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Numerical Solutions to the Rosenau–Kawahara Equation for Shallow Water Waves via Pseudo–Compact Methods

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Abstract : This paper presents two linear finite difference schemes for the so–called Rosenau–Kawahara equation, modified from a linear scheme by Hu et al. in 2014, under a pseudo–compact method. Existence and uniqueness of solutions generated by both schemes are proved. It is shown that the first scheme possesses some conservation properties for mass and energy, whereas the other proposed scheme provides only mass conservation. Some discussions on stability are given, which reveal that numerical solutions are stable with respect to $\|\cdot\|_\infty$. It is also shown that pseudo–compactness allows some terms in the schemes to reach fourth–order convergence, even though the numerical solutions is of second–order convergence overall. Furthermore, numerical simulations are illustrated confirming that our schemes induce some improvements over the existing scheme by Hu et al. on precision and cost.

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Keywords : Rosenau–Kawahara equation; finite difference method; convergence; stability.

1 Introduction

The computation regarding nonlinear waves has been recently of great interest. Shallow water waves, in particular, are one of the main aspects in oceanography and atmospheric science. Equatorial waves, a type of waves trapped near the equator, can also be explained by shallow water equations. It has been realized that equatorial waves are one of the key ingredients in the study of tropical climates and the El Niño Southern Oscillation (ENSO). Besides, theory of shallow water waves has been linked to efficiency of wave energy converters. As demand for energy consumption increases, sea waves are one of the most promising sustainable sources. Wave energy converters are typically installed in deep water due to greater energy production. However, nearshore devices can reduce installation cost, maintenance cost, and power losses in the cable. Although the amount of energy generated at sites from offshore to nearshore is reduced, it is claimed that the exploitable amount from both sites may not differ much. In [1], Folley and Whittaker compared exploitable energy reduced from devices placed in deep and shallow water at two sites in the Scottish seacoast, and found that the reduction was only 7% and 22%. This paves the way for future research on efficiency improvements of nearshore converters, and suggests that better understanding in nonlinear shallow water waves may help develop new technologies for renewable and sustainable energy production.

There have appeared to be a huge number of publications contributing to the theory of nonlinear waves through various mathematical models, see [2]– [5] for the Korteweg–de Vries (KdV) equation; see [6]– [12] for the regularized long wave (RLW) equation; see [13]– [18] for the Rosenau equation; see [19] for the Rosenau–KdV equation; see [20]– [21] for the Rosenau–RLW equation; to name but a few.

The Rosenau equation

$$u_t + u_{xxxxt} + u_x + uu_x \;=\; 0\,, \tag{1.1}$$

has been used by many authors in the study of shallow water waves. It turns out that seeking an analytic solution to (1.1) is not an easy task. In [22], Zuo proposed an idea of adding some viscous terms $+u_{xxx}$ and $-u_{xxxxx}$ to the Rosenau equation (1.1), and gave an exact periodic and solitary wave solution

$$u(x,t) = \left(-\frac{35}{12} + \frac{35}{156}\sqrt{205}\right)\operatorname{sech}^4\left[\frac{1}{12}\sqrt{-13+\sqrt{205}}\left(x - \frac{1}{13}\sqrt{205}\,t\right)\right] \tag{1.2}$$

to this modified equation. This gives rise to another nonlinear wave model, the so–called Rosenau–Kawahara equation, to which the viscous terms $+\beta u_{xxx}$ and

$-\gamma u_{xxxxx}$ are introduced. Numerous techniques have been developed in order to achieve better results. In [23], He's principle, Variational Iteration Method (VIM), was employed by Labidi and Bitwas to perform the integration of the Rosenau–Kawahara equation. Finite difference schemes have also come into play. In [24], a three–level linear and a two–level nonlinear Crank-Nicolson schemes were shown, by Hu et al., to be conservative. It was also shown that both schemes provided stability results and convergence of second order. Recently, in [25], a semi–explicit (semi–implicit) linearized scheme was used to yield results for the generalized Rosenau–Kawahara equation. A comparison between semi-implicit and purely implicit schemes has been discussed in the work of Koley's, [26]. It turns out that purely implicit schemes is a more efficient tool. The reader may be also referred to [27]– [28] for more papers concerning the Rosenau–Kawahara equation, and to [29]– [30] for more details regarding some other finite difference methods in nonlinear wave studies.

This paper aims to present two linear schemes and establish numerical results regarding the Rosenau–Kawahara equation of the form

$$u_t + u_{xxxxt} + \alpha u_x + \beta u_{xxx} - \gamma u_{xxxxx} + \eta(u^2)_x = 0, \quad x \in (x_L, x_R), \quad t > 0 \tag{1.3}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad x \in [x_L, x_R] \tag{1.4}$$

and the boundary conditions:

$$\begin{aligned} u(x_L,t) = u_{xx}(x_L,t) = u_{xxxx}(x_L,t) = 0, \\ u(x_R,t) = u_{xx}(x_R,t) = u_{xxxx}(x_R,t) = 0 \end{aligned} \tag{1.5}$$

$t \in [0,T]$, via a pseudo–compact finite difference method. It is shown that the former scheme preserves mass whereas the latter conserves both mass and energy. Moreover, the numerical accuracy of the latter is improved, compared to Hu's linear scheme in [24].

The structure of the paper is as follows. Section 2 provides a brief description of finite difference method and collects some lemmas that will be used throughout the work. Sections 3 and 4 present our linear schemes, and also discuss their conservation properties, convergence, solvability and stability. Section 5 illustrates the numerical results and shows the improvement in accuracy. Last but not least, Section 6 is devoted for the conclusions.

2 Finite Difference Method

Let $\Omega = \{(x,t)|\ x_L \le x \le x_R,\ 0 \le t \le T\} \subset \mathbb{R} \times \mathbb{R}_0^+$ be the domain of the solutions to the Rosenau–Kawahara equation (1.3) with initial and boundary conditions (1.4) and (1.5). Discretize the domain Ω in space and time by constant grid spacing

$$\tau = \frac{T}{N} \qquad \text{and} \qquad h = \frac{x_R - x_L}{M},$$

respectively, where N and M are integers. This defines the set of computational grid points denoted by

$$\Omega_h = \{(x_i, t_n)\,|x_i = x_L + ih,\ t_n = n\tau,\ i = -1, 0, 1, \ldots, M, M+1,\ n = 0, 1, 2, \ldots, N\}.$$

Each grid point (x_i, t_n) will be tagged by the exact solution $u(x_i, t_n)$, and by the associated numerical solution $u_i^n \approx u(x_i, t_n)$. Moreover, the following finite difference approximations are given for convenient use:

$$(u_i^n)_x = \frac{u_{i+1}^n - u_i^n}{h}, \qquad (u_i^n)_{\bar{x}} = \frac{u_i^n - u_{i-1}^n}{h}, \qquad (u_i^n)_{\hat{x}} = \frac{u_{i+1}^n - u_{i-1}^n}{2h},$$

$$(u_i^n)_t = \frac{u_i^{n+1} - u_i^n}{\tau}, \qquad (u_i^n)_{\bar{t}} = \frac{u_i^n - u_i^{n-1}}{\tau}, \qquad (u_i^n)_{\hat{t}} = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau},$$

$$\bar{u}_i^n = \frac{u_i^{n+1} + u_i^{n-1}}{2}, \qquad \langle u^n, v^n \rangle = h \sum_{i=1}^{M-1} u_i^n v_i^n, \qquad \|u^n\|^2 = \langle u^n, u^n \rangle$$

and $\|u^n\|_\infty = \max_{1 \le i \le M-1} |u_i^n|$.

Let us now recall some Sobolev spaces relevant to this work. Denote by

$$H^k(\Omega) = \{u \in L^2(\Omega)\,|\quad \frac{\partial^i u}{\partial x^i} \in L^2(\Omega), i = 0, 1, ..., k\}$$

the vector space containing square-integrable real-valued functions whose all spatial partial derivatives (up to order k) are also square-integrable, and by

$$H_0^k(\Omega) = \{u \in H^k(\Omega)\,|\quad \frac{\partial^i u}{\partial x^i} = 0 \text{ on } \partial\Omega, \quad i = 0, 1, ..., k-1\}$$

the subspace of $H^2(\Omega)$ containing functions whose all spatial partial derivatives (up to order k) vanish on the boundary. Both spaces are equipped with the usual Sobolev H^k–norm denoted by $\|\cdot\|_{H^k}$. For $k = 0$, the Sobolev space $H^0(\Omega) = L^2(\Omega)$ enjoys the L^2–norm $\|\cdot\|_{L^2}$ and the inner product $(\cdot, \cdot)$. Also, the norm of $L^\infty(\Omega)$ is denoted by $\|\cdot\|_{L^\infty}$.

The following lemmas will be used later on. The proofs can be found in [24], [31]. It is worth remarking that the positive constant C in our calculation, independent of h and τ, may have different values for different occurrences.

Now let us set

$$Z_h^0 = \{u = (u_i)|u_{-1} = u_0 = u_1 = u_2 = u_{M-2} = u_{M-1} = u_M = u_{M+1} = 0,$$
$$i = -1, 0, 1, \ldots, M, M+1\}.$$

Lemma 2.1. *For any two mesh functions $u, v \in Z_h^0$, we have*

$$\begin{aligned}(u_{\hat{x}}, v) &= -(u, v_{\hat{x}}),\\ (u_x, v) &= -(u, v_{\bar{x}}),\\ (v, u_{x\bar{x}}) &= -(v_x, u_x),\\ (u, u_{x\bar{x}}) &= -(u_x, u_x) = -\|u_x\|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}(u, u_{xx\bar{x}\bar{x}}) &= \|u_{x\bar{x}}\|^2\\ (u, u_{xxx\bar{x}\bar{x}\bar{x}}) &= -\|u_{xx\bar{x}}\|^2.\end{aligned}$$

Lemma 2.2. *For $u, v \in Z_h^0$ be any mesh function. Then, the following hold;*

1. $(u(v)_{\hat{x}}, v) = (uv, v_{\hat{x}})$.
2. $\|u^n_{xx\bar{x}}\|^2 \le \dfrac{4}{h^2}\|u^n_{x\bar{x}}\|^2$.
3. $\|u^n_{x\bar{x}\hat{x}}\|^2 \le \dfrac{4}{h^2}\|u^n_{\bar{x}\hat{x}}\|^2$.

Proof. (1) Consider

$$\begin{aligned}(u(v)_{\hat{x}}, v) &= h\sum_{i=1}^{M-1}[u(v)_{\hat{x}}v]\\ &= h\sum_{i=1}^{M-1}[uv(v)_{\hat{x}}]\\ &= (uv, v_{\hat{x}}).\end{aligned}$$

(2) We can reduce $\|u^n_{xx\bar{x}}\|^2$ to the relation

$$\|u^n_{xx\bar{x}}\|^2 \le \frac{2}{h}\sum_{i=1}^{M-1}\left[[(u^n_{i+1})_{x\bar{x}}]^2 + [(u^n_i)_{x\bar{x}}]^2\right] = \frac{4}{h^2}\|u^n_{x\bar{x}}\|^2.$$

(3) We can estimate

$$\|u^n_{x\bar{x}\hat{x}}\|^2 \le \frac{2}{h}\sum_{i=1}^{M-1}\left[[(u^n_{i+1})_{\bar{x}\hat{x}}]^2 + [(u^n_i)_{\bar{x}\hat{x}}]^2\right] = \frac{4}{h^2}\|u^n_{\bar{x}\hat{x}}\|^2.$$

□

Lemma 2.3. (Discrete Sobolev's inequality [31]) *There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \le C_1\|u^n\| + C_2\|u^n_x\|.$$

Lemma 2.4. (Discrete Gronwall's inequality [31]) *Suppose that $\omega(k)$ and $\rho(k)$ are nonnegative functions and $\rho(k)$ is nondecreasing. If $C > 0$ and*

$$\omega(k) \leq \rho(k) + C\tau \sum_{l=0}^{k-1} \omega(l) \quad for\ all\ k\,,$$

then

$$\omega(k) \leq \rho(k) e^{C\tau k} \quad for\ all\ k\,.$$

Lemma 2.5. (Hu et.al. [24]) *Suppose that $u_0 \in H_0^2[x_L, x_R]$, then the solution u^n of* (1.3)–(1.5) *satisfies*

$$\|u\|_{L_2} \leq C, \qquad \|u_x\|_{L_2} \leq C, \qquad \|u_{xx}\|_{L_2} \leq C, \qquad \|u\|_{L_\infty} \leq C.$$

3 A Linear Conservative Difference Scheme 1

We now propose a linear finite difference scheme for the Rosenau–Kawahara equation (1.3) with conditions (1.4) and (1.5):

$$(u_i^n)_{\hat{t}} + \frac{h^2}{3}(u_i^n)_{x\bar{x}\hat{t}} + (u_i^n)_{xx\bar{x}\bar{x}\hat{t}} + \frac{h^2}{6}(u_i^n)_{xxx\bar{x}\bar{x}\bar{x}\hat{t}} + \alpha(\bar{u}_i^n)_{\hat{x}} + s_1(\bar{u}_i^n)_{x\bar{x}\hat{x}} - s_2(\bar{u}_i^n)_{xx\bar{x}\bar{x}\hat{x}} + \frac{\eta}{3}\left[u_i^n(\bar{u}_i^n)_{\hat{x}} + (u_i^n\bar{u}_i^n)_{\hat{x}}\right] = 0; \tag{3.1}$$

$1 \leq i \leq M-1$, $1 \leq n \leq N-1$, where

$$u_i^0 = u_0(x_i), \quad -2 \leq i \leq M+2, \tag{3.2}$$

$$u_0^n = u_M^n = 0, \;\; (u_0^n)_{\hat{x}} = (u_M^n)_{\hat{x}} = 0, \;\; (u_0^n)_{x\bar{x}} = (u_M^n)_{x\bar{x}} = 0, \quad 1 \leq n \leq N, \tag{3.3}$$

where $s_1 = \beta + \frac{\alpha h^2}{3}$ and $s_2 = \gamma - \frac{\beta h^2}{12}$.

Before we proceed, let us state a fact concerning conservation of mass and energy at each time step, deduced by the scheme.

Theorem 3.1. *Suppose $u_0 \in H_0^2[x_L, x_R]$. Using the scheme* (3.1)–(3.3), *the discrete mass Q^n and the discrete energy E^n are conserved. That is,*

$$Q^n = \frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1} + u_i^n\right) + \frac{\eta h\tau}{6}\sum_{i=1}^{M-1} u_i^n \left(u_i^{n+1}\right)_{\hat{x}} = Q^{n-1} = \cdots = Q^0 \tag{3.4}$$

and

$$\begin{aligned} E^n &= (\|u^n\|^2 + \|u^{n-1}\|^2) - \frac{h^2}{3}(\|u_x^n\|^2 + \|u_x^{n-1}\|^2) + (\|u_{x\bar{x}}^n\|^2 + \|u_{x\bar{x}}^{n-1}\|^2) \\ &\quad - \frac{h^2}{6}(\|u_{x\bar{x}x}^n\|^2 + \|u_{x\bar{x}x}^{n-1}\|^2) \\ &= E^{n-1} = \cdots = E^0. \end{aligned} \tag{3.5}$$

Proof. We first multiply (3.1) by h and sum all the terms up from $i = 1$ to $M - 1$. The boundary conditions then provide

$$\begin{aligned} &h\sum_{i=1}^{M-1}\frac{\left(u_i^{n+1}-u_i^{n-1}\right)}{2\tau}+\frac{h}{3}\sum_{i=1}^{M-1}u_i^n\left(\bar{u}_i{}^n\right)_{\hat{x}} \\ &= h\sum_{i=1}^{M-1}\frac{\left(u_i^{n+1}-u_i^{n-1}\right)}{2\tau}+\frac{h}{6}\sum_{i=1}^{M-1}u_i^n\left(u_i^{n+1}\right)_{\hat{x}}-\frac{h}{6}\sum_{i=1}^{M-1}u_i^{n-1}\left(u_i^n\right)_{\hat{x}} \\ &= 0. \end{aligned}$$

This implies the conservation of mass (3.4).

For the conservation of energy, we first take the inner product of (3.1) with $2\bar{u}^n$, and then use Lemmas 2.1 and 2.2.

$$(\bar{u}^n_{\hat{x}}, \bar{u}^n) = 0, (\bar{u}^n_{x\bar{x}\hat{x}}, \bar{u}^n) = 0, (\bar{u}^n_{xx\bar{x}\bar{x}\hat{x}}, \bar{u}^n) = 0, \tag{3.6}$$

We thus obtain

$$\begin{aligned} (\|u^{n+1}\|^2-\|u^{n-1}\|^2)&-\frac{h^2}{3}(\|u_x^{n+1}\|^2-\|u_x^{n-1}\|^2)+(\|u_{x\bar{x}}^{n+1}\|^2-\|u_{x\bar{x}}^{n-1}\|^2) \\ &-\frac{h^2}{6}(\|u_{xx\bar{x}}^{n+1}\|^2-\|u_{xx\bar{x}}^{n-1}\|^2) \\ &=-\frac{2\eta}{3}\left(\left[u^n(\bar{u}^n)_{\hat{x}}+(u^n\bar{u}^n)_{\hat{x}}\right],\bar{u}^n\right) \\ &=-\frac{2\eta}{3}\left[(u^n(\bar{u}^n)_{\hat{x}},\bar{u}^n)+((u^n\bar{u}^n)_{\hat{x}},\bar{u}^n)\right] \\ &=-\frac{2\eta}{3}\left[(u^n(\bar{u}^n)_{\hat{x}},\bar{u}^n)-((u^n\bar{u}^n),\bar{u}^n_{\hat{x}})\right] \\ &=0. \end{aligned} \tag{3.7}$$

By the definition of E^n, we have

$$\begin{aligned} E^n=(\|u^n\|^2+\|u^{n-1}\|^2)&-\frac{h^2}{3}(\|u_x^n\|^2+\|u_x^{n-1}\|^2)+(\|u_{x\bar{x}}^n\|^2+\|u_{x\bar{x}}^{n-1}\|^2) \\ -\frac{h^2}{6}(\|u_{x\bar{x}x}^n\|^2+\|u_{x\bar{x}x}^{n-1}\|^2),& \end{aligned}$$

implying that

$$\begin{aligned} E^{n+1}-E^n=(\|u^{n+1}\|^2-\|u^{n-1}\|^2)&-\frac{h^2}{3}(\|u_x^{n+1}\|^2-\|u_x^{n-1}\|^2) \\ +(\|u_{x\bar{x}}^{n+1}\|^2-\|u_{x\bar{x}}^{n-1}\|^2)&-\frac{h^2}{6}(\|u_{xx\bar{x}}^{n+1}\|^2-\|u_{xx\bar{x}}^{n-1}\|^2)=0, \end{aligned}$$

which proves (3.5). □

3.1 Solvability

The following theorem guarantees that our scheme can produce a unique solution.

Theorem 3.2. *The finite difference scheme* (3.1)–(3.3) *is uniquely solvable.*

Proof. We will prove the theorem by induction on time levels n. Notice, first of all, that u^0 is obtained uniquely from the initial conditions, and that u^1 is computed by a second–order method.

Now, suppose that $u^0, u^1, u^2, ..., u^n$ are solved uniquely. By considering (3.1) for u^{n+1}, we have

$$\frac{1}{2\tau}u_i^{n+1} + \frac{1}{2\tau}\frac{h^2}{3}(u_i^{n+1})_{x\bar{x}} + \frac{1}{2\tau}(u_i^{n+1})_{xx\bar{x}\bar{x}} + \frac{1}{2\tau}\frac{h^2}{6}(u_i^{n+1})_{xxx\bar{x}\bar{x}\bar{x}} + \frac{\alpha}{2}(u_i^{n+1})_{\hat{x}} + \frac{s_1}{2}(u_i^{n+1})_{x\bar{x}\hat{x}} - \frac{s_2}{2}(u_i^{n+1})_{xx\bar{x}\bar{x}\hat{x}} + \frac{\eta}{6}\left[u_i^n(u_i^{n+1})_{\hat{x}} + \left(u_i^n u_i^{n+1}\right)_{\hat{x}}\right] = 0. \quad (3.8)$$

By taking the inner product of (3.8) with u^{n+1} and using the identities

$$(u_{\hat{x}}^{n+1}, u^{n+1}) = 0, \quad (u_{x\bar{x}\hat{x}}^{n+1}, u^{n+1}) = 0, \quad (u_{xx\bar{x}\bar{x}\hat{x}}^{n+1}, u^{n+1}) = 0, \quad (3.9)$$

we obtain

$$\|u^{n+1}\|^2 - \frac{h^2}{3}\|u_x^{n+1}\|^2 + \|u_{x\bar{x}}^{n+1}\|^2 - \frac{h^2}{6}\|u_{xx\bar{x}}^{n+1}\|^2 + \frac{\eta}{6}\left(\left[u_i^n(u_i^{n+1})_{\hat{x}} + \left(u_i^n u_i^{n+1}\right)_{\hat{x}}\right], u_i^{n+1}\right) = 0.$$

Lemmas 2.1 and 2.2 then imply

$$\begin{aligned}\left(\left[u_i^n(u_i^{n+1})_{\hat{x}} + \left(u_i^n u_i^{n+1}\right)_{\hat{x}}\right], u_i^{n+1}\right) &= \left(u_i^n(u_i^{n+1})_{\hat{x}}, u_i^{n+1}\right) + \left(\left(u_i^n u_i^{n+1}\right)_{\hat{x}}, u_i^{n+1}\right) \\ &= \left(u_i^n(u_i^{n+1})_{\hat{x}}, u_i^{n+1}\right) - \left(u_i^n u_i^{n+1}, (u_i^{n+1})_{\hat{x}}\right) \\ &= 0.\end{aligned}$$

It follows from Cauchy–Schwarz inequality that

$$\left\|u_x^{n+1}\right\|^2 = -(u^{n+1}, u_{x\bar{x}}^{n+1}) \leq \frac{1}{2}\left\|u^{n+1}\right\|^2 + \frac{1}{2}\left\|u_{x\bar{x}}^{n+1}\right\|^2. \quad (3.10)$$

Next, applying Lemma 2.2 to (3.8), we get

$$(\frac{1}{3} - \frac{h^2}{6})\|u^{n+1}\|^2 + (\frac{1}{3} - \frac{h^2}{6})\|u_{x\bar{x}}^{n+1}\|^2 = 0.$$

Therefore, (3.8) has the only one solution; that is, the scheme (3.1) u^{n+1} is uniquely solvable. This completes the proof of Theorem 3.2. □

3.2 Convergence and Stability

Let us now have some discussion on the convergence and stability of the scheme (3.1)–(3.3). Let $v_i^n = v(x_i, t_n)$ be the solution to (1.3)–(1.5). Set $e_i^n = v_i^n - u_i^n$. Then, we obtain the following truncation error:

$$r_i^n = (e_i^n)_{\hat{t}} + \frac{h^2}{3}(e_i^n)_{x\bar{x}\hat{t}} + (e_i^n)_{xx\bar{x}\bar{x}\hat{t}} + \frac{h^2}{6}(e_i^n)_{xxx\bar{x}\bar{x}\bar{x}\hat{t}} + \alpha(\bar{e}_i^n)_{\hat{x}} + s_1(\bar{e}_i^n)_{x\bar{x}\hat{x}}$$
$$- s_2(\bar{e}_i^n)_{xx\bar{x}\bar{x}\hat{x}} + \frac{\eta}{3}\left[v_i^n(\bar{v}_i^n)_{\hat{x}} + (v_i^n\bar{v}_i^n)_{\hat{x}}\right] - \frac{\eta}{3}\left[u_i^n(\bar{u}_i^n)_{\hat{x}} + (u_i^n\bar{u}_i^n)_{\hat{x}}\right]. \quad (3.11)$$

Using Taylor expansion, it is easy to see that

$$r_i^n = O(\tau^2 + h^2) \quad \text{as} \quad \tau, h \to 0.$$

The following theorem is required for the proof of convergence and stability of our scheme.

Theorem 3.3. *Suppose that $u_0 \in H_0^2[x_L, x_R]$. Then, the solution u^n to (3.1)–(3.3) satisfies*

$$\|u^n\| \le C, \quad \|u_x^n\| \le C, \quad \|u_{x\bar{x}}^n\| \le C\,,$$

that is,

$$\|u^n\|_\infty \le C, \quad \|u_x^n\|_\infty \le C \quad (n = 1, 2, 3, ..., N).$$

Proof. From (3.5), we know that

$$E^n = (\|u^n\|^2 + \|u^{n-1}\|^2) - \frac{h^2}{3}(\|u_x^n\|^2 + \|u_x^{n-1}\|^2) + (\|u_{x\bar{x}}^n\|^2 + \|u_{x\bar{x}}^{n-1}\|^2)$$
$$- \frac{h^2}{6}(\|u_{x\bar{x}x}^n\|^2 + \|u_{x\bar{x}x}^{n-1}\|^2) = E^0.$$

According to Cauchy-Schwarz inequality and Lemma 2.2, we have

$$(\frac{1}{3} - \frac{h^2}{6})\left[\|u^n\|^2 + \|u^{n-1}\|^2 + \|u_{x\bar{x}}^n\|^2 + \|u_{x\bar{x}}^{n-1}\|^2\right] \le C.$$

It follows that

$$\|u^n\| \le C \text{ and } \|u_{x\bar{x}}^n\| \le C$$

and, by (3.10), hence

$$\|u_x^n\| \le C\,.$$

Then, by Lemma 2.3, we finally obtain

$$\|u^n\|_\infty \le C \quad \text{and} \quad \|u_x^n\|_\infty \le C \qquad (n = 1, 2, 3, ..., N). \quad \square$$

The following shows that our scheme produces a solution with convergence rate $O(\tau^2 + h^2)$.

Theorem 3.4. *Suppose $u_0 \in H_0^2[x_L, x_R]$. Then, the solution u^n converges (with respect to $\|\cdot\|_\infty$) to the solution to the problem* (1.3)–(1.5) *with rate of convergence is $O(\tau^2 + h^2)$.*

Proof. By taking the inner product on both sides of (3.11) with $2\bar{e}^n = (e^{n+1} + e^{n-1})$ and by using

$$(\bar{e}^n_{\hat{x}}, \bar{e}^n) = 0, \quad (\bar{e}^n_{x\bar{x}\hat{x}}, \bar{e}^n) = 0, \quad (\bar{e}^n_{xx\bar{x}\bar{x}\hat{x}}, \bar{e}^n) = 0, \tag{3.12}$$

we get

$$\begin{aligned}
&(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) - \frac{h^2}{3}(\|e^{n+1}_x\|^2 - \|e^{n-1}_x\|^2) + (\|e^{n+1}_{x\bar{x}}\|^2 - \|e^{n-1}_{x\bar{x}}\|^2) \\
&\quad - \frac{h^2}{6}(\|e^{n+1}_{x\bar{x}x}\|^2 - \|e^{n-1}_{x\bar{x}x}\|^2) \\
&= 2\tau(r^n, 2\bar{e}^n) - \frac{\eta\tau}{3}\left([v^n(\bar{v}^n)_{\hat{x}} + (v^n\bar{v}^n)_{\hat{x}}], 2\bar{e}^n\right) + \frac{\eta\tau}{3}\left([u^n(\bar{u}^n)_{\hat{x}} + (u^n\bar{u}^n)_{\hat{x}}], 2\bar{e}^n\right).
\end{aligned} \tag{3.13}$$

Due to Cauchy-Schwarz inequality, Lemma 2.1, Theorem 3.1, and Lemma 2.5, we obtain the inequalities:

$$\|e^n_x\| = -(e^n, e^n_{x\bar{x}}) \leq \frac{1}{2}\left(\|e^n\|^2 + \|e^n_{x\bar{x}}\|^2\right), \tag{3.14}$$

$$\begin{aligned}
&-\frac{\eta}{3}\left([v^n(\bar{v}^n)_{\hat{x}} + (v^n\bar{v}^n)_{\hat{x}}], 2\bar{e}^n\right) + \frac{\eta}{3}\left([u^n(\bar{u}^n)_{\hat{x}} + (u^n\bar{u}^n)_{\hat{x}}], 2\bar{e}^n\right) \\
&\quad \leq C(\|e^{n-1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}_{x\bar{x}}\|^2 + \|e^{n+1}_{x\bar{x}}\|^2).
\end{aligned} \tag{3.15}$$

and

$$(r^n, 2\bar{e}^n) \leq \|r^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right). \tag{3.16}$$

Applying (3.14)–(3.16) to(3.13) and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) - \frac{h^2}{3}(\|e^{n+1}_x\|^2 - \|e^{n-1}_x\|^2) + (\|e^{n+1}_{x\bar{x}}\|^2 - \|e^{n-1}_{x\bar{x}}\|^2) \\
&\quad - \frac{h^2}{6}(\|e^{n+1}_{xx\bar{x}}\|^2 - \|e^{n-1}_{xx\bar{x}}\|^2) \qquad (3.17) \\
&\quad \leq 2\tau\|r^n\|^2 + \tau C(\|e^{n-1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}_{x\bar{x}}\|^2 + \|e^{n+1}_{x\bar{x}}\|^2).
\end{aligned} \tag{3.18}$$

Let us set

$$\begin{aligned}
E^n =&(\|e^n\|^2 + \|e^{n-1}\|^2) - \frac{h^2}{3}(\|e^n_x\|^2 + \|e^{n-1}_x\|^2) + (\|e^n_{x\bar{x}}\|^2 + \|e^{n-1}_{x\bar{x}}\|^2) \\
&- \frac{h^2}{6}(\|e^n_{xx\bar{x}}\|^2 + \|e^{n-1}_{xx\bar{x}}\|^2).
\end{aligned}$$

Using (3.14) and Lemma 2.2, (3.17) can be rewritten as

$$E^{n+1} - E^n \le 2\tau\|r^n\|^2 + \tau C(E^{n+1} + E^n),$$

implying that

$$(1 - 2\tau C)\left(E^{n+1} - E^n\right) \le 2\tau\|r^n\|^2 + 2\tau C E^n.$$

If τ is sufficiently small, which satisfies $1 - 2C\tau > 0$, then

$$E^{n+1} - E^n \le \tau C\|r^n\|^2 + \tau C E^n. \tag{3.19}$$

Since (3.19) holds for every n, it can be summed up from $k = 1$ to n to become

$$E^{n+1} \le E^1 + C\tau \sum_{k=1}^{n} \left\|r^k\right\|^2 + C\tau \sum_{k=1}^{n} E^k. \tag{3.20}$$

Thus, we can use a second–order method to compute u^1 such that

$$E^1 \le O(\tau^2 + h^2)^2,$$

and

$$\tau \sum_{k=1}^{n} \left\|r^k\right\|^2 \le n\tau \max_{0\le l\le n-1} \left\|r^l\right\|^2 \le T \cdot O(\tau^2 + h^2)^2.$$

By Lemma 2.3, we obtain $E^n \le O(\tau^2 + h^2)^2$. Next, applying Lemma 2.2 to (3.13), we have

$$(1-\frac{h^2}{6})\|e^n\|^2+(1-\frac{h^2}{6})\|e^{n-1}\|^2+(\frac{1}{3}-\frac{h^2}{6})\|e^n_{x\bar{x}}\|^2+(\frac{1}{3}-\frac{h^2}{6})\|e^{n-1}_{x\bar{x}}\|^2 \le O(\tau^2 + h^2)^2.$$

That is,

$$\|e^n\| \le O(\tau^2 + h^2) \quad \text{and} \quad \|e^n_{x\bar{x}}\| \le O(\tau^2 + h^2).$$

Using (3.14), we therefore obtain

$$\|e^n_x\| \le O(\tau^2 + h^2).$$

and thus, by and Lemma 2.3,

$$\|e^n\|_\infty \le O(\tau^2 + h^2)$$

as desired. □

Theorem 3.5. *Under the conditions of Theorem* 3.4, *the solution* u^n *obtained by the scheme* (3.1)–(3.3) *is stable with respect to* $\|\cdot\|_\infty$.

4 A Linear Conservative Difference Scheme 2

Let us now be concerned with another linear finite difference scheme that is proposed to solve the Rosenau–Kawahara equation (1.3)–(1.5).

$$(u_i^n)_{\hat{t}}+(u_i^n)_{xx\bar{x}\bar{x}\hat{t}}+\alpha(\bar{u}_i^n)_{\hat{x}}+\beta(\bar{u}_i^n)_{x\bar{x}\hat{x}}-\gamma(\bar{u}_i^n)_{xx\bar{x}\bar{x}\hat{x}}+\eta\left[(u_i^n)^2\right]_{\hat{x}}+\frac{\eta h^2}{6}\left[(u_i^n)^2\right]_{x\bar{x}\hat{x}}=0; \tag{4.1}$$

$1 \le i \le M-1$, $1 \le n \le N-1$, where

$$u_i^0 = u_0(x_i), \quad -2 \le i \le M+2, \tag{4.2}$$

$$u_0^n = u_M^n = 0, \;\; (u_0^n)_{\hat{x}} = (u_M^n)_{\hat{x}} = 0, \;\; (u_0^n)_{x\bar{x}} = (u_M^n)_{x\bar{x}} = 0, \quad 1 \le n \le N. \tag{4.3}$$

Theorem 4.1. *Suppose that $u^n \in Z_h^0$. Using the scheme (4.1)–(4.3), the discrete mass Q^n is conserved; that is,*

$$Q^n = \frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1}+u_i^n\right) = Q^{n-1} = \cdots = Q^0. \tag{4.4}$$

Proof. Multiplying Eq. (4.1) by h, summing up for i from $i=0$ to $M-1$ and using the boundary conditions, it follows that

$$\frac{h}{2}\sum_{i=1}^{M-1}\left(u_i^{n+1}-u_i^{n-1}\right)=0.$$

Thus, $Q^n - Q^{n-1} = 0$, yielding the conservation law (4.4). □

Theorem 4.2. *Suppose $u_0 \in H_0^2[x_L, x_R]$. Then, the solution u^n to (4.1)–(4.3) satisfies $\|u^n\| \le C$ and $\|u_{xx}^n\| \le C$, which yields $\|u^n\|_\infty \le C$.*

Proof. The theorem will be argued by induction on the time levels n. First of all, it follows from the initial condition (4.2) that $u^0 \le C$. The first–level approximation u^1 can be computed directly by a second–order method. Hence, $\left\|u^1\right\| \le C$ and $\left\|u^1\right\|_\infty \le C$. Now, we assume that

$$\left\|u^k\right\|_\infty \le C \;\text{ for }\; k = 0, 1, 2, \ldots, n. \tag{4.5}$$

Taking the inner product of (4.1) with $2\bar{u}^n$, and using equalities (3.6) and Lemma 2.1, we obtain

$$\begin{aligned}&(\|u^{n+1}\|^2 - \|u^{n-1}\|^2) + (\|u_{x\bar{x}}^{n+1}\|^2 - \|u_{x\bar{x}}^{n-1}\|^2)\\ &= -2\tau(\eta[(u_j^n)^2]_{\hat{x}}, 2\bar{u}^n) - 2\tau((\frac{\eta h^2}{6})[(u_j^n)^2]_{x\bar{x}\hat{x}}, 2\bar{u}^n).\end{aligned}$$

By Cauchy–Schwarz inequality and direct computation, it gives

$$\|u_{\hat{x}}^n\| \le \|u_x^n\|,$$

and

$$\|u_x^n\|^2 \le \frac{1}{2}(\|u^n\|^2 + \|u_{x\bar{x}}^n\|^2).$$

From (4.5), Cauchy–Schwarz inequality again together with Lemma 2.1, we get

$$\begin{aligned}\left(\left[(u^n)^2\right]_{\hat{x}}, 2\bar{u}^n\right) &= -h\sum_{i=1}^{M-1}(u_i^n)^2\left(u_i^{n+1}+u_i^{n-1}\right)_{\hat{x}}\\ &\le C\left(\|u^n\|^2 + \frac{1}{4}\left\|u^{n-1}\right\|^2 + \frac{1}{4}\left\|u^{n+1}\right\|^2 + \frac{1}{4}\left\|u_{x\bar{x}}^{n+1}\right\|^2 + \frac{1}{4}\left\|u_{x\bar{x}}^{n-1}\right\|^2\right).\end{aligned}$$

The boundary conditions (4.3) and Lemma 2.1 provide

$$\|u_{\hat{x}}^n\|^2 \le \frac{1}{2}(\|u_x^n\|^2 + \|u_{\bar{x}}^n\|^2) = \|u_x^n\|^2 \tag{4.6}$$

and

$$\|u_{x\hat{x}}^n\|^2 = \|u_{\bar{x}\hat{x}}^n\|^2 \le \|u_{x\bar{x}}^n\|^2. \tag{4.7}$$

Using (4.6), (4.7) and Lemma 2.2, we thus obtain

$$\begin{aligned}\left([(u^n)^2]_{x\bar{x}\hat{x}}, 2\bar{u}^n\right) &= -h\sum_{i=1}^{M-1}(u_i^n)^2\left(u_i^{n+1}+u_i^{n-1}\right)_{x\bar{x}\hat{x}}\\ &= -h\sum_{i=1}^{M-1}(u_i^n)(u_i^n)\left(u_i^{n+1}+u_i^{n-1}\right)_{x\bar{x}\hat{x}}\\ &\le C\left(\|u^n\|^2 + \frac{1}{2}\left\|u_{x\bar{x}\hat{x}}^{n+1}\right\|^2 + \frac{1}{2}\left\|u_{x\bar{x}\hat{x}}^{n-1}\right\|^2\right)\\ &\le C\left(\|u^n\|^2 + \frac{2}{h^2}\left\|u_{x\bar{x}}^{n+1}\right\|^2 + \frac{2}{h^2}\left\|u_{x\bar{x}}^{n-1}\right\|^2\right).\end{aligned}$$

Let us now set

$$B^n = (\|u^n\|^2 + \|u^{n-1}\|^2) + (\|u_{x\bar{x}}^n\|^2 + \|u_{x\bar{x}}^{n-1}\|^2).$$

This follows that

$$\begin{aligned}B^{n+1} - B^n &\le \tau C(\|u^n\|^2 + \|u^{n-1}\|^2 + \|u^{n+1}\|^2 + \|u_{x\bar{x}}^{n+1}\|^2 + \|u_{x\bar{x}}^{n-1}\|^2)\\ &\le \tau C(B^{n+1} + B^n).\end{aligned}$$

If τ is sufficiently small, which satisfies $\tau \leq \dfrac{k-2}{kC}$ for $k>2$, then

$$B^{n+1} \leq \frac{(1+\tau C)}{(1-\tau C)} B^n \leq (1+\tau kC)\, B^n \leq \exp(kCT)\, B^0.$$

Hence,

$$(\|u^{n+1}\|^2 + \|u^n\|^2) + (\|u^{n+1}_{x\bar{x}}\|^2 + \|u^n_{x\bar{x}}\|^2) \leq C.$$

Therefore, $\left\|u^{n+1}\right\| \leq C$ and $\left\|u^{n+1}_{x\bar{x}}\right\| \leq C$, which yield $\left\|u^{n+1}\right\|_\infty \leq C$ by Lemma 2.3. □

4.1 Solvability

We now prove that the scheme gives a unique solution to the Rosenau–Kawahara equation.

Theorem 4.3. *The finite difference scheme* (4.1)–(4.3) *is uniquely solvable.*

Proof. The proof will be done by induction on the time levels n. First, we can determine u^0 uniquely by the initial conditions. Next, a second–order method will give u^1. For the inductive procedure, suppose that $u^0, u^1, u^2, ..., u^n$ are solved uniquely. Considering (4.1) for u^{n+1}, we have

$$\frac{1}{2\tau}u_i^{n+1} + \frac{1}{2\tau}(u_i^{n+1})_{xx\bar{x}\bar{x}} + \alpha\frac{1}{2}(u_i^{n+1})_{\hat{x}} + \beta\frac{1}{2}(u_i^{n+1})_{x\bar{x}\hat{x}} - \gamma\frac{1}{2}(u_i^{n+1})_{xx\bar{x}\bar{x}\hat{x}} = 0. \quad (4.8)$$

By taking the inner product of (4.8) with u^{n+1} and using (3.9), we obtain

$$\|u^{n+1}\|^2 + \|u^{n+1}_{x\bar{x}}\|^2 = 0.$$

Therefore, (4.8) has the only one solution u^{n+1}; that is, (4.1) is uniquely solvable. This completes the proof of Theorem 4.3. □

4.2 Convergence and Stability

Next, we deal with the convergence and stability of the scheme (4.1)–(4.3). Let $v_i^n = v(x_i, t_n)$ be the solution to (1.3)–(1.5). Set $e_i^n = v_i^n - u_i^n$. Then, we obtain

$$\begin{aligned} r_i^n = {} & (e_i^n)_{\hat{t}} + (e_i^n)_{xx\bar{x}\bar{x}\hat{t}} + \alpha(\bar{e}_i^n)_{\hat{x}} + \beta(\bar{e}_i^n)_{x\bar{x}\hat{x}} - \gamma(\bar{e}_i^n)_{xx\bar{x}\bar{x}\hat{x}} \\ & + \eta[(v_i^n)^2_{\hat{x}} - (u_i^n)^2_{\hat{x}}] + (\frac{\eta h^2}{6})[(v_i^n)^2_{x\bar{x}\hat{x}} - (u_i^n)^2_{x\bar{x}\hat{x}}]. \end{aligned} \quad (4.9)$$

where r_i^n denotes the truncation error. By Taylor expansion, it follows that $r_i^n = O(\tau^2 + h^2)$ as $\tau, h \to 0$.

Theorem 4.4. *Suppose $u_0 \in H_0^2[x_L, x_R]$. Then, the solution u^n produced by the scheme* (4.1)–(4.3) *converges (with respect to $\|\cdot\|_\infty$) to the solution to the problem* (1.3)–(1.5) *with rate of convergence is $O(\tau^2 + h^2)$.*

Proof. By taking the inner product on both sides of (4.9) with $2\bar{e}^n = (e^{n+1}+e^{n-1})$ and by using (3.12), we get

$$
\begin{aligned}
&(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) + (\|e^{n+1}_{x\bar{x}}\|^2 - \|e^{n-1}_{x\bar{x}}\|^2) \qquad (4.10)\\
&= 2\tau(r^n, 2\bar{e}^n) - 2\tau\eta((v_i^n)^2_{\hat{x}} - (u_i^n)^2_{\hat{x}}, 2\bar{e}^n) - 2\tau(\frac{\eta h^2}{6})((v_i^n)^2_{x\bar{x}\hat{x}} - (u_i^n)^2_{x\bar{x}\hat{x}}, 2\bar{e}^n).
\end{aligned}
$$

By Cauchy–Schwarz inequality, Lemma 2.1, Theorem 4.1, and Lemma 2.5, we obtain the following inequalities

$$
\begin{aligned}
([(v_i^n)^2]_{\hat{x}} - [(u_i^n)^2]_{\hat{x}}, 2\bar{e}^n) &= 2h\sum_{i=1}^{M-1}[(v_i^n)^2_{\hat{x}} - (u_i^n)^2_{\hat{x}}]\bar{e}_i^n\\
&= -2h\sum_{i=1}^{M-1}[(v_i^n)^2 - (u_i^n)^2](\bar{e}_i^n)_{\hat{x}}\\
&= -2h\sum_{i=1}^{M-1}[(v_i^n) - (u_i^n)][(v_i^n) + (u_i^n)](\bar{e}_i^n)_{\hat{x}}\\
&= -2h\sum_{i=1}^{M-1}[e^n][(v_i^n) + (u_i^n)](\bar{e}_i^n)_{\hat{x}}\\
&\le C(\|e^n\|^2 + \|(\bar{e}^n)_{\hat{x}}\|^2)\\
&\le C(\|e^n\|^2 + \|e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}_{x\bar{x}}\|^2 + \|e^{n+1}_{x\bar{x}}\|^2) \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
([(v_i^n)^2]_{x\bar{x}\hat{x}} - [(u_i^n)^2]_{x\bar{x}\hat{x}}, 2\bar{e}^n) &= -2h\sum_{i=1}^{M-1}[(v_i^n)^2_{\hat{x}} - (u_i^n)^2_{\hat{x}}](\bar{e}_i^n)_{x\bar{x}}\\
&= -\sum_{i=1}^{M-1}[[(v_{i+1}^n)^2 - (v_{i-1}^n)^2] - [(u_{i+1}^n)^2 - (u_{i-1}^n)^2]](\bar{e}_i^n)_{x\bar{x}}\\
&= -\sum_{i=1}^{M-1}[e_{i+1}^n[(v_{i+1}^n) + (u_{i+1}^n)] - e_{i-1}^n[(v_{i-1}^n) + (u_{i-1}^n)]](\bar{e}_i^n)_{x\bar{x}}\\
&\le C\frac{1}{h}[h\sum_{i=1}^{M-1}[|e_{i+1}^n||(\bar{e}_i^n)_{x\bar{x}}|] + C\frac{1}{h}[h\sum_{i=1}^{M-1}|e_{i-1}^n||(\bar{e}_i^n)_{x\bar{x}}|]\\
&\le \frac{1}{h}C(\|e^n\|^2 + \|(\bar{e}^n)_{x\bar{x}}\|^2)\\
&\le \frac{1}{h}C(\|e^n\|^2 + \|e^{n-1}_{x\bar{x}}\|^2 + \|e^{n+1}_{x\bar{x}}\|^2) \qquad (4.12)
\end{aligned}
$$

and

$$
(r^n, 2\bar{e}^n) \le \|r^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right). \qquad (4.13)
$$

Utilizing (4.11)–(4.13), (4.10) becomes

$$\begin{aligned}&(\|e^{n+1}\|^2-\|e^{n-1}\|^2)+(\|e^{n+1}_{x\bar{x}}\|^2-\|e^{n-1}_{x\bar{x}}\|^2)\\&\quad\le 2\tau\|r^n\|^2+\tau C(\|e^{n-1}\|^2+\|e^{n}\|^2+\|e^{n+1}\|^2+\|e^{n-1}_{x\bar{x}}\|^2+\|e^{n+1}_{x\bar{x}}\|^2).\end{aligned}\tag{4.14}$$

Again, Cauchy-Schwarz inequality and Lemma 2.1 give

$$\|e^n_{\hat{x}}\|\le\|e^n_x\|,\tag{4.15}$$

and hence

$$\|e^n_x\|=-\left(e^n,e^n_{x\bar{x}}\right)\le\frac{1}{2}\left(\|e^n\|^2+\|e^n_{x\bar{x}}\|^2\right).\tag{4.16}$$

Setting

$$D^n=\|e^n\|^2+\|e^{n-1}\|^2+\|e^n_{x\bar{x}}\|^2+\|e^{n-1}_{x\bar{x}}\|^2$$

and using (4.15), (4.16) and Lemma 2.2, we can rewrite (4.14) as

$$D^{n+1}-D^n\le 2\tau\|r^n\|^2+\tau C(D^{n+1}+D^n).$$

This implies that

$$(1-2\tau C)\left(D^{n+1}-D^n\right)\le\tau\|r^n\|^2+2\tau CD^n.$$

If τ is small enough satisfying $1-2C\tau>0$, then

$$D^{n+1}-D^n\le\tau C\|r^n\|^2+\tau CD^n\tag{4.17}$$

for every n. Summing up (4.17) from $k=1$ to n gives

$$D^{n+1}\le D^1+C\tau\sum_{k=1}^{n}\left\|r^k\right\|^2+C\tau\sum_{k=1}^{n}D^k.\tag{4.18}$$

Thus, we can use a second–order method to compute u^1 such that

$$D^1\le O(\tau^2+h^2)^2,$$

and

$$\tau\sum_{k=1}^{n}\left\|r^k\right\|^2\le n\tau\max_{0\le l\le n-1}\left\|r^l\right\|^2\le T\cdot O(\tau^2+h^2)^2.$$

By Lemma 2.4, we obtain $E^n\le O(\tau^2+h^2)^2$. That is,

$$\|e^n\|\le O(\tau^2+h^2)\quad\text{and}\quad\|e^n_{x\bar{x}}\|\le O(\tau^2+h^2).$$

From (4.16), it follows that

$$\|e^n_x\|\le O(\tau^2+h^2)$$

and, by Lemma 2.3, we have

$$\|e^n\|_\infty\le O(\tau^2+h^2).$$

The proof is now complete. □

Theorem 4.5. *Under the conditions of Theorem 4.4, the solution u^n obtained by the scheme (4.1)–(4.3) is stable with respect to $\|\cdot\|_\infty$.*

5 Numerical Experiments

In this section, we will show the performance our schemes presented earlier in the case $\alpha = \beta = \gamma = 1$ and $\eta = \frac{1}{2}$. We will compare further our numerical results with the existing scheme proposed by J. Hu et al., see [24]. For convenience, our schemes (3.1) and (4.1) will be named Scheme I and Scheme II, respectively, and Hu's scheme will be called Scheme III.

5.1 Error and Rate of Convergence

We first verify the error and order of convergence of the presented schemes. Numerical experiments are illustrated using various step sizes in space and time. Let v_h be an approximate solution obtained by the schemes and u be an exact solution to the Rosenau–Kawahara equation. The error of approximation is simply $e_h = u - v_h$. In addition, the rate of convergence is computed by

$$\text{Rate} = \log_2\left(\frac{\|e_h\|}{\|e_{h/2}\|}\right).$$

We implement the soliton solutions on the domain $\Omega = [-50, 100]$ at the final time $T = 10$ with the initial condition

$$u(x,0) = \left(-\frac{32}{12} + \frac{35}{156}\sqrt{205}\right) \times \operatorname{sech}^4\left(\frac{1}{12}\sqrt{-13+\sqrt{205}}\,x\right). \tag{5.1}$$

Table 1: Error of approximation with respect to $\|\cdot\|_\infty$ of Scheme I, Scheme II, and Scheme III at $T = 10$ using various step sizes.

$h = \tau$	Scheme I		Scheme II		Scheme III	
	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate
0.5	2.086310×10^{-3}	n/a	3.264703×10^{-3}	n/a	3.594368×10^{-3}	n/a
0.25	5.309411×10^{-4}	1.974329811	8.391734×10^{-4}	1.959910898	9.169029×10^{-4}	1.970897257
0.125	1.334714×10^{-4}	1.992021187	2.120559×10^{-4}	1.984524325	2.306178×10^{-4}	1.991265090
0.0625	3.341600×10^{-5}	1.997919683	5.325164×10^{-5}	1.993546761	5.775198×10^{-5}	1.997561556

Table 2: Error of approximation with respect to $\|\cdot\|_2$ of Scheme I, Scheme II, and Scheme III at $T = 10$ using various step sizes.

$h = \tau$	Scheme I		Scheme II		Scheme III	
	$\|e\|_2$	Rate	$\|e\|_2$	Rate	$\|e\|_2$	Rate
0.5	6.467452×10^{-3}	n/a	9.621914×10^{-3}	n/a	1.050832×10^{-2}	n/a
0.25	1.640375×10^{-3}	1.979171780	2.458968×10^{-3}	1.968270944	2.674182×10^{-3}	1.974362478
0.125	4.117395×10^{-4}	1.994221897	6.196686×10^{-4}	1.988484190	6.718586×10^{-4}	1.992868117
0.0625	1.030601×10^{-4}	1.998245962	1.554175×10^{-4}	1.995347905	1.682122×10^{-4}	1.997875290

The error of approximation and rate of convergence with respect to $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are presented in Table 1 and Table 2, respectively. Our results indicate that the rate of convergence of each scheme is of second order in both space and time. However, the approximation error of Scheme I is reduced approximately 37% from the Scheme II and 42% from the scheme III.

Table 3: Comparison of the error of approximation of Scheme I, Scheme II, and Scheme III at $T = 100$ using $h = \tau$.

Numerical Scheme		$\|e\|_\infty$	$\|e\|_2$	CPU time (s)
Scheme I	$h = 0.5$	1.477357×10^{-2}	4.888249×10^{-2}	3.149645
Scheme II		2.066746×10^{-2}	6.796953×10^{-2}	2.898694
Scheme III		2.260739×10^{-3}	7.438533×10^{-2}	3.361335
Scheme I	$h = 0.25$	3.812803×10^{-3}	1.254717×10^{-2}	27.635016
Scheme II		5.390120×10^{-3}	1.758850×10^{-2}	23.926143
Scheme III		5.930241×10^{-3}	1.934925×10^{-2}	27.209961
Scheme I	$h = 0.125$	9.608151×10^{-4}	3.158538×10^{-3}	222.666841
Scheme II		1.362574×10^{-3}	4.436043×10^{-3}	205.035816
Scheme III		1.500739×10^{-3}	4.887029×10^{-3}	223.237495

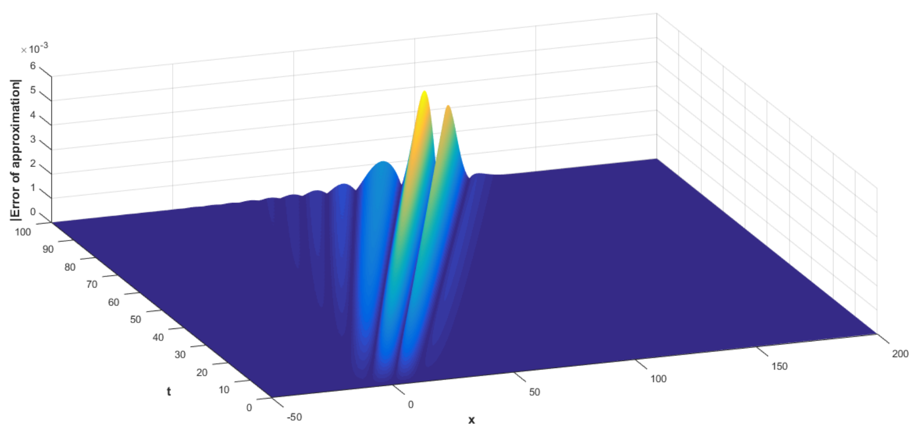

Figure 1: Error distribution of Scheme I on the time interval $0 \le t \le 100$ using $h = \tau = 0.25$.

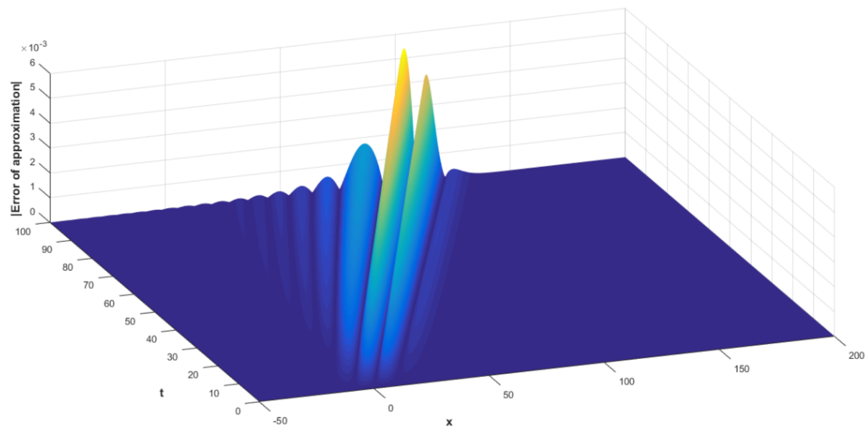

Figure 2: Error distribution of Scheme II on the time interval $0 \le t \le 100$ using $h = \tau = 0.25$.

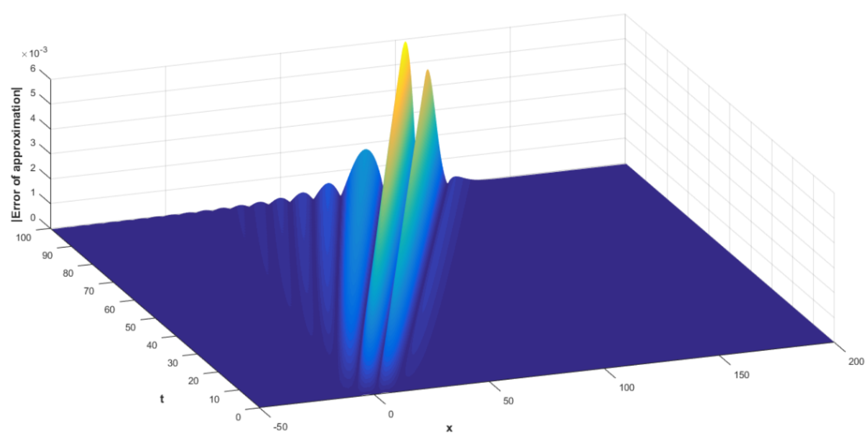

Figure 3: Error distribution of Scheme III on the time interval $0 \le t \le 100$ using $h = \tau = 0.25$.

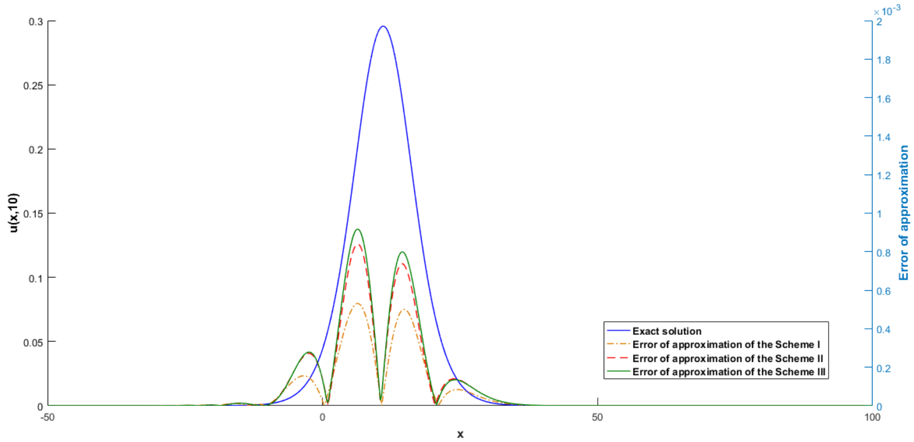

Figure 4: Exact solution and error of approximation of Scheme I, Scheme II and Scheme III at $T = 10$ using $h = \tau = 0.25$.

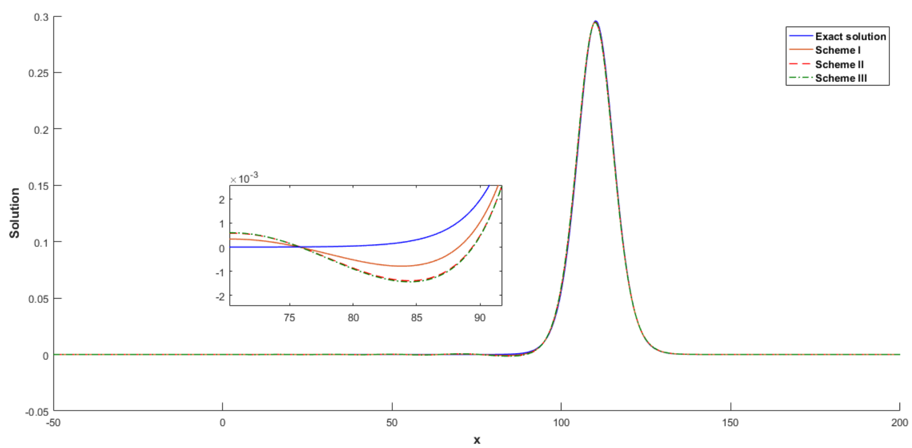

Figure 5: Exact solution and numerical solutions at $T = 100$ with step size $h = \tau = 0.25$.

We further observe prolonged behavior of the soliton solution by tracking the numerical solution until $T = 100$ on the domain $\Omega = [-50, 200]$. The error and computational cost (CPU time) are illustrated in Table 3. Notice that Scheme I performs best in terms of accuracy. However, an advantage of Scheme II is that it does not need a matrix reconstruction in each time step of computation, yielding the lowest computational cost that is reduced about 8% from the others. Figures 1–3 exhibit the distribution of approximation errors occurred from the three schemes. Observe that, in each time step, all schemes generate a similar characteristic of the errors. Figure 4 shows the error patterns together with the exact solution, which hints that the errors occur near the wave peaks with small fluctuation on the left tails. In fact, the fluctuation increases as the time increases. This causes the oscillation to appear on the left tails of our numerical solutions, as shown in Figure 5. Since Scheme I can reduce the error of approximation, it also produces less left-tail oscillation. In other words, Scheme I is more stable than the others.

5.2 Conservation of Mass and Energy

By Theorems 3.1 and 4.1, the quantities Q^n and E^n are theoretically conserved. More precisely, the formulae (3.4) and (3.5) guarantee conservation in each time step for Scheme I, and so does the formula (4.4) for Scheme II. To confirm this, our numerical investigation is implemented on the domain $\Omega \times T = [-50, 200] \times [0, 100]$ using step sizes $h = \tau = 0.5$ and $h = \tau = 0.25$. The numerical results obtained from both schemes are illustrated in Tables 4–6, indicating that the quantities Q^n and E^n are conserved with high accuracy.

Table 4: Conservative quantity Q of soliton solution from Scheme I on $\Omega = [-50, 200]$ at various times.

t	$h = \tau = 0.5$		$h = \tau = 0.25$	
	Q^n	$\lvert Q^0 - Q^n\rvert$	Q^n	$\lvert Q^0 - Q^n\rvert$
0	4.12120624293163	n/a	4.12089587363922	n/a
20	4.12122082087774	1.45779×10^{-5}	4.12093751058313	2.67624×10^{-6}
40	4.12130091258107	9.46696×10^{-5}	4.12095223369831	1.73994×10^{-5}
60	4.12130530603033	9.90631×10^{-5}	4.12095397274229	1.91384×10^{-5}
80	4.12129069413532	8.44512×10^{-5}	4.12095026453351	1.54302×10^{-5}
100	4.12127811180463	7.18689×10^{-5}	4.12094767914707	1.28448×10^{-5}

Table 5: Conservative quantity Q of soliton solution from Scheme II on $\Omega = [-50, 200]$ at various times.

t	$h = \tau = 0.5$		$h = \tau = 0.25$	
	Q^n	$\lvert Q^0 - Q^n\rvert$	Q^n	$\lvert Q^0 - Q^n\rvert$
0	4.12089587162373	n/a	4.12089589231287	n/a
20	4.12089536762160	5.04002×10^{-7}	4.12089618772231	2.95409×10^{-7}
40	4.12094544253396	4.95709×10^{-5}	4.12090527604004	9.38373×10^{-6}
60	4.12088990933825	5.96229×10^{-6}	4.12089568110516	2.11208×10^{-7}
80	4.12089504819607	8.23428×10^{-7}	4.12089482601477	1.06630×10^{-6}
100	4.12090784998133	1.19784×10^{-5}	4.12089314640128	2.74591×10^{-6}

Table 6: Conservative quantity E of soliton solution from Scheme I on $\Omega = [-50, 200]$ at various times.

t	$h = \tau = 0.5$		$h = \tau = 0.25$	
	E^n	$\|E^0 - E^n\|$	E^n	$\|E^0 - E^n\|$
0	0.835063957830431	n/a	0.835916655770310	n/a
20	0.835063957832700	2.26907×10^{-12}	0.835916655772429	2.11897×10^{-12}
40	0.835063957802984	2.74469×10^{-11}	0.835916655775377	5.06695×10^{-12}
60	0.835063957824593	5.83800×10^{-12}	0.835916655777922	7.61191×10^{-12}
80	0.835063957831566	1.13498×10^{-12}	0.835916655781469	1.11590×10^{-11}
100	0.835063957794769	3.56619×10^{-11}	0.835916655783726	1.34159×10^{-11}

Next, let us compare the conservation properties of Scheme I with Scheme III. The discrete mass ($\tilde{Q}$) and energy ($\tilde{E}$) are provided by

$$\tilde{Q}^n = h \sum_{i=1}^{N-1} u_i^n , \qquad \tilde{E}^n = \|u^n\|^2 + \|u_{x\bar{x}}^n\|^2 \tag{5.2}$$

Our experiment is again processed on $\Omega \times [0, T] = [-50, 200] \times [0, 100]$ using the several step sizes. The quantities of wave mass ($\tilde{Q}^n$) and energy ($\tilde{E}^n$) are calculated in each time step. Comparisons of these quantities between Schemes I and III are given in Tables 7 and 8. It can be seen from the tables that Scheme I performs better to maintain the wave mass, whereas Scheme III is noticeably more powerful to level the energy. Nevertheless, Scheme I can produce at least 6-digit energy conservation for $h = \tau = 0.5$ and at least 7-digit conservation for $h = \tau = 0.25$. These mass and energy fluctuations gets smaller as the mesh size decreases. This phenomenon can be explained in terms of the following conservative invariant law

$$E^n \to \tilde{E}^n \quad \text{and} \quad Q^n \to \tilde{Q}^n \qquad \text{as} \qquad h, \tau \to 0.$$

Table 7: Mass of soliton on $\Omega = [-50, 200]$ at various times using $h = \tau$.

t	Scheme I		Scheme III	
	Mass ($\tilde{Q}$)	$\|\tilde{Q}^0 - \tilde{Q}^n\|$	Mass ($\tilde{Q}$)	$\|\tilde{Q}^0 - \tilde{Q}^n\|$
$h = 0.5$				
0	4.12089590001200	n/a	4.12089590001200	n/a
20	4.12087376981871	2.21302×10^{-5}	4.12085949889725	3.64011×10^{-5}
40	4.12087006230423	2.58377×10^{-5}	4.12084330921174	5.25908×10^{-5}
60	4.12086399622980	3.19038×10^{-5}	4.12082446889683	7.14311×10^{-5}
80	4.12085554366322	4.03563×10^{-5}	4.12081628945948	7.96106×10^{-5}
100	4.12085028148892	4.56185×10^{-5}	4.12081296710255	8.29329×10^{-5}
$h = 0.25$				
0	4.12089590447742	n/a	4.12089590447742	n/a
20	4.12089333227784	2.57220×10^{-6}	4.12089137960091	4.52488×10^{-6}
40	4.12089403312682	1.87135×10^{-6}	4.12088978919473	6.11528×10^{-6}
60	4.12089356800729	2.33647×10^{-6}	4.12088674898706	9.15549×10^{-6}
80	4.12089204182397	3.86265×10^{-6}	4.12088554485882	1.03596×10^{-5}
100	4.12089111245420	4.79202×10^{-6}	4.12088516889890	1.07356×10^{-5}

Table 8: Energy of soliton on $\Omega = [-50, 200]$ at various times using $h = \tau$.

t	Scheme I		Scheme III	
	Energy ($\tilde{E}$)	$\lvert\tilde{E}^0 - \tilde{E}^n\rvert$	Energy ($\tilde{E}$)	$\lvert\tilde{E}^0 - \tilde{E}^n\rvert$
$h = 0.5$				
0	0.836200484217486	n/a	0.836200484217486	n/a
20	0.836200007917385	4.76300×10^{-7}	0.836200484216087	1.39910×10^{-12}
40	0.836199124415244	1.35980×10^{-6}	0.836200484216873	6.13065×10^{-13}
60	0.836198297681637	2.18654×10^{-6}	0.836200484215122	2.36400×10^{-12}
80	0.836197618268678	2.86595×10^{-6}	0.836200484214289	3.19700×10^{-12}
100	0.836197083275489	3.40094×10^{-6}	0.836200484214749	2.73703×10^{-12}
$h = 0.25$				
0	0.836201029067217	n/a	0.836201029067217	n/a
20	0.836200998914977	3.01522×10^{-8}	0.836201029064854	2.36300×10^{-12}
40	0.836200943334864	8.57324×10^{-8}	0.836201029066681	5.36016×10^{-13}
60	0.836200891256782	1.37810×10^{-7}	0.836201029066314	9.03055×10^{-13}
80	0.836200848426374	1.80641×10^{-7}	0.836201029065430	1.78701×10^{-12}
100	0.836200814710434	2.14357×10^{-7}	0.836201029065492	1.72506×10^{-12}

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