *Let $V$ be given by* (2.2)*, and $F$, $G$, $F_1$, $G_1$ be given by Lemma* 2.3*, $n$, $m$, $k$, $s$ be positive integers. If $V \neq 0$, then*

$$[m(n+s)-1]\overline{N}(r,\infty;f) \leq N(r,\infty;V)+S(r,f).$$

*Proof.* From the definitions of $F$, $G$ and $V$ we see that if $z_0$ ($\alpha(z_0) \neq 0,\infty$) is a pole of $f$ with multiplicity $p$ ($\geq 1$) then $z_0$ is a zero of $\frac{F'}{F-1}-\frac{F'}{F}$ with multiplicity $m(n+s)p-1$ and a zero of $\frac{G'}{G-1}-\frac{G'}{G}$ with multiplicity $m[(n+s)p+k]-1$. Thus, $z_0$ is a zero of $V$ with multiplicity $q \geq m(n+s)-1$. Since $m(r,V) = S(r,f)$, from (2.2) we obtain

$$\begin{aligned} [m(n+s)-1]\overline{N}(r,\infty;f) &\leq N(r,0;V)+S(r,f) \leq T(r,V)+S(r,f) \\ &\leq N(r,\infty;V)+S(r,f). \end{aligned}$$

This proves the lemma. □

**Lemma 2.8.** *Let $U$ be given by* (2.3)*, and $F$, $G$, $F_1$, $G_1$ be given by Lemma* 2.3*. If $n$, $m$, $k$, $s$ are positive integers such that $s \geq n > k+1$, and $U \neq 0$, then*

$$[m(n-k)-1]\overline{N}(r,0;f)+[m(s-k)-1]\overline{N}(r,a;f) \leq N(r,\infty;U)+S(r,f),$$

*where $a$ is a nonzero constant.*

*Proof.* From the definitions of $F$, $G$ and $U$ we see that if $z_1$ ($\alpha(z_1) \neq 0,\infty$) is a zero of $f$ with multiplicity $p$ ($\geq 1$) then $z_1$ is a zero of $\frac{F'}{F-1}$ with multiplicity $mnp-1$ and a zero of $\frac{G'}{G-1}$ with multiplicity $m(np-k)-1$. Thus, $z_1$ is a zero of $U$ with multiplicity at least $m(n-k)-1$.

Let $z_2$ ($\alpha(z_2) \neq 0,\infty$) is an $a$-point of $f$ with multiplicity $q$ ($\geq 1$). Then $z_2$ is a zero of $\frac{F'}{F-1}$ with multiplicity $msq-1$ and a zero of $\frac{G'}{G-1}$ with multiplicity $m(sq-k)-1$. Therefore, $z_2$ is a zero of $U$ with multiplicity at least $m(s-k)-1$. Using the fact that $m(r,U) = S(r,f)$, from (2.3) we obtain

$$\begin{aligned} [m(n-k)-1]\overline{N}(r,0;f)+[m(s-k)-1]\overline{N}(r,a;f) &\leq N(r,0;U)+S(r,f) \\ &\leq T(r,U)+S(r,f) \\ &\leq N(r,\infty;U)+S(r,f). \end{aligned}$$

This proves the lemma. □

**Lemma 2.9.** *Let $H$ be given by* (2.1)*, and $F$, $G$, $F_1$, $G_1$ be given as in Lemma* 2.3*. If $n$, $m$, $k$, $s$ are positive integers such that $n > k+1$, and*

$$\overline{N}(r,\infty;f) \; = \; \overline{N}(r,0;f) \; = \; \overline{N}(r,a;f) \; = \; S(r,f), \tag{2.12}$$

*and $H = 0$, then*

$$f^n(f-a)^s = t(f^n(f-a)^s)^{(k)},$$

*where $t^m = 1$, and $f$ assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, for some $i \in \{0,1,2, \ldots, s\}$; $a$, $c$ are two nonzero constants and $\lambda^{km} = 1$.*

*Proof.* Since $H = 0$, from (2.1), integrating twice we obtain

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{2.13}$$

where $A\ (\neq 0)$ and $B$ are constants. From (2.13) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF-(B+1)}.$$

We now discuss the following three cases.

**Case (i)** We assume that $B \neq 0, -1$. Then from (2.13) we have

$$\overline{N}\left(r, \frac{B+1}{B}; F\right) = \overline{N}(r,\infty;G).$$

Using second main theorem of Nevanlinna, Lemma 2.2 and the assumptions of the lemma we obtain

$$\begin{aligned} m(n+s)T(r,f) &= T(r,F) + S(r,F) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r, \frac{B+1}{B}; F\right) + S(r,F) \\ &\leq \overline{N}(r,0;f) + \overline{N}(r,a;f) + 2\overline{N}(r,\infty;f) + S(r,f) \\ &\leq S(r,f), \end{aligned}$$

a contradiction.

**Case (ii)** Assume that $B = 0$. Then from (2.13) we have

$$\overline{N}\left(r, \frac{A-1}{A}; F\right) = \overline{N}(r,0;G).$$

Now we consider the following two subcases.

**Subcase (i)** Let $A \neq 1$. Using second main theorem of Nevanlinna, Lemma 2.2 and the assumptions of the lemma we obtain

$$\begin{aligned} m(n+s)T(r,f) &= T(r,F)+S(r,F) \\ &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{A-1}{A};F\right)+\overline{N}(r,\infty;F)+S(r,F) \\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F) \\ &\leq \overline{N}(r,0;f)+\overline{N}(r,a;f)+N_{k+1}(r,0;f^n(f-a)^s) \\ &\quad +(k+1)\overline{N}(r,\infty;f)+S(r,f) \\ &\leq (k+2)\overline{N}(r,0;f)+(k+2)\overline{N}(r,a;f) \\ &\quad +(k+1)\overline{N}(r,\infty;f)+S(r,f) \\ &\leq S(r,f), \end{aligned}$$

a contradiction.

**Subcase (ii)** Let $A = 1$. Then we have $F = G$ that is

$$f^n(f-a)^s = t(f^n(f-a)^s)^{(k)},$$

where $t^m = 1$. Then the conclusion follows from Lemma 2.4.

**Case (iii)** Let $B = -1$. Then from (2.13) we have

$$G = \frac{(A+1)F-A}{F}.$$

Arguing similarly as in Case (ii) we obtain $FG = 1$, that is

$$f^n(f-a)^s(f^n(f-a)^s)^{(k)} = t\alpha^2(z)$$

where $t^m = 1$. Now

$$\begin{aligned} 2T\left(r,\frac{f^n(f-a)^s}{\alpha}\right) &= T\left(r,\frac{(f^n(f-a)^s)^2}{\alpha^2}\right) \\ &= T\left(r,\frac{t\alpha^2}{(f^n(f-a)^s)^2}\right)+S(r,f) \\ &= T\left(r,\frac{(f^n(f-a)^s)^{(k)}}{(f^n(f-a)^s)}\right)+S(r,f) \\ &\leq N_k(r,0;f^n(f-a)^s)+k\overline{N}(r,\infty;f)+S(r,f) \\ &\leq k[\overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}(r,\infty;f)]+S(r,f) \\ &\leq S(r,f), \end{aligned}$$

a contradiction. This proves the lemma. □

# 3 Proof of Theorems

## 3.1 Proof of Theorem 1.13

Let $F$, $G$, $F_1$, $G_1$ be given as in Lemma 2.3. We follow the idea proposed in [9]. If $V = 0$ or $U = 0$, we get the conclusion of the theorem from Lemmas 2.5 and 2.6. Now we assume that $V \neq 0$ and $U \neq 0$. Since $E_l(1, F) = E_l(1, G)$, from the definitions of $V$ and $U$ we have

$$\begin{aligned} N(r,\infty;V) \quad &\leq \quad \overline{N}(r,0;G) + \overline{N}(r,1;F \mid\geq l+1) + \overline{N}(r,1;G \mid\geq l+1) \\ &\quad\quad +S(r,f), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} N(r,\infty;U) \quad &\leq \quad \overline{N}(r,\infty;f) + \overline{N}(r,1;F \mid\geq l+1) + \overline{N}(r,1;G \mid\geq l+1) \\ &\quad\quad +S(r,f). \end{aligned} \tag{3.2}$$

Now

$$\begin{aligned} \overline{N}(r,1;F \mid\geq l+1) \quad &\leq \quad \frac{1}{l}N\left(r,\infty;\frac{F}{F'}\right) \leq \frac{1}{l}N\left(r,\infty;\frac{F'}{F}\right) + S(r,f) \\ &\leq \quad \frac{1}{l}\overline{N}(r,0;F) + \frac{1}{l}\overline{N}(r,\infty;F) + S(r,f) \\ &\leq \quad \frac{1}{l}\left[\overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f)\right] \\ &\quad\quad +S(r,f), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \overline{N}(r,1;G \mid\geq l+1) \quad &\leq \quad \frac{1}{l}N\left(r,\infty;\frac{G}{G'}\right) \leq \frac{1}{l}N\left(r,\infty;\frac{G'}{G}\right) + S(r,f) \\ &\leq \quad \frac{1}{l}\overline{N}(r,0;G) + \frac{1}{l}\overline{N}(r,\infty;G) + S(r,f) \\ &\leq \quad \frac{k+1}{l}\left[\overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f)\right] \\ &\quad\quad +S(r,f). \end{aligned} \tag{3.4}$$

From (3.1) - (3.4) we obtain

$$\begin{aligned} N(r,\infty;V) \quad &\leq \quad \overline{N}(r,0;G) + \frac{k+2}{l}\left[\overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f)\right] \\ &\quad\quad +S(r,f) \\ &\leq \quad \frac{kl+\gamma}{l}\overline{N}(r,0;f) + \frac{kl+\gamma}{l}\overline{N}(r,a;f) + \frac{kl+k+2}{l}\overline{N}(r,\infty;f) \\ &\quad\quad +S(r,f), \end{aligned} \tag{3.5}$$

and

$$N(r,\infty;U) \leq \frac{k+2}{l}\overline{N}(r,0;f) + \frac{k+2}{l}\overline{N}(r,a;f) + \frac{\gamma}{l}\overline{N}(r,\infty;f) + S(r,f), \tag{3.6}$$

where $\gamma = k+l+2$. From Lemma 2.7 and (3.5) we obtain

$$\begin{aligned}[m(n+s)-1]\overline{N}(r,\infty;f) &\leq N(r,\infty;V)+S(r,f)\\ &\leq \frac{kl+\gamma}{l}\overline{N}(r,0;f)+\frac{kl+\gamma}{l}\overline{N}(r,a;f)\\ &\quad +\frac{kl+k+2}{l}\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

From this we obtain

$$\begin{aligned}[lm(n+s)-(kl+\gamma)]\overline{N}(r,\infty;f) &\leq (kl+\gamma)[\overline{N}(r,0;f)+\overline{N}(r,a;f)]\\ &\quad +S(r,f). \end{aligned} \tag{3.7}$$

From Lemma 2.8 and (3.6) we obtain

$$\begin{aligned}&[m(n-k)-1]\overline{N}(r,0;f)+[m(s-k)-1]\overline{N}(r,a;f)\\ \leq\ & N(r,\infty;U)+S(r,f)\\ \leq\ & \frac{k+2}{l}\overline{N}(r,0;f)+\frac{k+2}{l}\overline{N}(r,a;f)+\frac{\gamma}{l}\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

From this we obtain

$$\begin{aligned}[lm(n-k)-\gamma]\overline{N}(r,0;f) &\leq \gamma\overline{N}(r,\infty;f)+[\gamma-lm(s-k)]\overline{N}(r,a;f)\\ &\quad +S(r,f).\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\begin{aligned}&[\{lm(n+s)-(kl+\gamma)\}\{lm(n-k)-\gamma\}-\gamma(kl+\gamma)]\overline{N}(r,\infty;f)\\ \leq\ & lm(n-s)(kl+\gamma)\overline{N}(r,a;f)+S(r,f),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&[\{lm(n+s)-(kl+\gamma)\}\{lm(n-k)-\gamma\}-\gamma(kl+\gamma)]\overline{N}(r,0;f)\\ \leq\ & lm[\gamma(n+s)-lm(n+s)(s-k)+(s-k)(kl+\gamma)]\overline{N}(r,a;f)\\ & +S(r,f).\end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10) we get

$$\begin{aligned}&[\{lm(n+s)-(kl+\gamma)\}\{lm(n-k)-\gamma\}-\gamma(kl+\gamma)]\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}\\ &\leq lm[\gamma(n+s)-lm(n+s)(s-k)+(n-k)(kl+\gamma)]\overline{N}(r,a;f)+S(r,f).\end{aligned}$$

From this we obtain

$$\begin{aligned}&[\{lm(n+s)-(kl+\gamma)\}\{lm(n-k)-\gamma\}-\gamma(kl+\gamma)]\\ &\qquad\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,a;f)\}\leq l^2m^2(n^2-s^2)\overline{N}(r,a;f)+S(r,f)\\ &\leq\ S(r,f).\end{aligned}$$

Since by (1.1)

$$\{lm(n+s)-(kl+\gamma)\}\{lm(n-k)-\gamma\}-\gamma(kl+\gamma) > 0$$

we obtain from above

$$\overline{N}(r,0;f) = \overline{N}(r,\infty;f) = \overline{N}(r,a;f) = S(r,f). \tag{3.11}$$

Now we consider the following two cases.

**Case 1.** Let $H \not\equiv 0$. Then by Lemma 2.3 and (3.11) we obtain $T(r,f) = S(r,f)$, a contradiction.

**Case 2.** Let $H \equiv 0$. Then from Lemma 2.9, we obtain the conclusion of the theorem.
This completes the proof of Theorem 1.13.

## 3.2 Proof of Theorem 1.14

Since $f$ is an entire function, $\overline{N}(r,\infty;f) = S(r,f)$. If $U = 0$, we can get the conclusions of the Theorem from Lemma 2.6. If $U \not\equiv 0$, then from Lemma 2.8 and (3.11), we obtain

$$\{lm(n-k)-\gamma\}\overline{N}(r,0;f)+\{lm(s-k)-\gamma\}\overline{N}(r,a;f) \le S(r,f).$$

From this we deduce that

$$\begin{aligned}\{lm(n-k)-\gamma\}[\overline{N}(r,0;f)+\overline{N}(r,a;f)] &\le lm(n-s)\overline{N}(r,a;f)+S(r,f)\\ &\le S(r,f).\end{aligned}$$

Since by (1.2), $lm(n-k)-\gamma > 0$, from above we can conclude that

$$\overline{N}(r,0;f) = \overline{N}(r,a;f) = S(r,f).$$

Now using the same argument as in Cases 1 and 2 of the proof of Theorem 1.13, we obtain the conclusion of Theorem 1.14. This completes the proof of Theorem 1.14.

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