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# A Fixed Point Theorem for Generalized Lipschitzian Semigroups in Hilbert Spaces

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**Abstract :** In this work, we use the concept of a generalized Lipschitzian type condition for a semigroup of self mappings as employed in [1] to provide an existence theorem of a common fixed point for a left reversible semitopological semigroup of continuous generalized Lipschitzian mappings in a Hilbert space defined on a nonconvex domain. This result extends and improves a result of Downing and Ray in [2].

**Keywords :** common fixed point; generalized Lipschitzian mapping; left reversible semitopological semigroup; Hilbert space.
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## 1 Introduction

Let $E$ be a Banach space with norm $\|.\|$ and $U$ be a nonempty bounded subset of $E$. A mapping $T : U \to U$ is said to be a Lipschitzian mapping if for each $n \in \mathbb{N}$, there exists $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in U$. A lipschitzian mapping $T$ is said to be uniformly $k$-Lipschitzian if $k_n = k$ for all $n \in \mathbb{N}$, and asymptotically nonexpansive if $\lim_n k_n = 1$, respectively.

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These mappings were first studied by Goebel and Kirk in [3] and [4]. They proved that such mappings have a fixed point in a uniformly convex Banach space for the case of convex domain. Especially for a uniformly $k$-Lipschitzian mapping, the constant $k$ should have value less than $k_0$ for some $k_0 > 1$. (In a Hilbert space, $k_0 = \sqrt{5}/2$, see [4]). In [5], Lifschitz proved that a uniformly $k$-Lipschitzian mapping in a Hilbert space with $k < \sqrt{2}$ has a fixed point. For the others existence theorems of a fixed point for Lipschitzian mappings in a larger space than a uniformly convex Banach space, we refer the readers to see [6], [7], and [8]. Especially in [8], Lim and Xu proved the existence theorem of a uniformly $k$-Lipschitzian mapping defined on a nonconvex domain by using the concept of the property $(P)$.

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from $S$ to $S$ are continuous. A semitopological semigroup $S$ is left reversible if any two closed right ideals of $S$ have a nonvoid intersection. In this case, $(S, \geq)$ is a directed system when the binary relation " $\geq$ "on $S$ is defined by $b \geq a$ if and only if $b \cup \overline{bS} \subseteq a \cup \overline{aS}$. Left reversible semitopological semigroups include all commutative semigroups (e.g., $[0, \infty)$) and all semitopological semigroups which are left amenable as discrete semigroups, see [9].

In 1982, Downing and Ray [2] proved that a discrete semigroup of uniformly $k$-Lipschitzian mappings in a Hilbert space defined on a convex domain with $k < \sqrt{2}$ has a common fixed point. Later, this result was extended to a left reversible semitopological semigroup of uniformly $k$-Lipschitzian mappings by Ishihara and Takahashi [10]. Also, Ishihara [11] extended and improved the result of Ishihara and Takahashi [10] to the case of a left reversible semitopological semigroup of Lipschitzian mappings defined on a nonconvex domain. In 1990, Xu [12] extended the result of Ishihara and Takahashi [10] to a $p$-uniformly convex Banach space. We also note that the result of Ishihara and Takahashi [10] has been extended and improved by Gornicki [13], by investigating the structure of a common fixed point set of the semigroup in a $p$-uniformly convex Banach space.

Imdad *et al.* [14] used the notion of generalized Lipschitzian as employed in [1], to establish the concept of a left reversible semitopological semigroup of continuous generalized Lipschitzian mappings. Let $U$ be a nonempty bounded subset of a Banach space $E$ and $S$ be a left reversible semitopological semigroup. A family $\mathcal{T} = \{T_s : s \in S\}$ of self mappings defined on $U$ is said to be a semigroup on $U$ if $\mathcal{T}$ satisfies the following properties:

(1) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in U$;

(2) the mapping $(s, x) \to T_sx$ from $S \times U$ into $U$ is continuous when $S \times U$ has the product topology.

Moreover, the semigroup $\mathcal{T}$ is said to be a generalized Lipschitzian semigroup on $U$ if $\mathcal{T}$ also satisfies the following property:

(3) for each $s \in S$, there exists $k_s > 0$ such that

$$\|T_s x - T_s y\| \leq k_s \max\left\{\|x-y\|, \frac{1}{2}\|x - T_s x\|, \frac{1}{2}\|y - T_s y\|, \frac{1}{2}\|x - T_s y\|, \frac{1}{2}\|y - T_s x\|\right\}$$

for all $x, y \in U$.

By using Example 5.3 in [15], we show that the class of generalized Lipschitzian semigroups properly includes the class of Lipschitzian semigroups.

**Example 1.1.** Let $E$ be the real line $\mathbb{R}$ and $S$ be a discrete space $[0, \infty)$. Let $U = [0,1]$, $b \in (0,1)$ and $\mathcal{T} = \{T_s : s \in S\}$ be a semigroup on $U$ defined by; for $s > 0$,

$$T_s x = \begin{cases} b^s x, & \text{if } x \in [0, \frac{1}{2}] \\ 0, & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

and

$$T_0 x = x, \quad x \in U.$$

Since for each $s > 0$, $T_s$ is discontinuous at $x = \frac{1}{2}$, then $\mathcal{T}$ is not a Lipschitzian semigroup. However, $\mathcal{T}$ is a generalized Lipschitzian semigroup. Indeed, for $s > 0$ we have

$$|T_s x - T_s y| = b^s|x-y| \leq 2b^s|x-y|$$

for all $x, y \in [0, \frac{1}{2}]$ and

$$|T_s x - T_s y| = 0 \leq 2b^s|x-y|$$

for all $x, y \in (\frac{1}{2}, 1]$. For $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$,

$$|T_s x - T_s y| = |b^s x - 0| = 2b^s\left(\frac{1}{2}|x - T_s y|\right).$$

Therefore, $\mathcal{T} = \{T_s : s \in S\}$ is a generalized Lipschitzian semigroup on $U$ with the constants $k_s$ are $2b^s$.

In [14], Imdad *et al.* proved the existence theorem of a common fixed point for a left reversible semitopological semigroup of continuous generalized Lipschitzian mappings in a $p$-uniformly convex Banach space defined on a nonconvex domain. But, we see from Remark 12 in [16] that the proof of Theorem 3.1 in [14] is unfortunately not correct because the inequality

$$\inf_{\lambda} \sup\{\|x_\iota - x_\kappa\| : \iota, \kappa \geq \lambda\} \leq \limsup_{\lambda} \limsup_{\kappa} \|x_\kappa - x_\lambda\|$$

is false.

Motivated by the above results, in this work, we use the concept of the property $(P)$ in the setting of a net and we employ a different technique than in the proof of Theorem 3.1 in [14], for showing existence theorem of a common fixed point for a left reversible semitopological semigroup of continuous generalized Lipschitzian mappings in a Hilbert space defined on a nonconvex domain. This result extends and improves Theorem 1 of Downing and Ray [2].

# 2 Preliminaries

Recall the concept and the notion of asymptotic center due to Edelstein [17]. Let $U$ be a nonempty subset of a Banach space $E$, $\Lambda$ be a directed set, and $\{x_\lambda\}_{\lambda\in\Lambda}$ be a bounded net in $E$. The asymptotic center of $\{x_\lambda\}$ with respect to $U$ is defined as the set

$$\mathcal{A}(\{x_\lambda\}, U) = \left\{ x \in U : \limsup_{\lambda} \|x_\lambda - x\| = \inf_{y\in U} \limsup_{\lambda} \|x_\lambda - y\| \right\}.$$

It is easily seen that if $E$ is reflexive and $U$ is closed convex, then $\mathcal{A}(\{x_\lambda\}, U)$ is nonempty. Moreover, it will be a singleton if $E$ is a Hilbert space.

In order to prove our main result, we need the following technical lemma.

**Lemma 2.1.** *Let $n > 1$ be any integer. Let us set $\{x_{1_\lambda}\}_{\lambda\in\Lambda}$, $\{x_{2_\lambda}\}_{\lambda\in\Lambda}$, $\cdots$, $\{x_{n_\lambda}\}_{\lambda\in\Lambda}$ as bounded nets of real numbers. Then*

$$\limsup_{\lambda} \max\{x_{1_\lambda}, x_{2_\lambda}, \cdots, x_{n_\lambda}\} = \max\left\{ \limsup_{\lambda} x_{1_\lambda}, \limsup_{\lambda} x_{2_\lambda}, \cdots, \limsup_{\lambda} x_{n_\lambda}\right\}.$$

*Proof.* Let $y_\lambda = \max\{x_{1_\lambda}, x_{2_\lambda}, \cdots, x_{n_\lambda}\}$ for all $\lambda \in \Lambda$. Then,

$$\limsup_{\lambda} y_\lambda \geq \max\left\{ \limsup_{\lambda} x_{1_\lambda}, \limsup_{\lambda} x_{2_\lambda}, \cdots, \limsup_{\lambda} x_{n_\lambda}\right\}.$$

On the other hand, without loss of generality, we may assume that

$$a_1 = \limsup_{\lambda} x_{1_\lambda} \geq a_2 = \limsup_{\lambda} x_{2_\lambda} \geq \cdots \geq a_n = \limsup_{\lambda} x_{n_\lambda}.$$

**Case 1** There exists $n_0 \in \{1, 2, \cdots, n-1\}$ such that $a_{n_0} > a_{n_0+1}$. Let

$$m_0 = \min\{n_0 \in \{1, 2, \cdots, n-1\} : a_{n_0} > a_{n_0+1}\}.$$

Then, we can find $\lambda_1 \in \Lambda$ such that

$$\max\left\{ \sup_{\lambda\geq\lambda_1} x_{i_\lambda} : i \in \{m_0+1, m_0+2, \cdots, n\}\right\} < \frac{a_{m_0} + a_{m_0+1}}{2}. \tag{2.1}$$

If $m_0 > 1$, let $\varepsilon > 0$ be fixed. Then, there exists $\lambda_2 \in \Lambda$ such that

$$\sup_{\lambda\geq\lambda_2} x_{i_\lambda} < a_1 + \varepsilon \quad \text{for all } i \in \{1, 2, \cdots, m_0\}. \tag{2.2}$$

Let $\lambda_0 \geq \max\{\lambda_1, \lambda_2\}$. From (2.1) and (2.2) we easily have

$$\limsup_{\lambda} y_\lambda \leq \sup_{\lambda\geq\lambda_0} y_\lambda < a_1 + \varepsilon. \tag{2.3}$$

Letting $\varepsilon \to 0$ into (2.3), we get the result.

We now prove for the case $m_0 = 1$. We note that for each $\lambda_2 \in \Lambda$,

$$\sup_{\lambda \geq \lambda_2} x_{1_\lambda} > \frac{a_1 + a_2}{2}.$$

So, by letting $\lambda_0 \geq \max\{\lambda_1, \lambda_2\}$ and next, by (2.1) we have

$$\limsup_{\lambda} y_\lambda \leq \sup_{\lambda \geq \lambda_0} y_\lambda \leq \sup_{\lambda \geq \lambda_2} x_{1_\lambda}.$$

It follows that

$$\limsup_{\lambda} y_\lambda \leq a_1.$$

**Case 2** $a_1 = a_2 = \cdots = a_n$. Let $\varepsilon > 0$ be fixed. Then, we can find $\lambda_3 \in \Lambda$ such that

$$\sup_{\lambda \geq \lambda_3} x_{i_\lambda} < a_1 + \varepsilon \quad \text{for all } i \in \{1, 2, , \cdots, n\}.$$

It follows that

$$\limsup_{\lambda} y_\lambda \leq \sup_{\lambda \geq \lambda_3} y_\lambda < a_1 + \varepsilon. \tag{2.4}$$

Thus, by letting $\varepsilon \to 0$ into (2.4), we get the result. □

We also need the following lemmas which were proved in [10] and [11], respectively.

**Lemma 2.2.** [10, Lemma] *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a bounded net in $H$ and $\{a\} = \mathcal{A}(\{x_\lambda\}, C)$. Then*

$$\inf_{y \in C} \limsup_{\lambda} \|x_\lambda - y\|^2 + \|a - x\|^2 \leq \limsup_{\lambda} \|x_\lambda - x\|^2$$

*for all $x \in C$.*

**Lemma 2.3.** [11, Lemma 2] *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $\{x_\lambda\}_{\lambda \in \Lambda}$ be a bounded net in $C$. Then*

$$\mathcal{A}(\{x_\lambda\}, C) \subseteq \bigcap_{\lambda} \overline{co}\{x_\kappa : \kappa \geq \lambda\},$$

*where $\overline{co}$ is the closed convex hull.*

We end this section by giving the following lemma.

**Lemma 2.4.** *Let $U$ be a nonempty bounded subset of a Banach space $E$ and $S$ be a left reversible semitopological semigroup. Let $\mathcal{T} = \{T_s : s \in S\}$ be a semigroup of self mappings on $U$. Then*

$$\limsup_{s} \limsup_{t} \|T_s T_t x - y\| \leq \limsup_{s} \|T_s x - y\|$$

*for all $x, y \in U$.*

*Proof.* We denote $r = \limsup_s \|T_s x - y\|$. Let $\varepsilon > 0$ be fixed. Then, we choose $s_0 \in S$ such that

$$\sup\{\|T_s x - y\| : s \geq s_0\} < r + \varepsilon. \tag{2.5}$$

Let $u \geq s_0$ be fixed. Then,

$$\begin{aligned}\limsup_t \|T_u T_t x - y\| &= \inf_q \sup\{\|a - y\| : a \in \overline{\{T_u T_t x : t \geq q\}}\} \\ &= \inf_q \sup\{\|a - y\| : a \in \overline{\{T_s x : s \geq uq\}}\} \\ &\leq \sup\{\|a - y\| : a \in \{T_s x : s \geq u\}\}. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) we get

$$\limsup_s \limsup_t \|T_s T_t x - y\| < r + \varepsilon. \tag{2.7}$$

So, by letting $\varepsilon \to 0$ into (2.7), we obtain the result. □

## 3 The Fixed Point Theorem

We now present the main result of this work.

**Theorem 3.1.** *Let $U$ be a nonempty bounded subset of a Hilbert space $H$ and $S$ be a left reversible semitopological semigroup. Suppose that $\mathcal{T} = \{T_s : s \in S\}$ is a generalized Lipschitzian semigroup on $U$ with $\limsup_s k_s < \sqrt{2}$ and $T_s$ is continuous for all $s \in S$. Suppose also that there exists a nonempty bounded closed convex subset $C$ of $U$ with the following property $(P)$:*

$$(P) \quad x \in C \Longrightarrow \omega_w(x) \subseteq C,$$

*where $\omega_w(x)$ is the weak $\omega$-limit set of $\mathcal{T}$ at $x$, i.e., the set*

$$\omega_w(x) = \left\{ y \in H : y = weak\text{-}\lim_\lambda T_{s_\lambda} x \text{ for some subset } \{s_\lambda\} \text{ of } S \right\}.$$

*Then there exists $z \in C$ such that $T_s z = z$ for all $s \in S$.*

*Proof.* We denote $k = \limsup_s k_s$. Let $B_s(x) = \{T_t x : t \geq s\}$ for $s \in S$ and $x \in C$. Let $x_0 \in C$ be fixed and then, taking $\{x_1\} = \mathcal{A}(\{B_s(x_0)\}, \overline{co}U)$. By Lemma 2.3, we get $x_1 \in \bigcap_s \overline{co} B_s(x_0)$. On the other hand, by Hahn-Banach Separation Theorem (see, Theorem 1.15(a) in [18]), we see that $\bigcap_s \overline{co} B_s(x_0) = \overline{co}\omega_w(x_0)$. Therefore, by the assumption, we get $x_1 \in C$. We may repeat this step and obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $C$ such that for every $n \in \mathbb{N}$,

$$\{x_n\} = \mathcal{A}(\{B_s(x_{n-1})\}, \overline{co}U) \subseteq \bigcap_s \overline{co} B_s(x_{n-1}). \tag{3.1}$$

Let $d_m = \limsup_s \|T_s x_m - x_{m+1}\|^2$ and $D_m = \limsup_s \|T_s x_m - x_m\|^2$ for $m \geq 0$. Let $n \geq 0$ be fixed. We may assume that $D_{n+1} > 0$, since otherwise, nothing that

$$\limsup_s \|T_s x - y\| = \inf_s \sup_t \|T_s T_t x - y\|$$

for all $x, y \in U$, then by using the continuity of $T_s$ at $x_{n+1}$, we see that $x_{n+1}$ is a common fixed point of $\mathcal{T}$ and hence the proof is finished.

We note that, by Lemma 2.1 and Lemma 2.4 we have

$$\begin{aligned}
&\limsup_s \limsup_t \|T_t x_n - T_s x_{n+1}\| \leq \limsup_s \limsup_t \|T_s T_t x_n - T_s x_{n+1}\| \\
&\leq \limsup_s \limsup_t k_s \max\left\{\|T_t x_n - x_{n+1}\|, \frac{1}{2}\|T_t x_n - T_s T_t x_n\|, \frac{1}{2}\|x_{n+1} - T_s x_{n+1}\|,\right. \\
&\quad \left.\frac{1}{2}\|T_t x_n - T_s x_{n+1}\|, \frac{1}{2}\|x_{n+1} - T_s T_t x_n\|\right\} \\
&\leq k \limsup_s \limsup_t \max\left\{\|T_t x_n - x_{n+1}\|, \frac{1}{2}(\|T_t x_n - x_{n+1}\| + \|x_{n+1} - T_s T_t x_n\|),\right. \\
&\quad \left.\frac{1}{2}\|x_{n+1} - T_s x_{n+1}\|, \frac{1}{2}\|T_t x_n - T_s x_{n+1}\|, \frac{1}{2}\|x_{n+1} - T_s T_t x_n\|\right\} \\
&\leq k \max\left\{(d_n)^{\frac{1}{2}}, (d_n)^{\frac{1}{2}}, \frac{1}{2}(D_{n+1})^{\frac{1}{2}}, \frac{1}{2}\limsup_s \limsup_t \|T_t x_n - T_s x_{n+1}\|, \frac{1}{2}(d_n)^{\frac{1}{2}}\right\} \\
&= k \max\left\{(d_n)^{\frac{1}{2}}, \frac{1}{2}(D_{n+1})^{\frac{1}{2}}, \frac{1}{2}(d_n)^{\frac{1}{2}}\right\} \\
&= k \max\left\{(d_n)^{\frac{1}{2}}, \frac{1}{2}(D_{n+1})^{\frac{1}{2}}\right\}. \qquad (3.2)
\end{aligned}$$

Therefore, from (3.1) and (3.2) we have

$$\begin{aligned}
D_{n+1} &= \limsup_s \|T_s x_{n+1} - x_{n+1}\|^2 \\
&\leq \limsup_s \limsup_t \|T_t x_n - T_s x_{n+1}\|^2 \\
&\leq k^2 \max\left\{d_n, \frac{1}{4} D_{n+1}\right\} \\
&= k^2 d_n. \qquad (3.3)
\end{aligned}$$

Now, by (3.1) and Lemma 2.2 we get for every $s \in S$,

$$\|T_s x_{n+1} - x_{n+1}\|^2 \leq \limsup_t \|T_t x_n - T_s x_{n+1}\|^2 - \limsup_t \|T_t x_n - x_{n+1}\|^2. \quad (3.4)$$

Therefore, by taking the limit superior into (3.4), we have from (3.2) and (3.3) that

$$\begin{aligned}D_{n+1} &\leq \limsup_s \limsup_t \|T_t x_n - T_s x_{n+1}\|^2 - d_n \\ &\leq k^2 \max\left\{d_n, \frac{1}{4}D_{n+1}\right\} - d_n \\ &\leq k^2 \max\left\{d_n, \frac{k^2}{4}d_n\right\} - d_n \\ &= (k^2-1)d_n. \end{aligned} \tag{3.5}$$

Thus, from (3.1) and (3.5) we obtain

$$D_{n+1} \leq (k^2-1)d_n = \eta d_n \leq \eta D_n,$$

where $\eta = k^2 - 1 < 1$ by the assumption. Consequently,

$$D_n \leq \eta D_{n-1} \leq \cdots \leq \eta^n D_0. \tag{3.6}$$

For every $s \in S$, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\leq (\|x_{n+1} - T_s x_n\| + \|T_s x_n - x_n\|)^2 \\ &\leq 2(\|T_s x_n - x_{n+1}\|^2 + \|T_s x_n - x_n\|^2). \end{aligned} \tag{3.7}$$

Therefore by taking the limit superior into (3.7) and then, by (3.1) we have

$$\|x_{n+1} - x_n\|^2 \leq 2(d_n + D_n) \leq 2D_n.$$

It follows from (3.6) that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty$, and hence $\{x_n\}$ is a Cauchy sequence in $C$. Let $z = \lim_n x_n$. Now for each $s \in S$,

$$\begin{aligned}\|z - T_s z\|^2 &\leq (\|z - T_t x_n\| + \|T_t x_n - T_s z\|)^2 \\ &\leq 2(\|z - T_t x_n\|^2 + \|T_t x_n - T_s z\|^2). \end{aligned} \tag{3.8}$$

Hence, by taking the limit superior into (3.8) we have

$$\begin{aligned}\|z - T_s z\|^2 &\leq 2\left(\limsup_t \|T_t x_n - z\|^2 + \limsup_t \|T_t x_n - T_s z\|^2\right) \\ &\leq 2\left(\limsup_t \|T_t x_n - z\|^2 + \limsup_t \|T_s T_t x_n - T_s z\|^2\right). \end{aligned} \tag{3.9}$$

Since

$$\limsup_t \|T_t x_n - z\|^2 \leq 2(D_n + \|x_n - z\|^2),$$

then by (3.6), it yields

$$\limsup_t \|T_t x_n - z\|^2 \to 0 \text{ as } n \to \infty.$$

By the continuity of $T_s$ at $z$ we also get

$$\limsup_t \|T_s T_t x_n - T_s z\|^2 \to 0 \text{ as } n \to \infty.$$

Thus, from (3.9) we have $T_s z = z$ for all $s \in S$. □

The following corollary extends and improves Theorem 1 in [2] to the case of a lipschitzian semigroup and it also has a nonconvex domain.

**Corollary 3.2.** *Let $U$ be a nonempty bounded subset of a Hilbert space $H$ and $S$ be a left reversible semitopological semigroup. Let $\mathcal{T} = \{T_s : s \in S\}$ be a Lipschitzian semigroup on $U$ with $\limsup_s k_s < \sqrt{2}$. Suppose that there exists a nonempty bounded closed convex subset $C$ of $U$ with the property $(P)$. Then there exists $z \in C$ such that $T_s z = z$ for all $s \in S$.*

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