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Fixed Point Theorems for Generalized Contractions with Triangular α-Orbital Admissible Mappings on Branciari Metric Spaces

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Abstract : The fixed point theorems and unique common fixed point theorems for generalized contractions with triangular f-α-admissible mappings on Branciari metric spaces are proven omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecreasing and continuous functions. We prove the unique common fixed point theorems for generalized contractions in the setting of partially ordered Branciari metric spaces using our main result. Moreover, we also present the example that supports our main result.

Keywords : Branciari metric spaces; common fixed points; triangular f-α-admissible mappings; weakly compatible mappings.
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1 Introduction and Preliminaries

One of the most important results in fixed point theory is the Banach contraction principle [1] because of its application in many branches of mathematics

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and mathematical sciences. The Banach contraction principle has been used and extended in many different directions. Recently, Branciari [2] introduced a class of generalized metric spaces by replacing triangular inequality by similar one which involves four or more points instead of three and improved Banach contraction principle. Any metric space is a generalized metric space but the converse is not true, for more details, see [3–14] and the related references contained therein. On the other hand, the common fixed point theorems are generalizations of fixed point theorems. There are many researchers are interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems.

In this paper, we prove the fixed point theorems and unique common fixed point theorems for the generalized contractions appeared in [15] omitting some conditions of $\psi \in \Psi_1$ using the set Ψ_2 introduced by [16]. The unique common fixed point theorem for generalized contractions in the setting of partially ordered Branciari metric spaces is proven using our main result. Moreover, we also present the example that supports our main result.

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ denote the set of all positive integers. We now recall some important definitions, lemmas and theorems.

Definition 1.1. [2] Let X be a nonempty set. We say that a mapping $d : X\times X \to [0,\infty)$ is a Branciari metric if for all $x, y \in X$ and for all distinct points $u, v \in X$ where each of them different from x and y, we have

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

If d is a Branciari metric, then (X, d) is called a Branciari metric space (or for short BMS). By the definition we see that a Branciari metric space is a generalization of a metric space.

Definition 1.2. Let (X, d) be a BMS, $\{x_n\}$ be a sequence in X, and $x \in X$. Then

(i) We say that $\{x_n\}$ is convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$ and denoted by $x_n \to x$ as $n \to \infty$.

(ii) We say that $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) We say that (X, d) is a complete BMS if and only if every Cauchy sequence in X converges to some element in X.

Lemma 1.3. [17] *Let (X, d) be a BMS, and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ converges to at most one point.*

In 2014, Rosa and Vetro [18] introduced the notion of f-α-admissible mappings as the following:

Definition 1.4. Let $T, f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. The mapping T is said to be an f-α-admissible mapping if for all $x, y \in X$,

$$\alpha(fx, fy) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1$$

If f is an identity mapping, then T is called to be an α-admissible mapping.

In 2014, Popescu [19] introduced the notion of triangular α-orbital admissible mappings.

Definition 1.5. [19] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is α-orbital admissible if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1.$$

Definition 1.6. [19] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is triangular α-orbital admissible if:

(i) T is α-orbital admissible;

(ii) for all $x, y \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply that $\alpha(x, Ty) \geq 1$.

Lemma 1.7. [19] *Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Suppose that T is a triangular α-orbital admissible mapping and assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Denote by Ψ_1 the set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(1) ψ is nondecreasing;

(2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \psi(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0;$$

(3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0} \frac{\psi(t)-1}{t^r} = l$.

Jleli et al. [15] established the following theorem by adding the continuity to a function $\psi \in \Psi_1$ on Branciari metric spaces.

Theorem 1.8. [15] *Let (X, d) be a complete BMS and $T : X \to X$. Suppose that there exist $\psi \in \Psi_1$ that is continuous and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \textit{ implies } \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T has a fixed point z in X and $\{T^n x_1\}$ converges to z.

Arshad et al. [20] extended the results proved by Jleli et al. [21] and [15] by using the concept of triangular α-orbital admissible mappings obtained in [19] by adding the continuity to a function $\psi \in \Psi_1$.

Theorem 1.9. [20] *Let (X,d) be a complete BMS, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_1$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \textit{ implies } \alpha(x,y) \cdot \psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$ and $\alpha(x_1,T^2x_1) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq 1$ for all $k \in \mathbb{N}$;

(v) ψ is continuous.

Then T has a fixed point z in X and $\{T^n x_1\}$ converges to z.

Li and Jiang [16] introduced Ψ_2 the set of all functions $\psi : (0,\infty) \to (1,\infty)$ which is nondecreasing and continuous. They also gave some examples illustrating the relationship between Ψ_1 and Ψ_2 as follows:

Example 1.10. [16] Let $f(t) = e^{te^t}$ for $t \geq 0$. Then $f \in \Psi_2$ but $f \notin \Psi_1$ since $\lim_{t\to 0} \frac{e^{te^t}-1}{t^r} = 0$ for each $r \in (0,1)$.

Example 1.11. [16] Let $g(t) = e^{t^a}$ for $t \geq 0$, where $a > 0$. If $a \in (0,1)$, then $g \in \Psi_1 \cap \Psi_2$. If $a = 1$, then $g \in \Psi_2$ but $g \notin \Psi_1$ since $\lim_{t\to 0} \frac{e^t-1}{t^r} = 0$ for each $r \in (0,1)$. If $a > 1$, then $g \in \Psi_2$ but $g \notin \Psi_1$ since $\lim_{t\to 0} \frac{e^{t^a}-1}{t^r} = 0$ for each $r \in (0,1)$.

Remark 1.12. From Example 1.10 and Example 1.11, we can conclude that $\Psi_2 \not\subset \Psi_1$ and $\Psi_1 \cap \Psi_2 \neq \varnothing$. Moreover, it is clear that if $\psi \in \Psi_1$ and ψ is continuous, then $\psi \in \Psi_2$.

Definition 1.13. Let $T, f : X \to X$. If $\omega = Tx = fx$ for some $x \in X$, then x is called a coincidence point of T and f, and ω is called a point of coincidence of T and f.

Definition 1.14. Let $T, f : X \to X$. The pair $\{T, f\}$ is said to be weakly compatible if $Tfx = fTx$ whenever $Tx = fx$ for some $x \in X$.

Proposition 1.15. [22] *Let $T, f : X \to X$ and $\{T, f\}$ is weakly compatible. If T and f have a unique point of coincidence $\omega = Tx = fx$, then ω is the unique common fixed point of T and f.*

2 Main Results

We now prove the existence of fixed point theorems for triangular α-orbital admissible mappings omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecreasing and continuous functions on $(0, \infty)$ to $(1, \infty)$.

Theorem 2.1. *Let (X, d) be a complete BMS, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Suppose that the following conditions hold:*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y) \cdot \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \tag{2.1}$$

where

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) T is continuous.

Then T has a fixed point.

Proof. Let $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define the iterative sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n, \text{ for all } n \in \mathbb{N}.$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T. We now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By condition (ii), we have $\alpha(x_1, Tx_1) \geq 1$. Using Lemma 1.7, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

From (2.1) and (2.2), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\psi(d(Tx_{n-1}, Tx_n)) \\ &\leq [\psi(R(x_{n-1}, x_n))]^{\lambda}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} R(x_{n-1}, x_n) &= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\Big\} \\ &= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $R(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by (2.3) we obtain that

$$\psi(d(x_n, x_{n+1})) \leq [\psi(d(x_n, x_{n+1}))]^{\lambda} < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $R(x_{n-1},x_n)=d(x_{n-1},x_n)$. Using (2.3), we have

$$\psi(d(x_n,x_{n+1}))\le[\psi(d(x_{n-1},x_n))]^{\lambda}<\psi(d(x_{n-1},x_n)).$$

Since ψ is nondecreasing, we have $d(x_n,x_{n+1})<d(x_{n-1},x_n)$. Hence the sequence $\{d(x_n,x_{n+1})\}$ is decreasing. Hence $\{d(x_n,x_{n+1})\}$ converges to a nonnegative real number. Thus there exists $r\ge 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1})=r$ and

$$d(x_n,x_{n+1})\ge r. \tag{2.4}$$

We will prove that $r=0$. Suppose that $r>0$. Since ψ is nondecreasing and by using (2.3) and (2.4), we obtain that

$$1<\psi(r)\le\psi(d(x_n,x_{n+1}))\le[\psi(d(x_{n-1},x_n))]^{\lambda}\le\cdots\le[\psi(d(x_0,x_1))]^{\lambda^n}, \tag{2.5}$$

for all $n\in\mathbb{N}$. Letting $n\to\infty$ in this inequality, we get that $\psi(r)=1$, which contradicts to the assumption that $\psi(t)>1$ for each $t>0$. Consequently, we have $r=0$ and therefore

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{2.6}$$

Suppose that there exist $n,p\in\mathbb{N}$ such that $x_n=x_{n+p}$. We prove that $p=1$. Assume that $p>1$. Using (2.1) and (2.2), we obtain that

$$\begin{aligned}\psi(d(x_n,x_{n+1}))&=\psi(d(x_{n+p},x_{n+p+1}))\\&=\psi(d(Tx_{n+p-1},Tx_{n+p}))\\&\le\alpha(x_{n+p-1},x_{n+p})\psi(d(Tx_{n+p-1},Tx_{n+p}))\\&\le[\psi(R(x_{n+p-1},x_{n+p}))]^{\lambda},\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}R(x_{n+p-1},x_{n+p})&=\max\Big\{d(x_{n+p-1},x_{n+p}),d(x_{n+p-1},Tx_{n+p-1}),d(x_{n+p},Tx_{n+p})\Big\}\\&=\max\Big\{d(x_{n+p-1},x_{n+p}),d(x_{n+p-1},x_{n+p}),d(x_{n+p},x_{n+p+1})\Big\}\\&=\max\Big\{d(x_{n+p-1},x_{n+p}),d(x_{n+p},x_{n+p+1})\Big\}.\end{aligned}$$

If $R(x_{n+p},x_{n+p+1})=d(x_{n+p},x_{n+p+1})$, then from (2.7) we obtain that

$$\begin{aligned}\psi(d(x_n,x_{n+1}))&=\psi(d(x_{n+p},x_{n+p+1}))\\&\le[\psi(d(x_{n+p},x_{n+p+1}))]^{\lambda}\\&<\psi(d(x_{n+p},x_{n+p+1})),\end{aligned}$$

which is a contradiction. Hence $R(x_{n+p},x_{n+p+1})=d(x_{n+p-1},x_{n+p})$. By (2.7), we obtain that

$$\begin{aligned}\psi(d(x_n,x_{n+1}))&=\psi(d(x_{n+p},x_{n+p+1}))\\&\le[\psi(d(x_{n+p-1},x_{n+p}))]^{\lambda}\\&<\psi(d(x_{n+p-1},x_{n+p})).\end{aligned}$$

Since ψ is nondecreasing, we have $d(x_n, x_{n+1}) < d(x_{n+p-1}, x_{n+p})$. By using (2.1), we get that

$$\begin{aligned}\psi(d(x_{n+p-1}, x_{n+p})) &\leq \alpha(x_{n+p-2}, x_{n+p-1})\psi(d(Tx_{n+p-2}, Tx_{n+p-1}))\\ &\leq [\psi(R(x_{n+p-2}, x_{n+p-1}))]^{\lambda}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}&R(x_{n+p-2}, x_{n+p-1})\\ &\quad= \max\Big\{d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-2}, Tx_{n+p-2}), d(x_{n+p-1}, Tx_{n+p-1})\Big\}\\ &\quad= \max\Big\{d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-1}, x_{n+p})\Big\}\\ &\quad= \max\Big\{d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-1}, x_{n+p})\Big\}.\end{aligned}$$

If $R(x_{n+p-2}, x_{n+p-1}) = d(x_{n+p-1}, x_{n+p})$, then by (2.8) we obtain that

$$\psi(d(x_{n+p-1}, x_{n+p})) \leq [\psi(d(x_{n+p-1}, x_{n+p}))]^{\lambda} < \psi(d(x_{n+p-1}, x_{n+p})),$$

which is a contradiction. Hence $R(x_{n+p-2}, x_{n+p-1}) = d(x_{n+p-2}, x_{n+p-1})$. By (2.8), we have

$$\psi(d(x_{n+p-1}, x_{n+p})) \leq [\psi(d(x_{n+p-2}, x_{n+p-1}))]^{\lambda} < \psi(d(x_{n+p-2}, x_{n+p-1})).$$

Since ψ is nondecreasing, we have $d(x_{n+p-1}, x_{n+p}) < d(x_{n+p-2}, x_{n+p-1})$. By continuing this process, we obtain the following inequality

$$d(x_n, x_{n+1}) < d(x_{n+p-1}, x_{n+p}) < d(x_{n+p-2}, x_{n+p-1}) < \ldots < d(x_n, x_{n+1}),$$

which is a contradiction and hence $p = 1$. We deduce that T has a fixed point. We can assume that $x_n \neq x_m$ for $n \neq m$. We now prove that $\{d(x_n, x_{n+2})\}$ is bounded. Since $\{d(x_n, x_{n+1})\}$ is bounded, there exists $M > 0$ such that

$$d(x_n, x_{n+1}) \leq M \text{ for all } n \in \mathbb{N}.$$

If $d(x_n, x_{n+2}) > M$ for all $n \in \mathbb{N}$, then from

$$\begin{aligned}R(x_{n-1}, x_{n+1}) &= \max\Big\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\Big\}\\ &= \max\Big\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\Big\}\\ &= d(x_{n-1}, x_{n+1}),\end{aligned}$$

and Lemma 1.7, we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+2})) &= \psi(d(Tx_{n-1}, Tx_{n+1}))\\ &\leq \alpha(x_{n-1}, x_{n+1})\psi(d(Tx_{n-1}, Tx_{n+1}))\\ &\leq [\psi(R(x_{n-1}, x_{n+1}))]^{\lambda}\\ &= [\psi(d(x_{n-1}, x_{n+1}))]^{\lambda}\\ &< \psi(d(x_{n-1}, x_{n+1})).\end{aligned}$$

This implies that $\{d(x_n, x_{n+2})\}$ is decreasing. Therefore $\{d(x_n, x_{n+2})\}$ is bounded. If $d(x_n, x_{n+2}) \leq M$ for some $n \in \mathbb{N}$, then from

$$\begin{aligned} R(x_n, x_{n+2}) &= \max\Big\{d(x_n, x_{n+2}), d(x_n, Tx_n), d(x_{n+2}, Tx_{n+2})\Big\} \\ &= \max\Big\{d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3})\Big\} \end{aligned}$$

and Lemma 1.7, we obtain that

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+3})) &= \psi(d(Tx_n, Tx_{n+2})) \\ &\leq \alpha(x_n, x_{n+2})\psi(d(Tx_n, Tx_{n+2})) \\ &\leq [\psi(R(x_n, x_{n+2}))]^{\lambda} \\ &\leq [\psi(M)]^{\lambda} \\ &< \psi(M). \end{aligned}$$

Therefore $d(x_{n+1}, x_{n+3}) < M$. This implies that $\{d(x_n, x_{n+2})\}$ is bounded. We next prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Suppose that $\lim_{n\to\infty} d(x_n, x_{n+2}) \neq 0$. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_k+2}) = a \text{ for some } a > 0.$$

Using (2.1) and Lemma 1.7, we have

$$\begin{aligned} \psi(d(x_{n_k}, x_{n_k+2})) &= \psi(d(Tx_{n_k-1}, Tx_{n_k+1})) \\ &\leq \alpha(x_{n_k-1}, x_{n_k+1})\psi(d(Tx_{n_k-1}, Tx_{n_k+1})) \\ &\leq [\psi(R(x_{n_k-1}, x_{n_k+1}))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} R(x_{n_k-1}, x_{n_k+1}) &= \max\Big\{d(x_{n_k-1}, x_{n_k+1}), d(x_{n_k-1}, Tx_{n_k-1}), d(x_{n_k+1}, Tx_{n_k+1})\Big\} \\ &= \max\Big\{d(x_{n_k-1}, x_{n_k+1}), d(x_{n_k-1}, x_{n_k}), d(x_{n_k+1}, x_{n_k+2})\Big\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain that

$$\psi(a) = \lim_{k\to\infty} \psi(d(x_{n_k}, x_{n_k+2})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k-1}, x_{n_k+1}))]^{\lambda} = [\psi(a)]^{\lambda} < \psi(a),$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.9}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{2.10}$$

This implies that

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.11}$$

By applying the rectangular inequality and using (2.10) and (2.11), we obtain that

$$\begin{aligned}\varepsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k}) \\ &< \varepsilon + d(x_{n_k-1}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (2.6) and (2.9), we get that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{2.12}$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned}R(x_{n_k}, x_{m_k}) &= \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, Tx_{n_k}), (x_{m_k}, Tx_{m_k})\Big\} \\ &= \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), (x_{m_k}, x_{m_k+1})\Big\}.\end{aligned}$$

By using (2.6) and (2.12), we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x_{m_k}) = \varepsilon. \tag{2.13}$$

By (2.12) and (2.13), there exists a positive integer k_0 such that

$$d(x_{n_k+1}, x_{m_k+1}) > 0 \quad \text{and} \quad R(x_{n_k}, x_{m_k}) > 0, \quad \text{for all} \quad k \geq k_0.$$

By Lemma 1.7 and using (2.1), we get that

$$\begin{aligned}\psi(d(x_{n_k+1}, x_{m_k+1})) &= \psi(d(Tx_{n_k}, Tx_{m_k})) \\ &= \psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq \alpha(x_{m_k}, x_{n_k})\psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq [\psi(R(x_{m_k}, x_{n_k}))]^{\lambda} \\ &= [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda},\end{aligned}$$

for all $n_k > m_k > k \geq k_0$. Letting $k \to \infty$ in this inequality, by (2.12), (2.13) and the continuity of ψ, we obtain that

$$\psi(\varepsilon) = \lim_{k\to\infty} \psi(d(x_{n_k+1}, x_{m_k+1})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda} = [\psi(\varepsilon)]^{\lambda} < \psi(\varepsilon),$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete BMS, it follows that $\{x_n\}$ converges to $x \in X$. Since T is continuous, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx.$$

Therefore x is a fixed point of T. $\square$

We now replace the continuity of T in Theorem 2.1 by some appropriate conditions to obtain the following theorem.

Theorem 2.2. *Let (X,d) be a complete BMS, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y) \cdot \psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda}, \tag{2.14}$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point.

Proof. As in the proof of Theorem 2.1, we can construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \text{ for all } n \in \mathbb{N},$$

$\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$. We can suppose that $x_{n_k} \neq Tx$. Applying inequality (2.14), we obtain that

$$\begin{aligned}\psi(d(Tx_{n_k}, Tx)) &\leq \alpha(x_{n_k}, x)\psi(d(Tx_{n_k}, Tx)) \\ &\leq [\psi(R(x_{n_k}, x))]^{\lambda},\end{aligned}$$

where

$$\begin{aligned}R(x_{n_k}, x) &= \max\Big\{d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}), d(x, Tx)\Big\} \\ &= \max\Big\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx)\Big\}.\end{aligned}$$

Taking the limit as $k \to \infty$ and since ψ is continuous, we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x) = d(x, Tx).$$

We will prove that $x = Tx$. Suppose that $x \neq Tx$. Therefore

$$d(x,Tx) \leq d(x, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, Tx).$$

It follows that

$$d(x,Tx) \leq \lim_{k\to\infty} d(x_{n_k}, Tx).$$

Since ψ is continuous and nondecreasing, we obtain that

$$\psi(d(x,Tx)) \leq \lim_{k\to\infty} \psi(d(x_{n_k},Tx)) \leq [\psi(d(x,Tx))]^{\lambda} < \psi(d(x,Tx)),$$

which is a contradiction. Thus $x = Tx$ and hence x is a fixed point of T. $\square$

We now present the example for supporting our main result.

Example 2.3. Let $X = \{0,1,2,3\}$. Define $d : X \times X \to [0,\infty)$ as follows:

$$\begin{aligned}
&d(x,x) = 0 \text{ for all } x \in X,\\
&d(0,2) = d(2,0) = d(0,3) = d(3,0) = d(2,3) = d(3,2) = 2,\\
&d(0,1) = d(1,0) = d(1,2) = d(2,1) = 4,\\
&d(1,3) = d(3,1) = 1, \text{ and}\\
&d(x,y) = |x-y|, \text{ otherwise.}
\end{aligned}$$

Therefore (X,d) is complete BMS but (X,d) is not a metric space because it lacks the triangular property as the following:

$$d(1,2) = 4 > 1 + 2 = d(1,3) + d(3,2).$$

Let $T : X \to X$ be the mapping defined by

$$Tx = \begin{cases} 1 \text{ if } x \neq 2 \\ 3 \text{ if } x = 2. \end{cases}$$

Let $\alpha : X \times X \to [0,\infty)$ be given by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in X \backslash \{2\} \\ \frac{3}{5} & \text{otherwise.} \end{cases}$$

Define a function $\psi : (0,\infty) \to (1,\infty)$ by $\psi(t) = e^t$. By Example 1.11, we obtain that $\psi \in \Psi_2$ but $\psi \notin \Psi_1$. We next illustrate that all conditions in Theorem 2.2 hold. Taking $x_1 = 1$, we have $\alpha(1,T1) = \alpha(1,1) = 1 \geq 1$. We next prove that T is α-orbital admissible. Let $x \in X$ such that $\alpha(x,Tx) \geq 1$. Therefore $x, Tx \in X\backslash\{2\}$ and then $x \in \{0,1,3\}$. By the definition of α, we obtain that

$$\begin{aligned}
\alpha(T0,T^2 0) &= \alpha(1,1) \geq 1,\\
\alpha(T1,T^2 1) &= \alpha(1,1) \geq 1,\\
\alpha(T3,T^2 3) &= \alpha(1,1) \geq 1.
\end{aligned}$$

It follows that T is α-orbital admissible. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$. By the definition of α, we have $x, y, Ty \in X\backslash\{2\}$. This yields

$$\alpha(0,1) \geq 1 \text{ and } \alpha(1,T1) \geq 1 \text{ imply } \alpha(0,T1) \geq 1,$$

$$\alpha(0,3) \geq 1 \text{ and } \alpha(3,T3) \geq 1 \text{ imply } \alpha(0,T3) \geq 1,$$
$$\alpha(1,3) \geq 1 \text{ and } \alpha(3,T3) \geq 1 \text{ imply } \alpha(1,T3) \geq 1,$$
$$\alpha(1,0) \geq 1 \text{ and } \alpha(0,T0) \geq 1 \text{ imply } \alpha(1,T0) \geq 1,$$
$$\alpha(3,0) \geq 1 \text{ and } \alpha(0,T0) \geq 1 \text{ imply } \alpha(3,T0) \geq 1,$$
$$\alpha(3,1) \geq 1 \text{ and } \alpha(1,T1) \geq 1 \text{ imply } \alpha(3,T1) \geq 1.$$

This implies that T is triangular α-orbital admissible. Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. By the definition of α, for each $n \in \mathbb{N}$, we get that $x_n \in X \backslash \{2\} = \{0,1,3\}$. We obtain that $x \in \{0,1,3\}$. Thus we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$. We next prove that (2.14) holds. Let $x, y \in X$ be such that $d(Tx, Ty) \neq 0$. So we consider the following cases:

- $x = 2$ and $y \in \{0,1,3\}$ or
- $y = 2$ and $x \in \{0,1,3\}$.

We divide the proof into three cases as follows:
(1) If $(x,y) \in \{(0,2),(2,0)\}$, then

$$R(0,2) = \max\Big\{d(0,2), d(0,1), d(2,3)\Big\} = \max\Big\{2,4,2\Big\} = 4.$$

This implies that

$$\psi(d(T0,T2)) = \psi(d(1,3)) = \psi(1) = e^1 \leq [e^4]^{0.3} = [\psi(4)]^{0.3} \leq [\psi(R(0,2))]^{0.3}.$$

Therefore

$$\alpha(0,2)\psi(d(T0,T2)) = \frac{3}{5}\psi(d(T0,T2)) \leq \psi(d(T0,T2)) \leq [\psi(R(0,2))]^{0.3}.$$

Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that

$$\alpha(2,0)\psi(d(T2,T0)) \leq [\psi(R(2,0))]^{0.3}.$$

(2) If $(x,y) \in \{(2,1),(1,2)\}$, then

$$R(2,1) = \max\Big\{d(2,1), d(2,3), d(1,1)\Big\} = \max\Big\{1,2,0\Big\} = 2.$$

This implies that

$$\psi(d(T2,T1)) = \psi(d(3,1)) = \psi(1) = e^1 \leq [e^2]^{0.7} = [\psi(2)]^{0.7} \leq [\psi(R(2,1))]^{0.7}.$$

Therefore

$$\alpha(2,1)\psi(d(T2,T1)) = \frac{3}{5}\psi(d(T2,T1)) \leq \psi(d(T2,T1)) \leq [\psi(R(2,1))]^{0.7}.$$

Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that

$$\alpha(1,2)\psi(d(T1,T2)) \leq [\psi(R(1,2))]^{0.7}.$$

(3) If $(x, y) \in \{(2,3), (3,2)\}$, then

$$R(2,3) = \max\Big\{d(2,3), d(2,3), d(3,1)\Big\} = \max\Big\{2, 2, 1\Big\} = 2.$$

This implies that

$$\psi(d(T2, T3)) = \psi(d(3,1)) = \psi(1) = e^1 \leq [e^2]^{0.7} = [\psi(2)]^{0.7} \leq [\psi(R(2,3))]^{0.7}.$$

Therefore

$$\alpha(2,3)\psi(d(T2, T3)) = \frac{3}{5}\psi(d(T2, T3)) \leq \psi(d(T2, T3)) \leq [\psi(R(2,3))]^{0.7}.$$

Since $d(x, y) = d(y, x)$ for all $x, y \in X$, we also obtain that

$$\alpha(3,2)\psi(d(T3, T2)) \leq [\psi(R(3,2))]^{0.7}.$$

It follows that if $x, y \in X$ and $d(Tx, Ty) \neq 0$, then

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}.$$

Hence all assumptions in Theorem 2.2 are satisfied and thus T has a fixed point which is $x = 1$.

We now introduce the notion of triangular f-α-admissible mappings and prove a key lemma that will be used for proving our results.

Definition 2.4. Let $T, f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be a triangular f-α-admissible mapping if

(i) T is an f-α-admissible mapping;

(ii) for all $x, y \in X, \alpha(fx, fy) \geq 1$ and $\alpha(fy, Ty) \geq 1$ imply $\alpha(fx, Ty) \geq 1$.

Lemma 2.5. *Let $T, f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ is a triangular f-α-admissible mapping and assume that there exists $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$. Define a sequence $\{fx_n\}$ by $fx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(fx_n, fx_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since T is a triangular f-α-admissible mapping and $\alpha(fx_1, Tx_1) \geq 1$, we have $\alpha(fx_2, fx_3) = \alpha(Tx_1, Tx_2) \geq 1$. By continuing this process, we obtain that

$$\alpha(fx_n, fx_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Suppose that $\alpha(fx_n, fx_m) \geq 1$. We will prove that $\alpha(fx_n, fx_{m+1}) \geq 1$ where $n < m$. Since T is triangular f-α-admissible and $\alpha(fx_m, fx_{m+1}) \geq 1$, we obtain that $\alpha(fx_n, fx_{m+1}) \geq 1$. Hence $\alpha(fx_n, fx_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$. $\square$

Theorem 2.6. *Let (X, d) be a BMS and $T, f : X \to X$ be such that $TX \subseteq fX$ where one of these two subsets of X being complete. Assume that $\alpha : X \times X \to [0, \infty)$ and suppose that the following conditions hold:*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \textit{ implies } \alpha(fx,fy)\cdot\psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda}, \quad (2.15)$$

where

$$R(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$;

(iii) T is a triangular f-α-admissible mapping;

(iv) T is continuous with respect to f;

(v) either $\alpha(fu,fv) \geq 1$ or $\alpha(fv,fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$.

Then T and f have a unique point of coincidence. Moreover, if the pair $\{T, f\}$ is weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$. Define the iterative sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \text{ for all } n \in \mathbb{N}.$$

Moreover, we assume that if $Tx_n = y_n = y_m = Tx_m$ for some $n \neq m$, then we choose $x_{n+1} = x_{m+1}$, this can be done since $TX \subseteq fX$. It follows that $y_{n+1} = y_{m+1}$. If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then y_{n_0+1} is a point of coincidence of T and f. Consequently, we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. By condition (ii), we have $\alpha(fx_1, Tx_1) \geq 1$. Using Lemma 2.5, we obtain that

$$\alpha(fx_n, fx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \quad (2.16)$$

From (2.15) and (2.16), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \alpha(fx_n, fx_{n+1})\psi(d(Tx_n, Tx_{n+1})) \\ &\leq [\psi(R(x_n, x_{n+1}))]^{\lambda}, \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} R(x_n, x_{n+1}) &= \max\left\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\right\} \\ &= \max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\right\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

If $R(x_n, x_{n+1}) = d(y_n, y_{n+1})$, then by (2.17) we obtain that

$$\psi(d(y_n, y_{n+1})) \leq [\psi(d(y_n, y_{n+1}))]^{\lambda} < \psi(d(y_n, y_{n+1})),$$

which is a contradiction. Hence $R(x_n, x_{n+1}) = d(y_{n-1}, y_n)$. Using (2.17), we have

$$\psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^{\lambda} < \psi(d(y_{n-1}, y_n)).$$

Since ψ is nondecreasing, we have $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$. Hence the sequence $\{d(y_n, y_{n+1})\}$ is decreasing. Hence $\{d(y_n, y_{n+1})\}$ converges to a nonnegative real number. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$ and

$$d(y_n, y_{n+1}) \geq r \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

We will prove that $r = 0$. Suppose that $r > 0$. Since ψ is nondecreasing and by using (2.17) and (2.18), we obtain that

$$1 < \psi(r) \leq \psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^{\lambda} \leq \cdots \leq [\psi(d(y_0, y_1))]^{\lambda^n}, \tag{2.19}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in this inequality, we get that $\psi(r) = 1$ which contradicts to the assumption that $\psi(t) > 1$ for each $t > 0$. Consequently, we have $r = 0$ and therefore

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.20}$$

Suppose that there exist $n, p \in \mathbb{N}$ such that $y_n = y_{n+p}$. We prove that $p = 1$. Assume that $p > 1$. By using (2.15), we obtain that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(y_{n+p}, y_{n+p+1})) \\ &= \psi(d(Tx_{n+p}, Tx_{n+p+1})) \\ &\leq \alpha(fx_{n+p}, fx_{n+p+1})\psi(d(Tx_{n+p}, Tx_{n+p+1})) \\ &\leq [\psi(R(x_{n+p}, x_{n+p+1}))]^{\lambda}, \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} &R(x_{n+p}, x_{n+p+1}) \\ &\quad = \max\Big\{d(fx_{n+p}, fx_{n+p+1}), d(fx_{n+p}, Tx_{n+p}), d(fx_{n+p+1}, Tx_{n+p+1})\Big\} \\ &\quad = \max\Big\{d(y_{n+p-1}, y_{n+p}), d(y_{n+p-1}, y_{n+p}), d(y_{n+p}, y_{n+p+1})\Big\} \\ &\quad = \max\Big\{d(y_{n+p-1}, y_{n+p}), d(y_{n+p}, y_{n+p+1})\Big\}. \end{aligned}$$

If $R(x_{n+p}, x_{n+p+1}) = d(y_{n+p}, y_{n+p+1})$, then from (2.21) we obtain that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(y_{n+p}, y_{n+p+1})) \\ &\leq [\psi(d(y_{n+p}, y_{n+p+1}))]^{\lambda} \\ &< \psi(d(y_{n+p}, y_{n+p+1})), \end{aligned}$$

which is a contradiction. Hence $R(x_{n+p}, x_{n+p+1}) = d(y_{n+p-1}, y_{n+p})$. By (2.21), we obtain that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(y_{n+p}, y_{n+p+1})) \\ &\leq [\psi(d(y_{n+p-1}, y_{n+p}))]^{\lambda} \\ &< \psi(d(y_{n+p-1}, y_{n+p})). \end{aligned}$$

Since ψ is nondecreasing, we have $d(y_n, y_{n+1}) < d(y_{n+p-1}, y_{n+p})$. Next, by using (2.15), we get that

$$\begin{aligned}\psi(d(y_{n+p-1}, y_{n+p})) &\leq \alpha(fx_{n+p-1}, fx_{n+p})\psi(d(Tx_{n+p-1}, Tx_{n+p}))\\ &\leq [\psi(R(x_{n+p-1}, x_{n+p}))]^{\lambda},\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}&R(x_{n+p-1}, x_{n+p})\\ &\quad= \max\Big\{d(fx_{n+p-1}, fx_{n+p}), d(fx_{n+p-1}, Tx_{n+p-1}), d(fx_{n+p}, Tx_{n+p})\Big\}\\ &\quad= \max\Big\{d(y_{n+p-2}, y_{n+p-1}), d(y_{n+p-2}, y_{n+p-1}), d(y_{n+p-1}, y_{n+p})\Big\}\\ &\quad= \max\Big\{d(y_{n+p-2}, y_{n+p-1}), d(y_{n+p-1}, y_{n+p})\Big\}.\end{aligned}$$

If $R(x_{n+p-1}, x_{n+p}) = d(y_{n+p-1}, y_{n+p})$, then by (2.22) we obtain that

$$\psi(d(y_{n+p-1}, y_{n+p})) \leq [\psi(d(y_{n+p-1}, y_{n+p}))]^{\lambda} < \psi(d(y_{n+p-1}, y_{n+p})),$$

which is a contradiction. Hence $R(x_{n+p-1}, x_{n+p}) = d(y_{n+p-2}, y_{n+p-1})$. By (2.22), we have

$$\psi(d(y_{n+p-1}, y_{n+p})) \leq [\psi(d(y_{n+p-2}, y_{n+p-1}))]^{\lambda} < \psi(d(y_{n+p-2}, y_{n+p-1})).$$

Since ψ is nondecreasing, we have $d(y_{n+p-1}, y_{n+p}) < d(y_{n+p-2}, y_{n+p-1})$. By continuing this process, we obtain the following inequality

$$d(y_n, y_{n+1}) < d(y_{n+p-1}, y_{n+p}) < d(y_{n+p-2}, y_{n+p-1}) < \ldots < d(y_n, y_{n+1}),$$

which is a contradiction and hence $p = 1$. We deduce that T and f have a point of coincidence. We can assume that $y_n \neq y_m$ for $n \neq m$. We now prove that $\{d(y_n, y_{n+2})\}$ is bounded. Since $\{d(y_n, y_{n+1})\}$ is bounded, there exists $M > 0$ such that

$$d(y_n, y_{n+1}) \leq M \text{ for all } n \in \mathbb{N}.$$

If $d(y_n, y_{n+2}) > M$ for all $n \in \mathbb{N}$, then from

$$\begin{aligned}R(x_n, x_{n+2}) &= \max\Big\{d(fx_n, fx_{n+2}), d(fx_n, Tx_n), d(fx_{n+2}, Tx_{n+2})\Big\}\\ &= \max\Big\{d(y_{n-1}, y_{n+1}), d(y_{n-1}, y_n), d(y_{n+1}, y_{n+2})\Big\}\\ &= d(y_{n-1}, y_{n+1}),\end{aligned}$$

and Lemma 2.5, we obtain that

$$\begin{aligned}\psi(d(y_n, y_{n+2})) &= \psi(d(Tx_n, Tx_{n+2}))\\ &\leq \alpha(fx_n, fx_{n+2})\psi(d(Tx_n, Tx_{n+2}))\\ &\leq [\psi(R(x_n, x_{n+2}))]^{\lambda}\\ &= [\psi(d(y_{n-1}, y_{n+1}))]^{\lambda}\\ &< \psi(d(y_{n-1}, y_{n+1})).\end{aligned}$$

This implies that $\{d(y_n, y_{n+2})\}$ is decreasing. Therefore $\{d(y_n, y_{n+2})\}$ is bounded. If $d(y_n, y_{n+2}) \leq M$ for some $n \in \mathbb{N}$, then from

$$\begin{aligned} R(x_{n+1}, x_{n+3}) &= \max\Big\{d(fx_{n+1}, fx_{n+3}), d(fx_{n+1}, Tx_{n+1}), d(fx_{n+3}, Tx_{n+3})\Big\} \\ &= \max\Big\{d(y_n, y_{n+2}), d(y_n, y_{n+1}), d(y_{n+2}, y_{n+3})\Big\}, \end{aligned}$$

and Lemma 2.5, we obtain that

$$\begin{aligned} \psi(d(y_{n+1}, y_{n+3})) &= \psi(d(Tx_{n+1}, Tx_{n+3})) \\ &\leq \alpha(fx_{n+1}, fx_{n+3})\psi(d(Tx_{n+1}, Tx_{n+3})) \\ &\leq [\psi(R(x_{n+1}, x_{n+3}))]^{\lambda} \\ &\leq [\psi(M)]^{\lambda} \\ &< \psi(M). \end{aligned}$$

It follows that $d(y_{n+1}, y_{n+3}) < M$. This implies that $\{d(y_n, y_{n+2})\}$ is bounded. We next prove that $\lim_{n\to\infty} d(y_n, y_{n+2}) = 0$. Suppose that $\lim_{n\to\infty} d(y_n, y_{n+2}) \neq 0$. So there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k\to\infty} d(y_{n_k}, y_{n_k+2}) = a \text{ for some } a > 0.$$

Using (2.1), we have

$$\begin{aligned} \psi(d(y_{n_k}, y_{n_k+2})) &= \psi(d(Tx_{n_k}, Tx_{n_k+2})) \\ &\leq \alpha(fx_{n_k}, fx_{n_k+2})\psi(d(Tx_{n_k}, Tx_{n_k+2})) \\ &\leq [\psi(R(x_{n_k}, x_{n_k+2}))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} R(x_{n_k}, x_{n_k+2}) &= \max\Big\{d(fx_{n_k}, fx_{n_k+2}), d(fx_{n_k}, Tx_{n_k}), d(fx_{n_k+2}, Tx_{n_k+2})\Big\} \\ &= \max\Big\{d(y_{n_k-1}, y_{n_k+1}), d(y_{n_k-1}, y_{n_k}), d(y_{n_k+1}, y_{n_k+2})\Big\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain that

$$\psi(a) = \lim_{k\to\infty} \psi(d(y_{n_k}, y_{n_k+2})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k}, x_{n_k+2}))]^{\lambda} = [\psi(a)]^{\lambda} < \psi(a),$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} d(y_n, y_{n+2}) = 0. \tag{2.23}$$

We now prove that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of $\{y_n\}$ such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon. \tag{2.24}$$

This implies that

$$d(y_{m_k}, y_{n_k-1}) < \varepsilon. \tag{2.25}$$

By applying the rectangular inequality and using (2.24) and (2.25), we obtain that

$$\begin{aligned}\varepsilon &\leq d(y_{m_k}, y_{n_k})\\ &\leq d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k})\\ &< \varepsilon + d(y_{n_k-1}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.20), we get that

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \varepsilon. \tag{2.26}$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned}R(x_{n_k}, x_{m_k}) &= \max\Big\{d(fx_{n_k}, fx_{m_k}), d(fx_{n_k}, Tx_{n_k}), (fx_{m_k}, Tx_{m_k})\Big\}\\ &= \max\Big\{d(y_{n_k-1}, y_{m_k-1}), d(y_{n_k-1}, y_{n_k}), (y_{m_k-1}, y_{m_k})\Big\}.\end{aligned}$$

By using (2.20) and (2.26), we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x_{m_k}) = \varepsilon. \tag{2.27}$$

By (2.26) and (2.27), there exists a positive integer k_0 such that

$$d(y_{n_k}, y_{m_k}) > 0 \quad \text{and} \quad R(x_{n_k}, x_{m_k}) > 0, \quad \text{for all} \quad k \geq k_0.$$

By Lemma 2.5 and using (2.15), we get

$$\begin{aligned}\psi(d(y_{n_k}, y_{m_k})) &= \psi(d(Tx_{n_k}, Tx_{m_k}))\\ &= \psi(d(Tx_{m_k}, Tx_{n_k}))\\ &\leq \alpha(fx_{m_k}, fx_{n_k})\psi(d(Tx_{m_k}, Tx_{n_k}))\\ &\leq [\psi(R(x_{m_k}, x_{n_k}))]^{\lambda}\\ &= [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda},\end{aligned}$$

for all $n_k > m_k > k \geq k_0$. Letting $k \to \infty$ in this inequality, by (2.26) and (2.27) and the continuity of ψ, we obtain that

$$\psi(\varepsilon) = \lim_{k\to\infty} \psi(d(y_{n_k}, y_{m_k})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda} = [\psi(\varepsilon)]^{\lambda} < \psi(\varepsilon),$$

which is a contradiction. Therefore $\{y_n\}$ is a Cauchy sequence in X. Assume that fX is a complete BMS. It follows that $\{y_n\}$ converges to $z \in fX$. Thus there exists $x \in X$ such that $fx \in fX$ and $\lim_{n\to\infty} y_n = fx$. Therefore $\lim_{n\to\infty} fx_{n+1} = fx$. Since T is continuous with respect to f, we have

$$fx = \lim_{n\to\infty} fx_{n+2} = \lim_{n\to\infty} Tx_{n+1} = Tx.$$

Therefore x is a coincidence point of T and f. In the case of completeness of TX, we obtain that $\{y_n\}$ converges to $z \in TX \subseteq fX$.

We now prove that the point of coincidence of T and f is unique. Suppose that u and v are two coincidence points of T and f. Therefore $Tu = fu$ and $Tv = fv$. We will show that $fu = fv$. Suppose that $fu \neq fv$. By (v), we have $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$. Suppose that $\alpha(fu, fv) \geq 1$. By condition (2.15), we obtain that

$$\psi(d(fu, fv)) = \psi(d(Tu, Tv)) \leq \alpha(fu, fv)\psi(d(Tu, Tv)) \leq [\psi(R(u, v))]^{\lambda},$$

where

$$R(u, v) = \max\{d(fu, fv), d(fu, Tu), d(fv, Tv)\} = d(fu, fv).$$

This implies that

$$\psi(d(fu, fv)) \leq [\psi(d(fu, fv))]^{\lambda} < \psi(d(fu, fv))$$

which is a contradiction. Thus $fu = fv$. This implies that T and f have a unique point of coincidence. Since the pair $\{T, f\}$ is weakly compatible and by Proposition 1.15, we have that T and f have a unique common fixed point. □

Theorem 2.7. *Let (X, d) be a BMS and $T, f : X \to X$ be such that $TX \subseteq fX$ where one of these two subsets of X being complete. Suppose that $\alpha : X \times X \to [0, \infty)$ and the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \ \textit{ implies } \ \alpha(fx, fy) \cdot \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \quad (2.28)$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$;

(iii) T is a triangular f-α-admissible mapping;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$;

(v) either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$.

Then T and f have a unique point of coincidence. Moreover, if the pair $\{T, f\}$ is weakly compatible, then T and f have a unique common fixed point.

Proof. As in the proof of Theorem 2.6, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = fx_{n+1} = Tx_n, \ \text{ for all } n \in \mathbb{N},$$

$\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = fx$. By (iv), there exists a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ such that $\alpha(fx_{n_k}, fx) \geq 1$ for all $k \in \mathbb{N}$. We can suppose that $fx_{n_k} \neq Tx$. Applying inequality (2.28), we obtain that

$$\psi(d(Tx_{n_k}, Tx)) \leq \alpha(fx_{n_k}, fx)\psi(d(Tx_{n_k}, Tx)) \leq [\psi(R(x_{n_k}, x))]^{\lambda},$$

where

$$\begin{aligned} R(x_{n_k}, x) &= \max\Big\{d(fx_{n_k}, fx), d(fx_{n_k}, Tx_{n_k}), d(fx, Tx)\Big\} \\ &= \max\Big\{d(y_{n_k-1}, fx), d(y_{n_k-1}, y_{n_k}), d(fx, Tx)\Big\}. \end{aligned}$$

Taking the limit as $k \to \infty$ and since ψ is continuous, we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x) = d(fx, Tx).$$

We will prove that $fx = Tx$. Suppose that $fx \neq Tx$. Therefore

$$d(fx, Tx) \leq d(fx, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) + d(Tx_{n_k}, Tx).$$

It follows that

$$d(fx, Tx) \leq \lim_{k\to\infty} d(Tx_{n_k}, Tx).$$

Since ψ is continuous and nondecreasing, we obtain that

$$\psi(d(fx, Tx)) \leq \lim_{k\to\infty} \psi(d(Tx_{n_k}, Tx)) \leq [\psi(d(fx, Tx))]^{\lambda} < \psi(d(fx, Tx)),$$

which is a contradiction. Thus $fx = Tx$. Let $z = fx = Tx$. Hence z is a point of coincidence for T and f. As in the proof of Theorem 2.6, we obtain that T and f have a unique point of coincidence. Since the pair $\{T, f\}$ is weakly compatible and by Proposition 1.15, then we have that T and f have a unique common fixed point. □

Let X be a nonempty set. If (X, d) is a BMS and $(X, \preceq)$ is a partially ordered set, then $(X, d, \preceq)$ is called a partially ordered BMS. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$. Let $(X, \preceq)$ be a partially ordered set and $T, f : X \to X$. A mapping T is called an f-nondecreasing mapping if $Tx \preceq Ty$ whenever $fx \preceq fy$ for all $x, y \in X$.

Using Theorem 2.7, we obtain the following theorem in the setting of partially ordered BMS spaces.

Theorem 2.8. *Let $(X, d, \preceq)$ be a partially ordered BMS and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX, d) is a complete BMS. Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$ with $fx \preceq fy$,

$$d(Tx, Ty) \neq 0 \;\; \textit{implies} \;\; \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \tag{2.29}$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) *T is f-nondecreasing;*

(iii) *there exists $x_1 \in X$ such that $fx_1 \preceq Tx_1$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*

(v) *fu and fv are comparable whenever $fu = Tu$ and $fv = Tv$.*

Then T and f have a unique point of coincidence. Moreover, if the pair $\{T, f\}$ is weakly compatible, then T and f have a unique common fixed point.

Proof. Define a mapping $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in X \text{ and } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

We first show that T is f-α-admissible. Let $x, y \in X$ such that $\alpha(fx, fy) \geq 1$. Therefore $fx \preceq fy$. Since T is f-nondecreasing, we have $Tx \preceq Ty$ and then $\alpha(Tx, Ty) \geq 1$. We next prove that T is a triangular f-α-admissible. Let $x, y \in X$ such that $\alpha(fx, fy) \geq 1$ and $\alpha(fy, Ty) \geq 1$. Then we have $fx \preceq fy$ and $fy \preceq Ty$. This implies that $fx \preceq Ty$. So $\alpha(fx, Ty) \geq 1$. Therefore T is a triangular f-α-admissible mapping. Since there exists $x_1 \in X$ such that $fx_1 \preceq Tx_1$, we have $\alpha(fx_1, Tx_1) \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. By definition of α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and hence $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$. Let $u, v \in X$ such that $fu = Tu$ and $fv = Tv$. Since fu and fv are comparable, then we have $fu \preceq fv$ or $fv \preceq fu$. This implies that $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$. Finally, we prove that (2.28) holds. Let $x, y \in X$ and $d(Tx, Ty) \neq 0$. If $\alpha(fx, fy) = 1$, then $fx \preceq fy$ and then (2.28) holds. If $\alpha(fx, fy) = 0$, then (2.28) holds. It follows that all assumptions of Theorem 2.7 hold. By Theorem 2.7, we obtain that T and f have a unique common fixed point. □

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