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A Generalization of the Trichotomy Principle

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Abstract : We write $X \preceq Y$ if there is an injection from a set X into a set Y and write $X \preceq^* Y$ if $X = \emptyset$ or there is a surjection from Y onto X. For any sets X and Y, $X \preceq Y$ implies $X \preceq^* Y$ but the converse cannot be proved without the Axiom of Choice (AC). The *Trichotomy Principle*, which states that for any sets X and Y, $X \preceq Y$ or $Y \preceq X$, is an equivalent form of AC. Surprisingly, the statement is still equivalent to AC when $\preceq$ is replaced by $\preceq^*$. Moreover, it has been shown that the k*-Trichotomy Principle*, which states that every family of sets which is of cardinality k contains two distinct sets X and Y such that $X \preceq Y$, is equivalent to AC when k is any natural number greater than 1. In this paper, we show that the statement is also equivalent to AC when $\preceq$ is replaced by $\preceq^*$.

Keywords : Axiom of Choice; Trichotomy Principle.
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1 Introduction

We write $X \preceq Y$ if there is an injection from a set X into a set Y and write $X \preceq^* Y$ if $X = \emptyset$ or there is a surjection from Y onto X. For any sets X and Y, $X \preceq Y$ implies $X \preceq^* Y$ but the converse cannot be proved without the Axiom of Choice (AC) [1].

Since the *cardinality* or the size of a set is the number of all elements of a set, if $X \preceq Y$, then the size of X is smaller than or equal to the size of Y. The *Trichotomy Principle* states that for any sets X and Y, $X \preceq Y$ or $Y \preceq X$. It is

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an equivalent form of AC [2]. Thus, without AC, we may not compare the sizes of arbitrary two sets. Obviously, the Trichotomy Principle implies that $X \preceq^* Y$ or $Y \preceq^* X$ for any sets X and Y and this statement seems to be weaker. Surprisingly, in fact, they are equivalent [3].

In [4], D. Feldman and M. Orhon extended the idea of the *Trichotomy Principle* to the *k-Trichotomy Principle*. It states that every family of sets which is of cardinality k contains two distinct sets X and Y such that $X \preceq Y$. This statement seems to be weaker than the Trichotomy Principle when k is a natural number greater than 2. However, it has been shown in [4] that it is also equivalent to AC. Thus it is natural to question that "is the statement still equivalent to AC when $\preceq$ is replaced by $\preceq^*$". This paper gives an affirmative answer to the question. Our result not only gives a surprise that the statement which appears to be weaker than AC is in fact as strong as AC, it also provides an alternative form of AC which will be useful for works concerning AC.

2 Preliminaries

This section gives some background on set theory. All basic notions and notations are used in the ordinary way. We use English capital letters (sometimes with subscripts) for sets. We write $\mathcal{P}(X)$ for the power set of X and $f[X]$ and $f^{-1}[X]$ for the image and the inverse image, respectively, of X under a function f.

All concepts and theorems in Sections 2.1 and 2.2 are standard in Zermelo-Fraenkel set theory (ZF). More details can be found in any set theory textbooks, for example [5].

2.1 Cardinal Numbers

Intuitively, the *cardinality* of a set is the number of all elements of a set. Its exact definition for an arbitrary set is not needed here. We only need to know that it is defined so that any two sets have the same cardinality if and only if there is a bijection between them. We say X is *equinumerous* to Y, denoted by $X \approx Y$, if there is a bijection from X onto Y. We denote the cardinality of X by $|X|$ and call $|X|$ a *cardinal* (*number*). Therefore for any sets X and Y,

$$|X| = |Y| \text{ if and only if } X \approx Y.$$

Each *natural number* is constructed so that it is the set of all smaller natural numbers, namely, $0 = \emptyset, 1 = \{0\}, 2 = \{0,1\}, 3 = \{0,1,2\}, \ldots$ and so on. Let ω denote the set of all natural numbers. We shall use k, l, m, and n for natural numbers.

A set is *finite* if it is equinumerous to a (unique) natural number. Otherwise, it is *infinite*.

For any sets X and Y, there are disjoint sets X' and Y' such that $X \approx X'$ and $Y \approx Y'$ (for example, $X' = X \times \{0\}$ and $Y' = Y \times \{1\}$). Thus, we will simply write $X \dot{\cup} Y$ for the union of some disjoint sets X' and Y' which are equinumerous to X and Y respectively and write lX for the disjoint union of l copies of X.

For any sets X and Y, we define $|X| + |Y| = |X \dot{\cup} Y|$ and $|X| \cdot |Y| = |X \times Y|$.
The operations $+$ and $\cdot$ on cardinal numbers are commutative and associative and $\cdot$ has distributive property over $+$.

We write $X \preceq Y$, if there is an injection from X into Y, and write $X \preceq^* Y$ if $X = \emptyset$ or there is a surjection from Y onto X. We write $X \prec Y$ if $X \preceq Y$ but $X \not\approx Y$, and $X \prec^* Y$ if $X \preceq^* Y$ but $X \not\approx Y$.

For any sets X and Y, if $X \preceq Y$, then $X \preceq^* Y$. The converse cannot be proved in ZF.

Two well-known theorems below will be needed later.

Theorem 2.1 (Cantor-Bernstein Theorem)**.** *For any sets X and Y, if $X \preceq Y$ and $Y \preceq X$, then $X \approx Y$.*

Theorem 2.2 (Cantor's Theorem)**.** *For any set X, $X \prec \mathcal{P}(X)$.*

Some other facts needed for our work are listed below.

For any sets X and Y,

1. if $X \preceq^* Y$, then $\mathcal{P}(X) \preceq \mathcal{P}(Y)$.
2. $\mathcal{P}(X \dot{\cup} Y) \approx \mathcal{P}(X) \times \mathcal{P}(Y)$ and so $\mathcal{P}(lX) \approx \underbrace{\mathcal{P}(X) \times \ldots \times \mathcal{P}(X)}_{l \text{ copies}}$.
3. if X and Y have at least two elements, then $X \cup Y \preceq X \times Y$.

We define $|X| \leq |Y|$ if $X \preceq Y$, $|X| < |Y|$ if $X \prec Y$, $|X| \leq^* |Y|$ if $X \preceq^* Y$, and $|X| <^* |Y|$ if $X \prec^* Y$.

The relation $\leq$ partially orders the class of cardinals. It cannot be proved in ZF that any two cardinals are comparable under $\leq$.

2.2 Well-Ordered Sets and Alephs

A relation R is a *well ordering* on a set X if R linearly orders X and every nonempty subset of X has an R-least element. A set is *well-ordered* if there is a well ordering on it.

We give some simple facts concerning well-ordered sets needed for later work below.

1. A subset of a well-ordered set and a finite union of well-ordered sets are well-ordered.
2. For any sets X and Y, if $X \preceq^* Y$ and Y is well-ordered, then $X \preceq Y$ and X can be well-ordered.

A set X is *transitive* if every member of X is also a subset of X.

An *ordinal* is a transitive set which is well-ordered by $\in$. Natural numbers and ω are ordinals. Note that every member of an ordinal is also an ordinal. We order ordinals by $\in$. This ordering is a well ordering on the class of ordinals. We denote the least uncountable ordinal by ω_1 and let ω_2 denote the least ordinal α such that $\omega_1 \prec \alpha$.

An important fact concerning well-ordered sets is that every well-ordered set is isomorphic to a unique ordinal. This guarantees that every well-ordered set is equinumerous to some ordinals. Thus the cardinality of a well-ordered set can be defined to be the least ordinal equinumerous to it.

For any well-ordered sets X and Y, $|X|$ and $|Y|$ are comparable and if X or Y is infinite, then $|X|+|Y| = \max\{|X|, |Y|\}$. As a result, $X \approx 2X$ if X is an infinite well-ordered set.

The cardinality of an infinite well-ordered set is called an *aleph*.

The following is an important theorem which can be proved in ZF.

Theorem 2.3 (Hartogs' Theorem)**.** *For any set X, there is a least aleph $\aleph$ such that $\aleph \not\leq |X|$.*

In an analogous way, for any set X, there exists a least aleph $\aleph$ such that $\aleph \not\leq^* |X|$. We denote such aleph by $\aleph^*(X)$.

2.3 The Axiom of Choice

The *Axiom of Choice* (AC) is an important axiom in mathematics and is independent from ZF. ZFC denotes ZF with AC. There are many equivalent forms of AC. We state only the forms that will be used in this work below.

Well-Ordering Theorem: every set can be well-ordered.

Trichotomy Principle: for any sets X and Y, $X \preceq Y$ or $Y \preceq X$.

Without AC, it cannot be proved that any two sets are comparable under $\preceq$. Moreover, for an infinite set X, we cannot guarantee whether $\omega \preceq X$ or not. Therefore, in the absence of AC, the following definitions are needed.

A set X is (*weakly*) *Dedekind infinite* if $\omega \preceq X$ ($\omega \preceq^* X$). Otherwise, X is (*weakly*) *Dedekind finite*.

Since $X \preceq Y$ implies $X \preceq^* Y$ for any sets X and Y, every Dedekind infinite set is weakly Dedekind infinite. Equivalently, every weakly Dedekind finite set is Dedekind finite. Without AC, a (weakly) Dedekind finite set needs not be finite. If AC holds, all these concepts of infinity are the same.

Some important properties of these two kinds of infinite sets are the following.

Theorem 2.4. [6, Proposition 4.2] *A set X is Dedekind infinite if and only if $X \approx X \cup A$ for any finite set A.*

Theorem 2.5. [6, Lemma 4.11] *A set X is weakly Dedekind infinite if and only if $\mathcal{P}(X)$ is Dedekind infinite.*

Theorem 2.6. [7, Theorem 4.1] *If there is a proper subset Y of X such that $X \preceq^* Y$, then X is weakly Dedekind infinite.*

For AC, see [6] and [8] for further details.

3 Main Results

As mentioned earlier, the *Trichotomy Principle*, which states that for any sets X and Y, $X \preceq Y$ or $Y \preceq X$, is an equivalent form of AC. The following is another form of the *Trichotomy Principle* which is also equivalent to AC.

Dual Trichotomy Principle: for any sets X and Y, $X \preceq^* Y$ or $Y \preceq^* X$.

The equivalence of the Dual Trichotomy Principle and the Trichotomy Principle is a surprise since for any sets X and Y, $X \preceq Y$ implies $X \preceq^* Y$ but the converse is not necessarily true. For example, $\omega_1 \preceq^* \mathbb{R}$ is provable in ZF but $\omega_1 \preceq \mathbb{R}$ is not [9, page 110]. Since "$\mathbb{R}$ can be well-ordered" is not provable in ZF [9, pages 386-391], nor is $\mathbb{R} \preceq \omega_1$. Therefore, in ZF, ω_1 and $\mathbb{R}$ are not comparable under $\preceq$.

In [4], a generalization of the *Trichotomy Principle* called the *k-Trichotomy Principle* was introduced.

k-Trichotomy Principle: every family of sets which is of cardinality k contains two distinct sets X and Y such that $X \preceq Y$.

Notice that the 2-*Trichotomy Principle* and the *Trichotomy Principle* are the same. It looks like when k becomes greater, the k-Trichotomy Principle becomes weaker. However, it has been shown in [4] that for any natural number $k > 2$, the *k-Trichotomy Principle* is also equivalent to AC.

We now extend the idea of the *k-Trichotomy Principle* to the *k-Dual Trichotomy Principle.*

k-Dual Trichotomy Principle: every family of sets which is of cardinality k contains two distinct sets X and Y such that $X \preceq^* Y$.

As discussed above, without AC, there are more pairs of sets which can be compared by $\preceq^*$ than those by $\preceq$. Thus the k-Dual Trichotomy Principle seems to be weaker than the k-Trichotomy Principle. Also, the principle when $k > 2$ appears to be weaker than the Dual Trichotomy Principle since there are sets that cannot be compared by $\preceq^*$ in ZF. For example, ω_2 and $\mathbb{R}$. If $\mathbb{R} \preceq^* \omega_2$, then $\mathbb{R}$ can be well-ordered which is not provable in ZF. If $\omega_2 \preceq^* \mathbb{R}$ is provable in ZF, then $\omega \prec \omega_1 \prec \omega_2 \preceq \mathbb{R}$ will be provable in ZFC. This contradicts the consistency of the Continuum Hypothesis with ZFC (a famous work by K. Gödel [10–12]). Thus $\{\omega_1, \omega_2, \mathbb{R}\}$ contains two elements that are comparable under $\preceq^*$, while $\{\omega_2, \mathbb{R}\}$ does not. However, when we consider all families of the same cardinality k, things turn out to be different from what it seems to be. We shall show that, in fact, these principles are equivalent.

Our goal is to show that for any natural number $k > 1$, the *k-Dual Trichotomy Principle* is equivalent to AC. Throughout this work, we fix k as a natural number greater than 1. In [4], they showed that the *k-Trichotomy Principle* implies that every infinite set is Dedekind infinite. Analogously, we first show that the *k-Dual Trichotomy Principle* implies that every infinite set is weakly Dedekind infinite.

Lemma 3.1. *Assume the k-Dual Trichotomy Principle and let $A_1, A_2, \ldots, A_k$ be sets such that $A_1 \preceq A_2 \preceq \cdots \preceq A_k$. Then there exist $m, n \le k$ where $n < m$ and a well-ordered set W such that $W \preceq A_m \preceq^* A_n \dot\cup W$.*

Proof. Define $\mu_k = \aleph^*(A_k)$ and $\mu_j = \aleph^*(\mu_{j+1})$ for all $1 \le j < k$. Then $\mu_1 > \mu_2 > \cdots > \mu_k$.

Consider the family $\{A_i \dot\cup \mu_i\}_{i=1}^k$ and apply the *k-Dual Trichotomy Principle*, we obtain $m \neq n$ and $A_m \dot\cup \mu_m \preceq^* A_n \dot\cup \mu_n$, i.e. there exists a surjective map $f : A_n \dot\cup \mu_n \to A_m \dot\cup \mu_m$.

Let $M = f^{-1}[\mu_m] \cap \mu_n$ and $A = f^{-1}[\mu_m \setminus f[M]]$. Then $f[M] \cup f[A] = \mu_m$. Since $A \subseteq A_n \preceq A_k$ and $\aleph^*(A_k) = \mu_k \le \mu_m$, $\mu_m \not\preceq^* A$. Hence $f[A] \not\approx \mu_m$ and so $|f[A]| < \mu_m$. Therefore $|f[M]| = \mu_m$ and thus $\mu_m \le^* |M| \le \mu_n$. This implies that $n < m$.

Let $N = f^{-1}[A_m] \cap \mu_n$. Since N is well-ordered, there is $W \subseteq N$ such that $W \approx f[N] \subseteq A_m$. Thus $W \preceq A_m \preceq^* A_n \dot\cup W$ where W is well-ordered. □

Lemma 3.2. *If X is a weakly Dedekind finite set, then so is lX for all $l \in \omega$.*

Proof. Let X be a weakly Dedekind finite set. The proof proceeds by induction. Assume that lX is weakly Dedekind finite but $(l+1)X$ is weakly Dedekind infinite. Then $\omega \preceq^* (l+1)X \approx lX \dot\cup X$, so there is a surjection $f : lX \dot\cup X \to \omega$. Since $\omega \not\preceq^* lX$ and every infinite subset of ω is equinumerous to ω, $f[lX]$ is finite. Since f is a surjection, $f[X]$ must be infinite. Thus $\omega \approx f[X] \preceq^* X$ but X is weakly Dedekind finite, a contradiction. □

Lemma 3.3. *The k-Dual Trichotomy Principle implies that every infinite set is weakly Dedekind infinite.*

Proof. Assume the *k-Dual Trichotomy Principle*. Suppose there is an infinite weakly Dedekind finite set A. Since $A \preceq 2A \preceq \cdots \preceq kA$, by Lemma 3.1, $W \preceq mA \preceq^* nA \dot\cup W$ for some $n < m$ and some well-ordered set W. By Lemma 3.2, mA is weakly Dedekind finite and so W must be finite (since $W \preceq mA$ where W is well-ordered). Since A is infinite and $n < m$, $nA \dot\cup W \approx X$ for some $X \subsetneq mA$ but $mA \preceq^* nA \dot\cup W$ where mA is weakly Dedekind finite. This contradicts Theorem 2.6. □

Next, we will show that the *k-Dual Trichotomy Principle* implies that every Dedekind infinite set can be well-ordered.

Lemma 3.4. *Let A be an infinite set, W an infinite well-ordered set, and $n > 0$. If $W \preceq nA$, then $W \preceq A$.*

Proof. Assume the statement holds for m and $W \preceq (m+1)A$. Then there is an injection $f : W \to mA \dot\cup A$. Let $X = f[W] \cap mA$ and $Y = f[W] \cap A$. Since W is an infinite well-ordered set and f is injective,

$$|W| = |f[W]| = |X \dot\cup Y| = |X| + |Y| = \max\{|X|, |Y|\}.$$

Hence $W \approx X \subseteq mA$ or $W \approx Y \subseteq A$. Thus $W \preceq mA$ or $W \preceq A$. By the induction hypothesis, we can conclude that $W \preceq A$. □

Lemma 3.5. *Assume the k-Dual Trichotomy Principle.*

If A is a Dedekind infinite set, then $\mathcal{P}(lA) \approx 2\mathcal{P}(lA) \approx \mathcal{P}(2lA)$ for some $l > 0$.

Proof. Let A be a Dedekind infinite set. Since $A \preceq 2A \preceq \cdots \preceq kA$, by Lemma 3.1, there exist $m, n \leq k$ and a well-ordered set W such that $W \preceq mA \preceq^* nA\dot{\cup}W$ where $0 < n < m$. Since A is Dedekind infinite, so is nA. If W is finite, then $nA\dot{\cup}W \approx nA$. Suppose W is infinite. Then $W \approx 2W$. Since $W \preceq mA$, by Lemma 3.4, $W \preceq A \preceq nA$. Hence there is a set X such that

$$nA \approx X\dot{\cup}W \approx X\dot{\cup}2W \approx (X\dot{\cup}W)\dot{\cup}W \approx nA\dot{\cup}W.$$

Thus $mA \preceq^* nA$ and so $\mathcal{P}(mA) \preceq \mathcal{P}(nA)$. Since $nA \preceq mA$, $\mathcal{P}(nA) \preceq \mathcal{P}(mA)$. By the *Cantor-Bernstein Theorem*, $\mathcal{P}(nA) \approx \mathcal{P}(mA)$.

Since $\mathcal{P}(mA) \approx \mathcal{P}(nA) \times \mathcal{P}((m-n)A) \approx \mathcal{P}(mA) \times \mathcal{P}((m-n)A)$,

$$\mathcal{P}(nA) \approx \mathcal{P}(mA) \times \underbrace{\mathcal{P}((m-n)A) \times \ldots \times \mathcal{P}((m-n)A)}_{n \text{ copies}} \approx \mathcal{P}((m+n(m-n))A).$$

Let $r = m + n(m-n)$. Then $\mathcal{P}(nA) \approx \mathcal{P}(rA)$ where $r > 2n$. Thus

$$\begin{aligned}\mathcal{P}((r-n)A) &= \mathcal{P}((n+r-2n)A)\\ &\approx \mathcal{P}(nA) \times \mathcal{P}((r-2n)A)\\ &\approx \mathcal{P}(rA) \times \mathcal{P}((r-2n)A)\\ &= \mathcal{P}(2(r-n)A).\end{aligned}$$

Let $l = r - n$. Then $\mathcal{P}(2lA) \approx \mathcal{P}(lA) \preceq 2\mathcal{P}(lA) \preceq \mathcal{P}(lA) \times \mathcal{P}(lA) \approx \mathcal{P}(2lA)$. Hence $\mathcal{P}(lA) \approx 2\mathcal{P}(lA) \approx \mathcal{P}(2lA)$. □

The following definition and Lemma 3.7 are by A. Blass [4, Appendix].

Definition 3.6. For any set X, define $Q(X) = Q^1(X) = X \times \mathcal{P}(X)$ and $Q^{i+1}(X) = Q(Q^i(X))$ for all $i \geq 1$.

Note that, by Cantor's Theorem, $X \prec Q(X)$ for any nonempty set X.

Lemma 3.7. [4, Lemma 7] *For any sets X and Y, if Y is well-ordered, then an injection from $Q(X)$ into $X\dot{\cup}Y$ induces a canonical well ordering of X.*

Lemma 3.8. *The k-Dual Trichotomy Principle implies that every Dedekind infinite set can be well-ordered.*

Proof. Fix a Dedekind infinite set A. By Lemma 3.5, there exists $l > 0$ such that $\mathcal{P}(lA) \approx 2\mathcal{P}(lA) \approx \mathcal{P}(2lA)$. Let $B = lA$. Then $\mathcal{P}(B) \approx 2\mathcal{P}(B) \approx \mathcal{P}(2B)$. It follows straightforwardly by induction that $\mathcal{P}(Q^p(B)) \approx \mathcal{P}(2Q^p(B))$ for all natural numbers $p \geq 1$.

Since $Q(B) \prec Q^2(B) \prec \cdots \prec Q^k(B)$, by Lemma 3.1, there exist $m, n \leq k$ where $n < m$ and a well-ordered set W such that $Q^m(B) \preceq^* Q^n(B)\dot{\cup}W$. Let $f\colon Q^n(B)\dot{\cup}W \to Q^m(B)$ be a surjection, $X = f[Q^n(B)]$ and $Y = Q^m(B) \setminus X$.

Then $Q^m(B) = X\dot{\cup}Y$ where $X \preceq^* Q^{m-1}(B)$ and Y is well-ordered since $f^{-1}[Y] \subseteq W$. Hence $X \prec \mathcal{P}(X) \preceq \mathcal{P}(Q^{m-1}(B))$ and so

$$\begin{aligned} Q(X) &= X \times \mathcal{P}(X) \\ &\preceq \mathcal{P}(Q^{m-1}(B)) \times \mathcal{P}(Q^{m-1}(B)) \\ &\approx \mathcal{P}(2Q^{m-1}(B)) \\ &\approx \mathcal{P}(Q^{m-1}(B)) \\ &\preceq Q^{m-1}(B) \times \mathcal{P}(Q^{m-1}(B)) = Q^m(B) = X\dot{\cup}Y. \end{aligned}$$

By Lemma 3.7, X can be well-ordered and so can $Q^m(B)$. Since $A \preceq B \prec Q(B) \preceq Q^m(B)$, A can be well-ordered. □

Now, we are ready to prove our main theorem.

Theorem 3.9. *The k-Dual Trichotomy Principle implies AC.*

Proof. Assume the *k-Dual Trichotomy Principle* and let A be an infinite set. By Lemma 3.3, A is weakly Dedekind infinite and hence $\mathcal{P}(A)$ is Dedekind infinite. By Lemma 3.8, $\mathcal{P}(A)$ can be well-ordered. Since $A \prec \mathcal{P}(A)$, A can also be well-ordered. □

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