



Error Bounds of Conjugate of a Periodic Signal by Almost Generalized Nörlund Means

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Abstract : In this paper, we determine the error bounds of conjugate signals between input periodic signals and processed output signals, whenever signals belong to $Lip(\alpha, r)$ - class and as a processor we have taken almost generalized Nörlund means using head bounded variation sequences and rest bounded variation sequences. The results obtained in this paper further extend several known results on linear operators.

Keywords : error bounds; conjugate Fourier series; almost generalized Nörlund means; $Lip(\alpha, r)$ - class.

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1 Introduction

A number of researchers Al-Saqabi et al. [1], Liu and Srivastava [2], Alzer et al. [3], Mohapatra [4, 5], Bor et al. [6], Choi and Srivastava [7], Deepmala and Piscoran [8], Ganguly and Dutta [9], Bala et al. [10] have established interesting results in sequences and series using different linear summability operators. The degree of approximation of a function belonging to various classes using different linear summability operators has been determined by several investigators like Bernstein [11], Alexits [12, 13], Sahney and Goel [14], Khan [15], Mishra et al. [16–19]. Chen and Hong [20] used Cesàro sum regularization technique for hyper singularity of dual integral equation. Summability of Fourier series is useful for engineering analysis. Sahney and Rao [21] and Khan [22] have studied the degree of approximation of functions belonging to $Lip(\alpha, r)$ by (N, p_n) & (N, p, q) means respectively. Summability techniques were also applied on some engineering problems: for example, Chen and Jeng [23] implemented the *Cesàro* sum of order $(C, 1)$ and $(C, 2)$, in order to accelerate the convergence rate to deal with the Gibbs phenomenon, for the dynamic response of a finite elastic body subjected to boundary traction. Recently, Mursaleen and Mohiuddine [24] discussed convergence methods for double sequences and their applications in various fields. Quereshi [25, 26] discussed the degree of approximation of function belonging to $Lip \alpha$ & $Lip(\alpha, r)$ by (N, p_n) means of conjugate series of a Fourier series. Braha [27] discussed the asymptotic representation of the best approximation for continuous 2π - periodic functions, using the module of smoothness. Summation-Integral operators play an important role modeling various physical and biological processes. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The theory of linear summability operators has had an important influence on the development of mathematical systems theory. On the other hand, mathematical systems theory serves as a direct source of motivation and new techniques for the theory of linear summability operators and its applications. The engineers and scientists use properties of Fourier approximation for designing digital filters. Application of summability theory in wavelet analysis can be seen in Lal and Kumar [28]. Singh [29], Pati [30] and Dikshit [31] have obtained the error bounds of conjugate signals by different summability methods. The purpose of this paper is to determine the error bounds of conjugate signals between input periodic signals and processed output signals, whenever signals belong to $Lip(\alpha, r)$ - class and as a processor by almost generalized Nörlund means using under conditions that $(p_n) \in HBVS$ and $(q_n) \in RBVS$. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let p denotes the sequence $\{p_n\}$, $p_{-1} = 0$. For two sequences p and q ,

$$\begin{aligned} P_n &:= p_0 + p_1 + \cdots + p_n, (P_{-1} = p_{-1} = 0), \\ Q_n &:= q_0 + q_1 + \cdots + q_n, (Q_{-1} = q_{-1} = 0), \end{aligned}$$

the convolution $(p * q)_n$ is defined by

$$R_n = (p * q)_n = \sum_{m=0}^n p_m q_{n-m} = \sum_{m=0}^n p_{n-m} q_m.$$

It is obvious that

$$P_n := (p * 1)_n = \sum_{m=0}^n p_m \text{ and } Q_n := (1 * q)_n = \sum_{m=0}^n q_m = \sum_{m=0}^n q_{n-m}.$$

When $(p * q)_n \neq 0$ for all n , the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^n p_{n-m} q_m s_m.$$

If $t_n^{p,q} \rightarrow s$ as $n \rightarrow \infty$, then the sequence $\{s_n\}$ is said to be summable to s by generalized Nörlund method (N, p, q) and is denoted by $s_n \rightarrow s(N, p, q)$ [32].

The necessary and sufficient conditions for a (N, p, q) method to be regular are

$$\sum_{m=0}^n |p_{n-m} q_m| = O(|(p * q)_n|)$$

and

$p_{n-m} = o(|(p * q)_n|)$, as $n \rightarrow \infty$, for every fixed $m \geq 0$ for which $q_m \neq 0$.

The (N, p, q) method reduces to Nörlund method (N, p_n) if $q_n = 1$ for all n . The (N, p, q) method reduces to Riesz method (\bar{N}, q_n) if $p_n = 1$ for all n .

In the special case when $p_n = \binom{n + \alpha - 1}{\alpha - 1}$, $\alpha > 0$, the method (N, p_n) reduces to the well-known method of summability (C, α) .

The particular case $p_n = (n + 1)^{-1}$ of the Nörlund mean is known as harmonic mean and is written as $(N, 1/n + 1)$.

Let f be a 2π - periodic signal (function) and Lebesgue integrable. The Fourier series of f is given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.1}$$

with n^{th} partial sum $s_n(f; x)$ called trigonometric polynomial of degree (or order) n , of the first $(n+1)$ terms of the Fourier series of f .

The conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

The conjugate function $\tilde{f}(x)$ is defined for almost every x by (see [33], Definition 1.10).

$$\tilde{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot t/2 dt = \lim_{h \rightarrow 0} \left(-\frac{1}{2\pi} \int_h^\pi \psi(t) \cot t/2 dt \right)$$

A signal (function) $f \in Lip \alpha$ if

$$f(x+t) - f(x) = O(|t|^\alpha) \quad \text{for } 0 < \alpha \leq 1, t > 0$$

and $f \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O(|t|^\alpha), \quad 0 < \alpha \leq 1, r \geq 1, t > 0.$$

If we take $r \rightarrow \infty$ then $Lip(\alpha, r) = Lip \alpha$.

L_∞ - norm of a function $f : R \rightarrow R$ is defined by $\|f\|_\infty = \sup \{|f(x)| : x \in R\}$.

L_r - norm of a function is defined by $\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r}$, $1 \leq r < \infty$.

The degree of approximation of a function $f : R \rightarrow R$ by trigonometric polynomial t_n of order n under sup norm $\|\cdot\|_\infty$ is defined by ([34]) $r \rightarrow \infty$

$$\|t_n - f\|_\infty = \sup \{|t_n(x) - f(x)| : x \in R\}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$[E_n(f)] = \min_n \|t_n - f\|_r.$$

We say that the conjugate series is said to be almost generalized Nörlund summable to the finite number s , if

$$\tilde{t}_{n,r}^{p,q} = \frac{1}{R_n} \sum_{m=0}^n p_m q_{n-m} \tilde{s}_{m,r} \rightarrow s \text{ as } n \rightarrow \infty$$

uniformly with respect to r , where

$$\tilde{s}_{m,r} = \frac{1}{m+1} \sum_{j=r}^{m+r} \tilde{s}_j.$$

We note that $t_{n,r}^{p,q}$ and $\tilde{t}_{n,r}^{p,q}$ are also trigonometric polynomials of degree (or order) n .

Here we defines two classes of sequences (see [35]).

A sequence $c = \{c_n\}$ of non-negative numbers is called Head Bounded Variation, or briefly $c \in HBVS$, if

$$\sum_{k=0}^{n-1} |c_k - c_{k+1}| \leq K(c) c_n, \forall n \in N, \quad (1.3)$$

or only for $n \leq N$ if the sequence c has only finite non-zero terms and the last non-zero term is c_N .

A sequence $c = \{c_n\}$ of non-negative numbers tending to zero is called of Rest Bounded Variation, or briefly $c \in RBVS$, if

$$\sum_{k=n}^{\infty} |c_k - c_{k+1}| \leq K(c) c_n, \forall n \in N, \tag{1.4}$$

where $K(c)$ is a constant depending only on c .

For an example of a sequence $c \in HBVS$, we take c to be any monotone increasing sequence.

Then

$$\sum_{k=0}^{n-1} |c_k - c_{k+1}| = \sum_{k=0}^{n-1} |c_{k+1} - c_k| = c_n - c_0 \leq K(c) c_n,$$

with $K(c) = 1$.

For an RBVS sequence we take c to be any monotone decreasing sequence with limit 0. Let m be any positive integer with $m > n$.

Then

$$\sum_{k=n}^m |c_k - c_{k+1}| = \sum_{k=n}^m |c_k - c_{k+1}| = c_n - c_{m+1},$$

and

$$\sum_{k=n}^{\infty} |c_k - c_{k+1}| = c_n \lim_m c_{m+1} = c_n - 0 = K(c) c_n,$$

with, again, $K(c) = 1$.

We shall use the following notations throughout the paper:

$$\psi_x(t) = \psi(t) = f(x+t) - f(x-t),$$

$$\tilde{G}_n(t) = \sum_{p=r}^{r+m} \frac{\cos(p+1/2)t \cos pt/2}{\sin(t/2)},$$

and M denotes a constant which may be different at each of its occurrence.

2 Known Result

Very recently Mishra et al. [18] determined the degree of approximation of a signal $f \in Lip(\alpha, r)$, ($r \geq 1$)– class by almost Riesz summability means of its Fourier series. Krasniqi [36] established the following theorem to estimate the error between the input signal $f(t)$ and signal obtained by passing through the almost generalized Nörlund mean $t_{n,r}^p(f(t); x)$.

Theorem 2.1. Let $(p_n) \in HBVS$ and $(q_n) \in RBVS$. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a 2π -periodic function, Lebesgue integrable and belonging to $Lip(\alpha, r)$, ($r \geq 1$)-class, then the degree of approximation of the function f by almost generalized Nörlund means of its Fourier series $t_{n,r}^{p,q}(f(t); x)$ is given by

$$\|f(t) - t_{n,r}^{p,q}(f(t); x)\|_r = O\left(R_n^{1/r-\alpha}\right), \forall n, \quad (2.1)$$

and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/R_n} \left(\frac{t|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O\left(R_n^{-1}\right), \quad (2.2)$$

$$\left[\int_{\pi/R_n}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O\left(R_n^{\delta-1}\right), \quad (2.3)$$

where δ is a finite quantity, generalized Nörlund means are regular and $r+s=rs$ such that $1 \leq r \leq \infty$.

3 Main Result

The object of this paper is to generalize the above result under much more general assumptions. We will measure the error between the input signal $\tilde{f}(t)$ and the processed output signal $\tilde{t}_{n,r}^{p,q}(\tilde{f}(t); x)$ by establishing the following theorem:

Theorem 3.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic, integrable in the sense of Lebesgue and belonging to $Lip(\alpha, r)$, ($r \geq 1$)-class, then the degree of approximation of function f by almost generalized Nörlund means of its conjugate series of its Fourier series i.e. $\tilde{t}_{n,r}^{p,q}(\tilde{f}(t); x)$ is given by

$$\|\tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(\tilde{f}(t); x)\|_r = O\left(R_n^{1/r-\alpha}\right), \forall n, \quad (3.1)$$

$\{p_n\} \in HBVS$ and $\{q_n\} \in RBVS$ and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/R_n} \left(\frac{|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O\left(R_n^{-1}\right), \quad (3.2)$$

$$\left[\int_{\pi/R_n}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O\left(R_n^\delta\right), \quad (3.3)$$

where δ is a finite quantity, generalized Nörlund means are regular and $r^{-1}+s^{-1}=1$ such that $1 \leq r \leq \infty$.

Proof. Let us write

$$\tilde{s}_n(f; x) = \sum_{k=1}^n B_k(x) \quad \text{and} \quad \tilde{t}_n(x) = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} \tilde{s}_k(x).$$

Then we have

$$\begin{aligned} \tilde{s}_k(f; x) - \tilde{f}(x) &= \frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{\cos(k + 1/2)t}{\sin(t/2)} dt \\ &= \frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{\cos(k + 1/2)t}{\sin(t/2)} dt + \eta_k, \end{aligned}$$

where, by the Riemann-Lebesgue theorem,

$$\eta_k = \frac{1}{\pi} \int_\delta^\pi \psi(t) \frac{\cos(k + 1/2)t}{\sin(t/2)} dt \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and now

$$\tilde{s}_{k,r}(f(t); x) - \tilde{f}(t) = \frac{2}{\pi(k+1)} \sum_{p=r}^{r+k} \int_0^\pi \psi(t) \frac{\cos(p + 1/2)t \cos pt/2}{\sin(t/2)} dt + \eta_{k,r},$$

where, for almost generalized Nörlund means of $\tilde{s}_{k,r}(f(t); x)$, we have

$$\begin{aligned} \tilde{t}_{n,r}^{p,q}(\tilde{f}(t); x) - \tilde{f}(t) &= \frac{1}{R_n} \sum_{m=0}^n p_m q_{n-m} \left\{ \tilde{s}_{m,r}(f(t); x) - \tilde{f}(t) \right\} \\ &= \frac{2}{\pi R_n} \int_0^\pi \psi(t) \sum_{m=0}^n \frac{p_m q_{n-m}}{(m+1)} \sum_{p=r}^{r+m} \frac{\cos(p + 1/2)t \cos pt/2}{\sin(t/2)} dt + \xi_{n,r} \\ &= \frac{2}{\pi R_n} \int_0^\pi \psi(t) \sum_{m=0}^n \frac{p_m q_{n-m}}{(m+1)} \tilde{G}_n(t) dt + \xi_{n,r} \\ &= \frac{2}{\pi R_n} \left[\int_0^{\pi/R_n} + \int_{\pi/R_n}^\pi \right] \int_0^\pi \psi(t) \sum_{m=0}^n \frac{p_m q_{n-m}}{(m+1)} \tilde{G}_n(t) dt + \xi_{n,r} \\ &= I_1 + I_2 \text{ (say),} \end{aligned} \tag{3.4}$$

Using Hölder inequality, $f(t) \in Lip(\alpha, s) \Rightarrow \psi(t) \in Lip(\alpha, s)$ on $[0, \pi]$, condition (3.2), inequalities

$$(\sin t/2)^{-1} \leq \pi/t, \quad \text{for } 0 < t \leq \pi, \quad |\cos nt| \leq 1. \tag{3.5}$$

$$\begin{aligned}
 |I_1| &\leq \frac{2}{\pi R_n} \left[\int_0^{\pi/R_n} \left(\frac{|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} \left[\int_0^{\pi/R_n} \left(\frac{1}{t^{-\alpha}} \left| \sum_{m=0}^n \frac{p_m q_{m-n}}{m+1} \tilde{G}_n(t) \right|^s \right) dt \right]^{1/s} \\
 &= O(R_n^{-1}) \left[\int_0^{\pi/R_n} \left(\frac{1}{t^{-\alpha}} \left| \sum_{m=0}^n \frac{p_m q_{m-n}}{m+1} (m+1)t^{-1} \right|^s \right) dt \right] \\
 &= O(R_n^{-1}) \left[\int_0^{\pi/R_n} R_n^s t^{(\alpha-1)s} dt \right]^{1/s} = O(1) \left[\int_0^{\pi/R_n} t^{(\alpha-1)s} dt \right]^{1/s} \\
 &= O(1) \cdot O\left(\frac{1}{R_n^{\alpha-1+\frac{1}{s}}}\right) = O\left(\frac{1}{R_n^{\alpha-(1-\frac{1}{s})}}\right) \\
 &= O\left(\frac{1}{R_n^{\alpha-\frac{1}{r}}}\right). \tag{3.6}
 \end{aligned}$$

$$|I_2| \leq \frac{2}{\pi R_n} \left[\int_{\pi/R_n}^{\pi} \left(\frac{t^{-\delta} |\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} \left[\int_{\pi/R_n}^{\pi} \left(t^{\alpha+\delta} \left| \sum_{m=0}^n \frac{p_m q_{m-n}}{m+1} \tilde{G}_n(t) \right|^s \right) dt \right]^{1/s}.$$

Now, using the fact that $f(t) \in Lip(\alpha, s) \Rightarrow \psi(t) \in Lip(\alpha, s)$ on $[0, \pi]$, conditions (3.3), (3.5) and $r^{-1} + s^{-1} = 1$, we obtain

$$\begin{aligned}
 |I_2| &\leq O(R_n^{\delta-1}) \left[\int_{\pi/R_n}^{\pi} \left(\frac{t^{\alpha+\delta}}{\sin t/2} \sum_{m=0}^n \frac{p_m q_{m-n}}{m+1} \sum_{p=r}^{r+m} \cos(p+1/2)t \cos(pt/2) \right)^s dt \right]^{1/s} \\
 &= O(R_n^{\delta-1}) \left[\int_{\pi/R_n}^{\pi} \left(\frac{t^{\alpha+\delta}}{\sin t/2} \sum_{m=0}^n p_m q_{m-n} \right)^s dt \right]^{1/s}.
 \end{aligned}$$

Also, since $p = \{p_n\} \in \text{HBVS}$, then by (1.3), we obtain

$$\begin{aligned}
 p_m - p_n &\leq |p_m - p_n| \leq \sum_{k=m}^{n-1} |p_k - p_{k+1}| \\
 &\leq \sum_{k=0}^{n-1} |p_k - p_{k+1}| \\
 &\leq K(p) p_n, \\
 &\Rightarrow p_m \leq (K(p) + 1)p_n, \forall m \in [0, n]. \tag{3.7}
 \end{aligned}$$

Also, since $q = \{q_n\} \in \text{RBVS}$, then by (1.4), we obtain

$$\begin{aligned} q_{n-m} &\leq \sum_{k=m}^{\infty} |q_{n-k} - q_{n-k-1}| \\ &\leq \sum_{k=0}^{\infty} |q_{n-k} - q_{n-k-1}| \\ &\leq K(q) q_n, \forall m \in [0, n]. \end{aligned} \tag{3.8}$$

Now using (3.7) and (3.8), we have

$$\begin{aligned} \sum_{m=0}^n |p_m q_{m-n}| &\leq \sum_{m=0}^n (K(p) + 1) p_n K(q) q_n \\ &= (K(p) + 1) K(q) (n + 1) p_n q_n \\ &= O(R_n). \end{aligned} \tag{3.9}$$

Subsequently, we get

$$\begin{aligned} |I_2| &= O(R_n^\delta) \left[\int_{\pi/R_n}^{\pi} (t^{\alpha+\delta-1})^s dt \right]^{1/s} \\ &= O(R_n^\delta) \left[\left(\frac{t^{(\alpha+\delta-1)s+1}}{(\alpha+\delta-1)s+1} \right)_{\pi/R_n}^{\pi} \right]^{1/s} \\ &= O(R_n^\delta) O(R_n^{-\alpha-\delta+1-1/s}) \\ &= O(R_n^{-\alpha+1/r}) \\ &= O\left(\frac{1}{(R_n^{\alpha-1/r})}\right). \end{aligned} \tag{3.10}$$

Using (3.6) and (3.10) in (3.4), we have

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right| = O\left(\frac{1}{R_n^{\alpha-1/r}}\right).$$

Now, using L_r - norm, we obtain,

$$\begin{aligned} \left\| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right\|_r &= \left[\int_0^{2\pi} \left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right|^r dt \right]^{1/r} \\ &= \left[\int_0^{2\pi} O\left(\frac{1}{R_n^{\alpha-1/r}}\right)^r dt \right]^{1/r} = O\left(\frac{1}{R_n^{\alpha-1/r}}\right). \end{aligned}$$

This completes the proof of Theorem 3.1. \square

4 Corollaries

The result of our theorem is more general rather than the results of any other previous proved theorems, which will enrich the literature of summability theory of infinite series. The following corollaries can be derived from our main theorem:

Corollary 4.1. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π - periodic, integrable in the sense of Lebesgue and belonging to $Lip\alpha$ class, then the degree of approximation of function f by almost generalized Nörlund means of its conjugate series is given by*

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right|_{\infty} = O(R_n^{-\alpha}), \forall n, \quad (4.1)$$

where $\{p_n\} \in HBVS$ and $\{q_n\} \in RBVS$ and $\psi(t)$ satisfies (2.2), (2.3) and $r^{-1} + s^{-1} = 1$ such that $1 \leq r \leq \infty$.

Proof. Putting $r \rightarrow \infty$ in Theorem 3.1, we have

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right|_{\infty} = O(R_n^{-\alpha}). \quad \square$$

Finally, we find that

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right| \leq \left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right|_{\infty} = O(R_n^{-\alpha}).$$

For $\forall n \geq 0$, we put $q_n = 1$ or $p_n = 1$ in Corollary 4.1, we obtain two cases

1. If $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π - periodic, integrable in the sense of Lebesgue and belonging to $Lip\alpha$ class, then the degree of approximation of function f by almost Riesz means of its conjugate series

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right| = O(P_n^{-\alpha}), \forall n, \quad (4.2)$$

where $\{p_n\} \in HBVS$ and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/P_n} \left(\frac{|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O(P_n^{-1}), \quad (4.3)$$

$$\left[\int_{\pi/P_n}^{\pi} \left(\frac{t^{-\delta} |\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O(P_n^\delta), \tag{4.4}$$

where δ is a finite quantity, Riesz means are regular and $r^{-1} + s^{-1} = 1$ such that $1 \leq r \leq \infty$.

2. If $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic, integrable in the sense of Lebesgue and belonging to $Lip\ \alpha$ - class, then the degree of approximation of function f by almost Nörlund means of its conjugate series

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right| = O(Q_n^{-\alpha}), \forall n, \tag{4.5}$$

$\{q_n\} \in RBVS$ and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/Q_n} \left(\frac{|\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O(Q_n^{-1}), \tag{4.6}$$

$$\left[\int_{\pi/Q_n}^{\pi} \left(\frac{t^{-\delta} |\psi(t)|^r}{t^\alpha} \right) dt \right]^{1/r} = O(Q_n^\delta), \tag{4.7}$$

where δ is a finite quantity, means are regular and $r^{-1} + s^{-1} = 1$ such that $1 \leq r \leq \infty$. Let $q_n = 1$ in Theorem 3.1 $\forall n \geq 0$, we get

Corollary 4.2. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic, integrable in the sense of Lebesgue and belonging to $Lip(\alpha, r)$, ($r \geq 1$) class, then the degree of approximation of function f by almost Riesz means of its conjugate series is given by*

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right| = O(P_n^{1/r-\alpha}), \forall n, \tag{4.8}$$

where $\{p_n\} \in HBVS$, $\psi(t)$ satisfies the following conditions (4.3) and (4.4), δ is a finite quantity, Riesz means are regular and $r^{-1} + s^{-1} = 1$ such that $1 \leq r \leq \infty$.

Let $p_n = 1$ in Theorem 3.1, $\forall n \geq 0$, we get

Corollary 4.3. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic, integrable in the sense of Lebesgue and belonging to $Lip(\alpha, p)$ class, then the degree of approximation of function f by almost Nörlund means of its conjugate series*

$$\left| \tilde{f}(t) - \tilde{t}_{n,r}^{p,q}(f(t); x) \right| = O(Q_n^{1/r-\alpha}), \forall n, \tag{4.9}$$

where $\{q_n\}$ in $RBVS$, $\psi(t)$ satisfies the following conditions (4.6) and (4.7), δ is a finite quantity, Nörlund means are regular and $r^{-1} + s^{-1} = 1$ such that $1 \leq r \leq \infty$.

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