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Rational Extensions of C(X) via Hausdorff Continuous Functions

R. Anguelov

Abstract : The ring operations and the metric on $C(X)$ are extended to the set $\mathbb{H}_{nf}(X)$ of all nearly finite Hausdorff continuous interval valued functions and it is shown that $\mathbb{H}_{nf}(X)$ is both rationally and topologically complete. Hence, the rings of quotients of $C(X)$ as well as their metric completions are represented as rings of Hausdorff continuous functions.

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1 Introduction

Let B be a commutative ring with identity and let A be a subring of B having the same identity. The ring B is called a *rational extension* or a *ring of quotients* of A if for every $b \in B$ the subring $b^{-1}A = \{a \in A : ba \in A\}$ is rationally dense in B, that is, $b^{-1}A$ does not have nonzero annihilators. Any ring A has a maximal rational extension $\mathcal{Q}(A)$. The ring $\mathcal{Q}(A)$ is also called a complete or total ring of quotients of A. The classical ring of quotients

$$\mathcal{Q}_{cl}(A) = \left\{ \frac{p}{q} : p, q \in A,\ q \text{ is not a zero divisor} \right\}$$

is, in general, a subring of $\mathcal{Q}(A)$. A ring without proper rational extension is called *rationally complete*.

It is well known that the ring $C(X)$ of all continuous real functions on a topological space X is not rationally complete. Our goal is to represent the rational extensions of $C(X)$ as rings of functions defined on the same domain, namely as rings of Hausdorff continuous (H-continuous) functions on X. The H-continuous functions are a special class of extended interval valued functions, that is, their range, or target set, is $\mathbb{I}\overline{\mathbb{R}} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\},\ \underline{a} \le \overline{a}\}$. Due to a certain minimality condition, they are not unlike the usual continuous functions. For instance, they are completely determined by their values on any dense subset of the domain. More precisely, for H-continuous functions f, g and a dense subset

D of X we have

$$f(x) = g(x),\ x \in D \implies f(x) = g(x),\ x \in X. \tag{1.1}$$

Within the realm of Set-Valued Analysis, the H-continuous functions can be identified with the minimal upper semi-continuous compact set-valued (usco) maps from X into $\overline{\mathbb{R}}$, [3].

We extend the ring structure on $C(X)$ to the set $\mathbb{H}_{nf}(X)$ of all nearly finite H-continuous functions following the approach in [5]. We show further that the ring $\mathbb{H}_{nf}(X)$ is rationally complete. Hence, it contains all rational extensions of $C(X)$. The maximal and classical rings of quotients are represented as subrings of $\mathbb{H}_{nf}(X)$.

Let $||\cdot||$ denote the supremum norm on the set $C_{bd}(X)$ of the bounded continuous functions. Then

$$\rho(f,g) = \left\| \frac{f-g}{1+|f-g|} \right\| \tag{1.2}$$

is a metric on $C(X)$, which can be extended further to $\mathcal{Q}(C(X))$. We prove that its completion $\overline{\mathcal{Q}}(C(X))$ with respect to this metric is exactly $\mathbb{H}_{nf}(X)$. The Dedekind completions $C(X)^{\#}$ and $C_{bd}(X)^{\#}$ of $C(X)$ and $C_{bd}(X)$ with respect to the pointwise partial order, being subrings of $\mathcal{Q}(C(X))$, also admit a convenient representation as rings of H-continuous functions. These results significantly improve earlier results of Fine, Gillman and Lambek [14], where the considered rings are represented as direct limits of rings of continuous functions on dense subsets of X and βX.

2 The ring of nearly finite Hausdorff continuous functions

We recall [16] that an interval function $f : X \to \mathbb{I}\overline{\mathbb{R}}$ is called *S-continuous* if its graph is a closed subset of $X \times \overline{\mathbb{R}}$. An interval function $f : X \to \mathbb{I}\overline{\mathbb{R}}$ is Hausdorff continuous (H-continuous) if it is an S-continuous function which is minimal with respect to inclusion, that is, if $\varphi : \Omega \to \mathbb{I}\overline{\mathbb{R}}$ is an S-continuous function then $\varphi \subseteq f$ implies $\varphi = f$. Here the inclusion is considered in a point-wise sense. We denote by $\mathbb{H}(X)$ the set of H-continuous functions on X.

Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{I}\overline{\mathbb{R}}$,

$$w(a) = \begin{cases} \overline{a} - \underline{a} & \text{if} \quad \underline{a}, \overline{a} \text{ finite}, \\ +\infty & \text{if} \quad \underline{a} < \overline{a} = +\infty \text{ or } \underline{a} = -\infty < \overline{a}, \\ 0 & \text{if} \quad \underline{a} = \overline{a} = \pm\infty, \end{cases}$$

is the width of a, while $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a. An interval a is called proper interval, if $w(a) > 0$ and point interval, if $w(a) = 0$. Identifying $a \in \overline{\mathbb{R}}$ with the point interval $[a, a] \in \mathbb{I}\overline{\mathbb{R}}$, we consider $\overline{\mathbb{R}}$ as a subset of $\mathbb{I}\overline{\mathbb{R}}$. H-continuous functions are similar to the usual real valued continuous real functions in that they assume proper interval values only on a set of First Baire category, that is, for every $f \in \mathbb{H}(X)$ the set $W_f = \{x \in X : w(f(x)) > 0\}$ is countable union of closed and nowhere dense set, [2]. Furthermore, f is continuous on $X \setminus W_f$. If X is a Baire space, $X \setminus W_f$ is also dense in X. Thus, in this case, f is completely determined by its point values. This approach is used for defining linear space operations [7] and ring operations [5] for H-continuous functions. Here we do not make any such assumption on X. Hence the approach is different.

For every S-continuous function g we denote by $\langle g \rangle$ the set of H-continuous functions contained in g, that is,

$$\langle g \rangle = \{f \in \mathbb{H}(\Omega) : f \subseteq g\}.$$

Identifying $\{f\}$ with f we have $\langle f \rangle = f$ whenever f is H-continuous. The S-continuous functions g such that the set $\langle g \rangle$ is a singleton, that is, it contains only one function, play an important role in the sequel. In analogy with the H-continuous functions, which are minimal S-continuous functions, we call these functions *quasi-minimal S-continuous functions* [3]. The following characterization of the quasi-minimal S-continuous functions is useful.

Theorem 1 *Let f be an S-continuous function on X. Then f is quasi- minimal S-continuous function if and only if for every $\varepsilon > 0$ the set*

$$W_{f,\varepsilon} = \{x \in X : w(f(x)) \geq \varepsilon\}$$

is closed and nowhere dense in X.

Proof. Let us assume that an S-continuous function f is not quasi-minimal. Then there exist H-continuous functions $\phi = [\underline{\phi}, \overline{\phi}]$ and $\psi = [\underline{\psi}, \overline{\psi}]$, $\phi \neq \psi$ such that $\phi \subseteq f$ and $\psi \subseteq f$. Due to the minimality property of H-continuous functions the set $\{x \in X : \phi(x) \cap \psi(x) = \emptyset\}$ is open and dense subset of X. Let $a \in X$ be such that $\phi(a) \cap \psi(a) = \emptyset$. Without loss of generally we may assume that $\overline{\psi}(a) < \underline{\phi}(a)$. Let $\varepsilon = \frac{1}{3}(\underline{\phi}(a) - \overline{\psi}(a))$. Using that $\overline{\phi}$ and $\underline{\phi}$ are respectively upper semi-continuous and lower semi-continuous functions, there exists an open neighborhood V of a such that $\overline{\psi}(x) < \overline{\psi}(a) + \varepsilon < \underline{\phi}(a) - \varepsilon < \underline{\phi}(x)$, $x \in V$. Then

$$w(f)(x) \geq \underline{\phi}(x) - \overline{\psi}(x) > \underline{\psi}(a) - \overline{\phi}(a) - 2\varepsilon = \varepsilon, \quad x \in V.$$

Hence $V \subset W_{f,\varepsilon}$, which implies that $W_{f,\varepsilon}$ is not nowhere dense. Therefore, if $W_{f,\varepsilon}$ is nowhere dense for every $\varepsilon > 0$ then f is quasi-minimal.

Now we prove the inverse implication, that is, that for any S-continuous quasi-minimal function f and $\varepsilon > 0$ the set $W_{f,\varepsilon}$ is closed and nowhere dense. Assume the opposite. Since for an S-continuous function f the set $W_{f,\varepsilon}$ is always closed, this means that there exists an S-continuous function $f = [\underline{f}, \overline{f}]$ and $\varepsilon > 0$ such that $W_{f,\varepsilon}$ is not nowhere dense. Hence there exists an open set V such that $V \subseteq W_{f,\varepsilon}$. Then there exist an H-continuous functions ϕ on V such that $\phi(x) \subseteq [\underline{f}(x), \overline{f}(x) - \varepsilon]$, $x \in V$. The we have $\phi(x) + \varepsilon \subseteq [\underline{f}(x), \overline{f}(x) - \varepsilon]$, $x \in V$. It is easy to see that the functions ϕ and $\phi+\varepsilon$ can both be extended from V to the whole space X so that they belong to $\langle f\rangle$. Hence f is not quasi-minimal.

The familiar operations of addition and multiplication on the set of real intervals is defined for $[\underline{a},\overline{a}],[\underline{b},\overline{b}]\in\mathbb{I}\,\mathbb{R}$ as follows:

$$[\underline{a},\overline{a}]+[\underline{b},\overline{b}]=\{a+b:a\in[\underline{a},\overline{a}],b\in[\underline{b},\overline{b}]\}=[\underline{a}+\underline{b},\overline{a}+\overline{b}],$$
$$[\underline{a},\overline{a}]\times[\underline{b},\overline{b}]=\{ab:a\in[\underline{a},\overline{a}],b\in[\underline{b},\overline{b}]\}=[\min\{\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}\},\max\{\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}\}],$$

Ambiguities related to $\pm\infty$ are resolved in a way which guarantees inclusion:

$$-\infty+(+\infty)=[-\infty,+\infty],\ 0\times(+\infty)=[0,+\infty],\ 0\times(-\infty)=[-\infty,0]$$

Point-wise operations for interval functions are defined in the usual way:

$$(f+g)(x)=f(x)+g(x)\ ,\quad (f\times g)(x)=f(x)\times g(x)\ ,\ x\in X. \tag{2.1}$$

It is easy to see that the set of the S-continuous functions is closed under the above point-wise operations while the set of H-continuous functions is not. In earlier works by Markov, Sendov and the author, see [5]–[8], it was shown that the algebraic operations on the set $\mathbb{H}_{ft}(X)$ of all finite H-continuous function can be defined in such a way that it is a linear space (the largest linear space of finite interval functions) and a ring. As mentioned above, these results were derived in the case when X is a Baire space. Here we extend these results in two ways:

(i) We assume that the domain X is an arbitrary completely regular topological space.

(ii) We consider the wider set $\mathbb{H}_{nf}$ of nearly finite H-continuous functions.

These generalizations are motivated by the aim of the paper: namely, constructing the rational extensions of $C(X)$ as rings of functions defined on the same domain. More precisely, the problem is considered in the same setting as in [14], that is, X a completely regular topological space. Furthermore, it is shown in the sequel that the rational extensions of $C(X)$ and their metric completions considered in [14] cannot be all represented within the realm of finite H-continuous function. Hence we need to considered the larger set $\mathbb{H}_{nf}(X)$.

Let us recall that an H-continuous function f is called *nearly finite* if the set

$$\Gamma_f=\{x\in X:-\infty\in f(x)\ or+\infty\in f(x)\}$$

is closed and nowhere dense. The set $\mathbb{H}_{nf}(X)$ has important applications in the Analysis of PDEs within the context of the Order Completion Method, [15]. It turns out that the solutions of large classes of systems of nonlinear PDEs can be assimilated with nearly finite H-continuous functions, see [10], [11]. The definition of the operations on $\mathbb{H}_{nf}(X)$ are based on the following theorem.

Theorem 2 *For any $f,g\in\mathbb{H}_{nf}(X)$ the functions $f+g$ and $f\times g$ are quasi-minimal S-continuous functions.*

Proof. The proofs for $f+g$ and $f \times g$ use similar ideas based on Theorem 1. We will present only the proof for $f \times g$ which is slightly more technical. Assume the opposite. In view of Theorem 1, this means that there exists $\varepsilon > 0$ and an open set V such that $V \subset W_{f\times g,\varepsilon}(f)$. Furthermore, since f and g are nearly finite, the set $V \setminus (\Gamma(f) \cup \Gamma(g))$ is also open and nonempty. Let $a \in V \setminus (\Gamma(f) \cup \Gamma(g))$. It is easy to see that the functions $|f|$ defined by $|f|(x) = |f(x)|$, $x \in X$, is an upper semi-continuous function. Therefore, there exists an open set D_1 such that $a \in D_1 \subset V \setminus (\Gamma(f) \cup \Gamma(g))$ and $|f(x)| < |f(a)| + 1$, $x \in D_1$. Similarly, there exists an open set D_2 such that $a \in D_2 \subset V \setminus (\Gamma(f) \cup \Gamma(g))$ and $|g(x)| < |g(a)| + 1$, $x \in D_2$. Denote $D = D_1 \cap D_2$. Then, using a well known inequality about the width of a product of intervals, see [1], for every $x \in D$ we obtain

$$\begin{aligned} \varepsilon &\leq w((f \times g)(x)) = w(f(x) \times g(x)) \ \leq \ w(f(x))|g(x)| + w(g(x))|f(x)| \\ &\leq w(f(x))(|g(a)|+1) + w(g(x))(|f(a)|+1) \ \leq \ (w(f(x)) + w(g(x))m, \end{aligned}$$

where $m = \max\{|f(a)| + 1\ ,\ |g(a)| + 1\}$. This implies

$$D \subset W_{f,\frac{\varepsilon}{2m}} \cup W_{g,\frac{\varepsilon}{2m}}.$$

Therefore, at least one of the sets $W_{f,\frac{\varepsilon}{2m}}$ and $W_{g,\frac{\varepsilon}{2m}}$ has a nonempty open subset. However, by Theorem 1 this is impossible. The obtained contradiction completes the proof.

Theorem 2 implies that for every $f, g \in \mathbb{H}_{nf}(X)$ the sets $[f+g]$ and $[f \times g]$ contain one element each. Then we define addition and multiplication on $\mathbb{H}_{nf}(X)$ by

$$f \oplus g = [f+g] \tag{2.2}$$

$$f \otimes g = [f \times g] \tag{2.3}$$

Here and in the sequel we denote the operations by $\oplus$ and $\otimes$ to distinguish them from the point-wise operations denoted earlier by $+$ and $\times$. Equivalently to (2.2)–(2.3), one can say that $f \oplus g$ is the unique H-continuous function contained in $f+g$ and that $f \otimes g$ is the unique H-continuous function contained in $f \times g$. It should be noted that the operations $\oplus$ and $\otimes$ may coincide with $+$ and $\times$ respectively for some values of the arguments. In particular,

$$f \oplus g = f + g, \ f \otimes g = f \times g \ \text{ for } \ f \in C(X), \ g \in \mathbb{H}_{nf}(X). \tag{2.4}$$

Theorem 3 *The set $\mathbb{H}_{nf}(X)$ is a commutative ring with respect to the operations $\oplus$ and $\otimes$.*

The proof will be omitted. It involves standard techniques and is partially discussed in [7] for the case of finite functions. The zero and the identity in $\mathbb{H}_{nf}(X)$ are the constant functions with values 0 and 1 respectively. We will denote them by 0 and 1 with the context showing whether we mean a constant function or the respective real number. The multiplicative inverse of $f \in \mathbb{H}_{nf}(X)$, whenever it exists, is denoted by $\frac{1}{f}$. The non-zero-divisors in the ring $\mathbb{H}_{nf}(X)$ can be characterized similarly to the ring $C(X)$. However, unlike $C(X)$ all non-zero-divisors are invertible. More precisely,

$$f \text{ is a non-zero-divisor} \Longleftrightarrow Z(f) \text{ is nowhere dense in } X \Longleftrightarrow \exists g \in \mathbb{H}_{nf}(X) : f \otimes g = 1 \quad (2.5)$$

where $Z(f)$ is the zero set of the function f given by $Z(f) = \{x \in X : 0 \in f(x)\}$. We denote the inverse of f, that is, the function g in (2.5) above, by $\frac{1}{f}$. If $f(x) = [\underline{f}(x), \overline{f}(x)$, $x \in X$, then we have

$$\frac{1}{f}(x) = \left[\frac{1}{\overline{f}(x)} \frac{1}{\underline{f}(x)}\right] \ , \quad x \in coz(f) = X \setminus Z(f). \quad (2.6)$$

Note that in view of the property (1.1), the equality (2.6) determines $\frac{1}{f}$ in a unique way because $coz(f)$ is dense in X.

Let D be an open subset of X. The restriction $f|_D$ of f on D is an H-continuous function on D, see [6]. More precisely,

$$f \in \mathbb{H}_{nf}(X) \implies f|_D \in \mathbb{H}_{nf}(D). \quad (2.7)$$

Since $\mathbb{H}_{nf}(D)$ is also a ring it is useful to remark that for any $f, g \in \mathbb{H}_{nf}(X)$ we have

$$(f \oplus g)|_D = f|_D \oplus g|_D \ , \ (f \otimes g)|_D = f|_D \otimes g|_D. \quad (2.8)$$

We will also use the following property, [6]:

> for any dense subset D of X and $g \in \mathbb{H}_{nf}(D)$ there exists a unique function $f \in \mathbb{H}_{nf}(X)$ such that $f|_D = g$. (2.9)

3 Representing the Rational Extensions of $C(X)$

The zero set $Z(f)$ and the cozero set $coz(f)$ of $f \in \mathbb{H}_{nf}(X)$ generalize the respective concepts for continuous function and play an important role in the ring $\mathbb{H}_{nf}(X)$ as suggested by (2.5) and (2.6). This is further demonstrated in the following useful lemma which extends the respective result in [14, Section 2.2].

Lemma 4 *Let $H_a(X)$ be a ring of H-continuous functions such that $C(X) \subseteq H_a(X) \subseteq \mathbb{H}_{nf}(X)$. An ideal P of $H_a(X)$ is rationally dense in $H_a(X)$ if and only if $coz(P)$ is a dense subset of X.*

Any ideal P of a ring A is also an A-module. The rational completeness of a ring can be characterized in terms of the A-homomorphisms from the rationally dense ideals of A to A as shown in the next theorem [13].

Theorem 5 *A ring A is rationally complete if for every rationally dense ideal P of A and an A-homomorphism $\varphi : P \to A$ there exists $s \in P$ such that $\varphi(p) = sp$, $p \in P$.*

In the sequel we refer to the A-homomorphism shortly as homomorphisms.

Theorem 6 *The ring $\mathbb{H}_{nf}(X)$ is rationally complete.*

Proof. We use Theorem 5. Let P be an ideal of $\mathbb{H}_{nf}(X)$ and $\varphi : P \to \mathbb{H}_{nf}(X)$ a homomorphism. Let $p \in P$. Consider the ring $\mathbb{H}_{nf}(coz(p))$. By (2.6), p is an invertible element of $\mathbb{H}_{nf}(coz(p))$. Since $\phi(p)|_D \in \mathbb{H}_{nf}(coz(p))$, see (2.7), we can consider the function $\psi_p = \frac{1}{p} \otimes \phi(p)|_D \in \mathbb{H}_{nf}(coz(p))$.

Now we define the function $\psi \in \mathbb{H}_{nf}(X)$ in the following way. For any $x \in coz(P)$ select $p \in P$ such that $0 \notin p(x)$. Then

$$\psi(x) = \psi_p(x)$$

It is easy to see that the definition does not depend on the function p. Indeed, let $q \in P$ be such that $0 \notin q(x)$. Since φ is a homomorphism we have

$$\varphi(p) \otimes q = p \otimes \varphi(q). \tag{3.1}$$

Denote $D = coz(p) \cap coz(q)$. Clearly D is an open neighborhood of x. Using (2.8) we have

$$\varphi(p)|_D \otimes q|_D = p|_D \otimes \varphi(q)|_D,$$

which implies

$$\frac{1}{p}|_D \otimes \varphi(p)|_D = \frac{1}{q}|_D \otimes \varphi(q)|_D.$$

Therefore $\phi_p(y) = \phi_q(y)$, $y \in D$. In particular, $\phi_p(x) = \phi_q(x)$.

Now ψ is defined on $coz(P)$ and it is easy to see that $\psi \in \mathbb{H}_{nf}(coz(P))$. Since $coz(P)$ is dense in X, see Lemma 4, using (2.9) the function ψ can be defined on the rest of the set X in a unique way so that $\psi \in \mathbb{H}_{nf}(X)$).

We will show that $\varphi(p) = \psi \otimes p$, $p \in P$. Let $p \in P$. We have

$$(\psi \otimes p)|_{coz(p)} = (\psi_p \otimes p|_{coz(p)}) = \left(\frac{1}{p}\right)|_{coz(p)} \otimes \varphi(p)|_{coz(p)} \otimes p|_{coz(p)} = \varphi(p)|_{coz(p)}$$

Then, using also (2.9), we obtain

$$(\psi \otimes p)(x) = \varphi(p)(x)\ , \quad x \in \overline{coz(p)}, \tag{3.2}$$

where $\overline{coz(p)}$ denotes the topological closure of the set $coz(p)$. Applying standard techniques based on the minimality property of H-continuous functions one can obtain that $\varphi(p)(x) = 0$ for $x \in X \setminus \overline{cos(p)} \subset Z(p)$. Then we have

$$\varphi(p)(x) = 0 \in \psi(x) \times p(x)\ , \quad x \in X \setminus \overline{coz(p)} \tag{3.3}$$

From (3.2) and (3.3) it follows

$$\varphi(p)(x) \subseteq \psi(x) \times p(x) \ , \quad x \in X. \tag{3.4}$$

By the definition of the operation $\otimes$ the inclusion (3.4) implies $\varphi(p) = \psi \otimes p$, which completes the proof.

The rational completeness of $\mathbb{H}_{nf}(X)$ implies that any rational extension of any subring of $\mathbb{H}_{nf}(X))$ is a subring of $\mathbb{H}_{nf}(X)$. In particular this applies to $C(X)$, where the respective maximal ring of quotients and classical ring of quotients are characterized in the next theorem. As in the classical theory we call a subset V of X a zero set if there exists $f \in C(X)$ such that $V = Z(f)$.

Theorem 7 (Representation Theorem) *The ring of quotients and the classical ring of quotients of $C(X)$ are the following subrings of $\mathbb{H}_{nf}(X))$:*

a) $\mathcal{Q}(C(X)) = \mathbb{H}_{nd}(X) = \{f \in \mathbb{H}(X) : \overline{W_f}$ *is nowhere dense*$\}$

b) $\mathcal{Q}_{cl}(C(X)) = \mathbb{H}_{sz}(X) = \{f \in \mathbb{H}(X) : W_f$ *is a subset of a nowhere dense zero set*$\}$

Proof. a) First we need to show that $\mathbb{H}_{nd}(X)$ is a ring of quotients of $C(X)$. In terms of the definition we have to prove that for any $\phi, \psi \in \mathbb{H}_{nd}(X))$, $\psi \neq 0$, there exists $f \in C(X)$ such that $\phi \otimes f \in C(X)$ and $\psi \otimes f \neq 0$. Since $\psi \neq 0$ the open set $coz(\psi)$ is not empty. Using that W_ϕ and Γ_ϕ are closed nowhere dense sets we have $coz(\psi) \setminus (W_\phi \cup \Gamma_\phi) \neq \emptyset$. Let $a \in coz(\psi) \setminus (W_\phi \cup \Gamma_\phi)$. By the complete regularity of X: (i) there exists a neighborhood V of x such that $\overline{V} \subset coz(\psi) \setminus (W_\phi \cap \Gamma_\phi)$; (ii) there exists a function $f \in C(X)$ such that $f(a) = 1$ and $f(x) = 0$ for $x \in X \setminus V$. We have that $\psi \times f$ does not have zeros in a neighborhood of a, Therefore $\psi \otimes f \neq 0$. We prove next that $\phi \otimes f \in C(X)$. Indeed, $\phi(x) \times f(x) = 0$ for x in the open set $(X \setminus \overline{V}) \setminus (W_\phi \cup \Gamma_\phi)$ which is dense in the open set $X \setminus \overline{V}$. Therefore $(\phi \otimes f)|_{X \setminus \overline{V}} = 0$, which implies $(\phi \otimes f)|_{X \setminus \overline{V}} \in C(X \setminus \overline{V})$. Obviously, $(\phi \otimes f)|_{X \setminus (W_\phi \cup \Gamma_\phi)} \in C(X \setminus (W_\phi \cup \Gamma_\phi)$. Hence, $\phi \otimes f \in C(X)$. Therefore, $\mathbb{H}_{nd}(X)$ is a ring of quotients of $C(X)$. It is the maximal ring of quotients of $C(X)$ if and only if it is rationally complete. The proof of the rational completeness of $\mathbb{H}_{nd}(X)$ is done in a similar way as for $\mathbb{H}_{nf}(X)$ and will be omitted.

b) Let $f, g \in C(X)$, $Z(g)$ - nowhere dense in X. Then using (2.5) we obtain $\dfrac{f}{g} = f \otimes \dfrac{1}{g} \in \mathbb{H}_{nf}(X)$. Moreover, by (2.6) we have $W_{\frac{f}{g}} \subseteq Z(g)$ which implies $\dfrac{f}{g} \in \mathbb{H}_{nd}(X)$. Therefore $\mathcal{Q}_{cl}(C(X)) \subseteq \mathbb{H}_{nd}(X)$. Now we will prove the inverse inclusion. Let $f \in \mathbb{H}_{nd}(X)$. Then there exists $g \in C(X)$ such that $Z(g)$ is nowhere dense on X and $W_f \subseteq Z(g)$. Consider the functions

$$\begin{aligned} \phi &= \frac{f \otimes g}{1 + f \otimes f} \\ \psi &= \frac{g}{1 + f \otimes f}. \end{aligned}$$

It easy to see that $|\phi| \leq |g|$ and $|\psi| \leq |g|$, which implies that $\Gamma_\phi = \Gamma_\psi = \emptyset$. Furthermore, since $\phi(x) = \psi(x) = 0$ for all $x \in W_f$ we have $W_\phi = W_\psi = \emptyset$. Hence, $\phi, \psi \in C(X)$. Since $Z(\psi) = Z(g)$ is nowhere dense in X, the function ψ is an invertible element of $\mathbb{H}_{nd}(X)$. Then $f = \dfrac{\phi}{\psi} \in \mathcal{Q}_{cl}(C(X))$, which completes the proof.

4 Representing the metric completions of $\mathcal{Q}(C(X))$ and $\mathcal{Q}_{cl}(C(X))$

The metric ρ on $C(X)$ given in (1.2) can be extended to $\mathbb{H}_{nf}(X)$ as follows

$$\rho(f,g) = \sup_{x \in X \setminus (\Gamma_f \cup \Gamma_g)} \frac{|f \ominus g|}{1 + |f \ominus g|}, \tag{4.1}$$

where $f \ominus g = f \oplus (-1)g$.

Theorem 8 *The set $\mathbb{H}_{nf}(X)$ is a complete metric space with respect to the metric ρ in (4.1).*

Proof. Verifying that ρ satisfies the axioms of a metric uses standard techniques and will be omitted. We will prove the completeness by using that $\mathbb{H}_{nf}(X)$ is a Dedekind complete latticet with respect to the usual point-wise order, see [10, 11]. Furthermore, $\mathbb{H}_{nf}(X)$ is also a vector lattice with respect to the addition $\oplus$ and the multiplication by constants. This can be shown similarly to [12], where the case of finite H-continuous functions is considered. The following implication for any $\varepsilon \in (0,1)$ is easy to obtain and is useful in the proof

$$\rho(f,g) < \varepsilon \Longleftrightarrow \frac{-\varepsilon}{1-\varepsilon} \leq f \ominus g \leq \frac{\varepsilon}{1-\varepsilon} \Longleftrightarrow g \ominus \frac{\varepsilon}{1-\varepsilon} \leq f \leq g \oplus \frac{\varepsilon}{1-\varepsilon}. \tag{4.2}$$

Let $(f_\lambda)_{\lambda \in \Lambda}$ be a Cauchy net on $\mathbb{H}_{nf}(X)$. There exists $\mu \in \Lambda$ such that $\rho(f_\lambda, f_\mu) < 0.5$. Then by (4.2) the net $(f_\lambda)_{\lambda \geq \mu}$ is bounded. Due to the Dedekind order completeness of $\mathbb{H}_{nf}(X)$ the following infima and suprema exist

$$\begin{aligned} \phi_\lambda &= \inf\{f_\nu : \nu \geq \lambda\}\ , \quad \lambda \geq \mu, \\ \psi_\lambda &= \sup\{f_\nu : \nu \geq \lambda\}\ , \quad \lambda \geq \mu, \\ \phi &= \sup\{\phi_\lambda : \lambda \geq \mu\} \\ \psi &= \inf\{\psi_\lambda : \lambda \geq \mu\} \end{aligned}$$

Let $\varepsilon \in (0,1)$. There exists λ_ε such that $\rho(f_\lambda, f_\nu) < \varepsilon$, $\lambda, \nu \geq \lambda_\varepsilon$. It follows from (4.2) that

$$f_{\lambda_\varepsilon} \ominus \frac{\varepsilon}{1-\varepsilon} \leq f_\nu \leq f_{\lambda_\varepsilon} \oplus \frac{\varepsilon}{1-\varepsilon},\ \nu \geq \lambda_\varepsilon.$$

Therefore,

$$f_{\lambda_\varepsilon} \ominus \frac{\varepsilon}{1-\varepsilon} \le \phi_\lambda \le \psi_\lambda \le f_{\lambda_\varepsilon} \oplus \frac{\varepsilon}{1-\varepsilon},\ \lambda \ge \lambda_\varepsilon,$$

which implies

$$0 \le \psi_\lambda \ominus \phi_\lambda \le \frac{2\varepsilon}{1-\varepsilon},\ \lambda \ge \lambda_\varepsilon. \tag{4.3}$$

Taking a supremum on λ and considering that ε is arbitrary we obtain $\phi = \psi$.

Further, from the inequalities

$$\phi_\lambda \le f_\lambda \le \psi_\lambda$$
$$\phi_\lambda \le \phi \le \psi_\lambda$$

and (4.3) we obtain

$$|f_\lambda \ominus \phi| \le |\psi_\lambda \ominus \phi_\lambda| \le \frac{2\varepsilon}{1-\varepsilon},\ \lambda \ge \lambda_\varepsilon.$$

or equivalently $\rho(f_\lambda, \phi) < \varepsilon$. This implies that $\lim_\lambda f_\lambda = \phi$, which completes the proof.

Since the ring $\mathbb{H}_{nf}(X)$ is rationally complete, see Theorem 6, as well as complete with respect to the metric (4.1), see Theorem 8, it contains all rings of quotients of $C(X)$ as well as their metric completions. In particular, representation of the metric completion $\overline{\mathcal{Q}(C(X))}$ of $\mathcal{Q}(C(X))$ is given in the next theorem.

Theorem 9 *The completion of the ring of quotients of* $C(X)$ *is* $\mathbb{H}_{nf}(X))$, *that is,*

$$\overline{\mathcal{Q}(C(X))} = \mathbb{H}_{nf}(X)$$

Proof. Since the completeness of $\mathbb{H}_{nf}(X))$ has already been proved, we only need to show that $\mathcal{Q}(C(X))$ is dense in $\mathbb{H}_{nf}(X)$. Using the representation of $\mathcal{Q}(C(X))$ given in Theorem 7, equivalently, we need to show that $\mathbb{H}_{nd}(X)$ is dense in $\mathbb{H}_{nf}(X))$. Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}_{nf}(X)$ and let $n \in \mathbb{N}$. We have

$$\overline{f}(x) - \frac{1}{n} \le \underline{f}(x) + \frac{1}{n}\ ,\quad x \in X \setminus (W_{f,\frac{1}{n}} \cup \Gamma_f). \tag{4.4}$$

Since the function on the left side of the inequality (4.4) is upper semi-continuous while the function on the right side is lower semi-continuous by the well known Theorem of Han there exists $f_n \in C(X \setminus (W_{f,\frac{1}{n}} \cap \Gamma_f))$ such that

$$\overline{f}(x) - \frac{1}{n} \le f_n(x) \le \underline{f}(x) + \frac{1}{n}\ ,\quad x \in X \setminus (W_{f,\frac{1}{n}} \cup \Gamma_f). \tag{4.5}$$

The set $X \setminus (W_{f,\frac{1}{n}} \cup \Gamma_f)$ is an open and dense subset of X because $W_{f,\frac{1}{n}}$ and Γ_f are closed nowhere dense sets. Hence f_n can be extended in a unique way to X so that it is H-continuous on X. Since this extended function may assume interval values or values involving $\pm\infty$ only on the closed nowhere dense set $W_{f,\frac{1}{n}} \cup \Gamma_f$ we have $f_n \in \mathbb{H}_{nd}(X)$. Moreover, it follows from the inequality (4.5) that

$$\rho(f, f_n) \le \sup_{x \in X \setminus (W_{f,\frac{1}{n}} \cup \Gamma_f)} |f \ominus f_n| \le \frac{1}{n}.$$

Hence, $\lim_{n\to\infty} f_n = f$. Therefore $\mathbb{H}_{nd}(X)$ is dense in $\mathbb{H}_{nf}(X)$.

5 Conclusion

This paper gives an application of a class of interval functions, namely, the Hausdorff continuous function, to the representation of the rational extensions of $C(X)$ as well as their metric completions. Traditionally, Interval Analysis is considered as part of Numerical Analysis due to its major applications to scientific computing. However, the study of the order, topological and algebraic structure of the spaces of interval functions led to some significant applications to other areas of mathematics, e.g. Approximation Theory [16], Analysis of PDEs [10, 11, 17], Real Analysis [2, 12]. The results presented here are in the same line of applications. It is shown that all rings of quotients of $C(X)$ and their matric completions are subrings of the ring $\mathbb{H}_{nf}(X)$ of nearly finite Hausdorff continuous functions. Thus, $\mathbb{H}_{nf}(X))$ is the largest set of functions to which the ring and metric structure of $C(X)$ can be extended in an unambiguous way.

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R. Anguelov
Department of Mathematics and Applied Mathematics
University of Pretoria, South Africa
e-mail : `roumen.anguelov@up.ac.za`