



# Quadratic Transformations of Copula Density and Probability Density

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**Abstract :** Studying transformations is a simple way to obtain new types of mappings in data analysis. In this work, we present a characterization of quadratic transformations of copula densities. Additionally, we also characterize quadratic transformations of probability densities. Both transformations correspond to their quadratic coefficients and the sets of their coefficients are always convex. Especially, the set of coefficients of quadratic transformations of probability densities is a non-polygon.

**Keywords :** construction; copula density; polynomial; probability density; quadratic.

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## 1 Introduction

One important topic in data analysis is the construction and characterization of data distribution and data representation. Over the years, several families of distribution functions have been introduced (for example, [1–3]). Data representation such as aggregation functions has also gain interests (e.g. [4, 5]). The most important construction theorem is Sklar’s Theorem which explains copula construction from joint distributions and joint-distribution construction from copulas (see more in [6]). In [7], de Amo, Carrillo, and Fernández-Sánchez describe all copulas in view of Sklar’s Theorem that use for characterizing copulas extend from an arbitrary subcopula. Another construction idea is via combining two copulas to

obtain a new copula. In [8], Gluing of copulas is a way of constructing  $n$ -copulas, by scaling and combining finitely  $n$ -copulas.

One common construction method is via transformations. For example, Kolesárová and Mesiar [9] discuss quadratic constructions of distinguished classes of certain aggregation functions, and Tasena [10] studies polynomial transformation of copulas generalized the work of [11] and [12]. Other constructions include, for example, [13–17]. Following this idea, we are interest in quadratic transformations of functions. Instead of copulas and aggregation functions, however, copula densities and probability densities will be considered. We found that results are quite different and interesting on their own.

This paper is organized as follows. In the next section, we will provide basic terminologies and notations of probability density and copula density used later on. In section 3, characterization of transformations of copula density is provided. In section 4, we will study transformations of probability density and give additional examples that its boundary area of parameters is nonlinear. For implicitly, a density on  $I^2$  will be considered where  $I = [0, 1]$ .

## 2 Preliminaries

Recall that a (probability) density is a function  $f : I^2 \rightarrow [0, \infty)$  such that  $\int_{I^2} f d\lambda = 1$ . A density  $f$  is associated to a random vector  $(X, Y)$  if

$$\mathbb{P}(X \leq x, Y \leq y) = \int_0^x \int_0^y f d\lambda$$

for all  $x, y \in I$ . In this case, the expectation of  $X$  and  $Y$  is defined via

$$\mathbb{E}X = \int_{I^2} x f(x, y) d\lambda \quad \text{and} \quad \mathbb{E}Y = \int_{I^2} y f(x, y) d\lambda$$

which always exist and that  $0 < \mathbb{E}X < 1$  and  $0 < \mathbb{E}Y < 1$ . For each  $\alpha$  and  $\beta$  in  $(0, 1)$ , denote  $\mathfrak{D}_{\alpha, \beta}$  the collection of all densities whose expectation with respect to the first and second coordinate are  $\alpha$  and  $\beta$ , respectively. Denote also  $\mathfrak{D} = \bigcup_{\alpha, \beta \in (0, 1)} \mathfrak{D}_{\alpha, \beta}$ . It can be seen that  $\mathfrak{D}$  is the collection of all densities on  $I^2$ .

To guarantee that  $\mathfrak{D}_{\alpha, \beta}$  is nonempty, consider the density given below.

**Example 2.1.** We will construct a mapping  $f \in \mathfrak{D}_{\alpha, \beta}$  as follow.

First, we construct a distribution density  $g : I \rightarrow \mathbb{R}$  with the expectation  $\alpha$ . For each  $x \in (0, 1)$  with  $x < \alpha$ , its reflection  $x' > \alpha$  is denoted by  $x' = 1 - \frac{x(1-\alpha)}{\alpha}$ . Let  $k = \frac{1-\alpha}{x}$  and  $h = \frac{\alpha}{(1-x')}$ . Hence, we obtain  $kx + h(1-x') = 1$  and two closed intervals  $A = [0, x]$  and  $B = [x', 1]$ . Consequently, the mapping  $g : I \rightarrow \mathbb{R}$  is defined by

$$g(t) = k \cdot \mathbf{1}_A(t) + h \cdot \mathbf{1}_B(t)$$

for each  $t \in I$  where  $\mathbf{1}_A(t) = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{if } t \in I \setminus A \end{cases}$ .

Thus,  $g$  is a distribution density since

$$\int_I g d\lambda = k \int_A 1 d\lambda + h \int_B 1 d\lambda = kx + h(1 - x') = 1.$$

Additionally, we have that

$$\begin{aligned} \int_I tg(t)d\lambda &= k \int_0^x t dt + h \int_{x'}^1 t dt \\ &= \frac{1 - \alpha}{x} \left( \frac{x^2}{2} \right) + \frac{\alpha}{(1 - x')} \left( \frac{1 - (x')^2}{2} \right) \\ &= \alpha. \end{aligned}$$

Therefore,  $g : [0, 1] \rightarrow \mathbb{R}$  is a distribution density whose expectation is  $\alpha$ .

Finally, for each  $(\alpha, \beta) \in (0, 1)^2$ , we obtain two distribution densities  $g_\alpha$  and  $g_\beta$  from  $[0, 1]$  to  $\mathbb{R}$  with expectations  $\alpha$  and  $\beta$  defined from above. Now the mapping  $f : I^2 \rightarrow (0, 1)$ , defined by  $f(s, t) = g_\alpha(s) \cdot g_\beta(t)$  for each  $(s, t) \in I^2$ , belongs to  $\mathfrak{D}_{\alpha, \beta}$ .

Recall that a copula is a mapping  $C : I^2 \rightarrow I$  satisfying the following conditions.

1.  $C(u, 0) = 0 = C(0, v)$  for all  $u, v \in I$ .
2.  $C(u, 1) = u = C(1, v)$  for all  $u, v \in I$ .
3.  $V([u_1, u_2] \times [v_1, v_2]) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$  for all  $[u_1, u_2] \times [v_1, v_2] \subseteq I^2$ .

Additionally, if a mapping  $c : I^2 \rightarrow \mathbb{R}$  satisfies

$$C(s, t) = \int_0^s \int_0^t c \, dudv$$

for all  $s, t \in I$ , then  $c$  is called a **density of the copula**  $C$ . It can be seen that the density of any copula is a probability density and a probability density of  $f$  is a copula density if

$$\int_0^1 \int_0^u f(s, t) d\lambda = \int_0^u \int_0^1 f(s, t) d\lambda = u$$

for all  $u \in I$ .

**Example 2.2** (Clayton copula [18]). For each arbitrary  $\theta \in (0, \infty)$ , the Clayton copula  $C_\theta : I^2 \rightarrow I$  is defined by

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

for each  $(u, v) \in I^2$ . So its density is in the form

$$\begin{aligned} c_\theta(u, v) &= \frac{\partial^2}{\partial v \partial u} C_\theta(u, v) = (1 + \theta)(uv)^{-(\theta-1)}(u^{-\theta} + v^{-\theta} - 1)^{-(1+\theta)/\theta} \\ &= (1 + \theta)(uv)^2 \left( \frac{1}{u^\theta + v^\theta - u^\theta v^\theta} \right)^{(1+\theta)/\theta} \end{aligned}$$

for each  $(u, v) \in (0, 1)^2$ . We have that  $\lim_{\theta \rightarrow \infty} c_\theta(u, v) = \infty$  for each  $(u, v) \in (0, 1)^2$  since  $\lim_{\theta \rightarrow \infty} \left( \frac{1}{u^\theta + v^\theta - u^\theta v^\theta} \right) = \infty$ . Thus, we obtain a density  $c$  of a copula with an arbitrary large value at a point  $(u, v) \in (0, 1)^2$ .

**Example 2.3.** Fixed  $(u, v) \in (0, 1)^2$ , let

$$h = \frac{1}{2}[C_{\max}(u, v) + C_{\min}(u, v)]$$

where  $C_{\max}(u, v) = \min(u, v)$  and  $C_{\min}(u, v) = \max\{u + v - 1, 0\}$ . Note that  $(u, v) \in (h, 1]^2 \cap \{(s, t) : s + t - 1 < h\}$ . We define a mapping  $C : [0, 1]^2 \rightarrow I$  by

$$C(s, t) = \begin{cases} \max\{s + t - 1, h\}, & \text{if } (s, t) \in [h, 1]^2 \\ C_{\max}(s, t), & \text{otherwise} \end{cases}$$

for each  $(s, t) \in I^2$ . We get that  $C$  is a copula and  $C(s, t) = h$  for each  $(s, t) \in [h, 1]^2 \cap \{(s, t) : s + t - 1 \leq h\}$ . Therefore, its density is zero at  $(u, v)$ .

### 3 Main Results

#### 3.1 Transformations of Copula Densities

Let  $\mathfrak{F}$  be the collection of all mappings from  $I^2$  to  $\mathbb{R}$ , and  $\mathfrak{C}_{\mathfrak{D}}$  be the collection of all densities of copulas. Given a quadratic polynomial  $P$  of three variables, we always obtain a mapping  $T : \mathfrak{C}_{\mathfrak{D}} \rightarrow \mathfrak{F}$  defined by

$$T(c)(u, v) = P(u, v, c(u, v))$$

for all  $(u, v) \in I^2$  and  $c \in \mathfrak{C}_{\mathfrak{D}}$ . Notice that every  $T(c)$  is in the form

$$T(c)(u, v) = a_1 + a_2u + a_3v + a_4c + a_5u^2 + a_6v^2 + a_7c^2 + a_8uv + a_9uc + a_{10}vc$$

where  $a_1, a_2, \dots, a_{10}$  are constants.

**Definition 3.1.** The mapping  $T$  is said to be a **transformation of copula densities (shortly, CD-transformation)** if  $T(c)$  is a density of a copula whenever  $c$  is a density of a copula.

To obtain necessary conditions of coefficients on quadratic polynomials observed the following.

**Lemma 3.2.** *If  $T$  is a CD-transformation, then*

- $a_5 = 0 = a_6 = a_7 = a_9 = a_{10}$ ,
- $2a_1 + a_3 + 2a_4 = 2$ ,
- $2a_3 + a_8 = 0$ ,
- $a_2 = a_3$ .

*Proof.* Assume that  $T$  is a CD-transformation; that is, for each density  $c$  of a copula, there is a copula  $C$  such that

$$T(c) = \frac{\partial^2}{\partial u \partial v} C.$$

First, we follow the condition on boundary of copula that  $v = 1$  and  $u = t$  being arbitrary on  $[0, 1]$ . We observe that

$$\begin{aligned} \int_0^1 \int_0^t a_1 \, dudv &= a_1 t, & \int_0^1 \int_0^t a_2 u \, dudv &= \frac{a_2}{2} t^2, & \int_0^1 \int_0^t a_3 v \, dudv &= \frac{a_3}{2} t, \\ \int_0^1 \int_0^t a_5 u^2 \, dudv &= \frac{a_5}{3} t^3, & \int_0^1 \int_0^t a_6 v^2 \, dudv &= \frac{a_6}{3} t, & \int_0^1 \int_0^t a_8 uv \, dudv &= \frac{a_8}{4} t^2. \end{aligned}$$

Moreover, we get that

$$\begin{aligned} \int_0^1 \int_0^t a_4 c \, dudv &= a_4 \int_0^1 [C_v]_0^t \, dv \\ &= a_4 \int_0^1 C_v(t, v) \, dv \\ &= a_4 C(t, 1) = a_4 t, \\ \int_0^1 \int_0^t a_9 uc \, dudv &= a_9 \int_0^1 \left\{ [uC_v]_{u=0}^{u=t} - \int_0^t C_v \, du \right\} \, dv \\ &= a_9 \left[ \int_0^1 t C_v(t, v) \, dv - \int_0^1 \int_0^t C_v \, du \, dv \right] \\ &= a_9 \left[ t [C_v(t, v)]_{v=0}^{v=1} - \int_0^t \int_0^1 C_v \, dv \, du \right], \end{aligned}$$

and

$$\int_0^1 \int_0^t a_{10}vc \, dudv = a_{10}t - a_{10} \int_0^1 C(t, v)dv.$$

Consequently, we obtain that

$$\begin{aligned} t = C(t, 1) &= \int_0^1 \int_0^t T(c)dudv \\ &= \left(a_1 + \frac{a_3}{2} + a_4 + \frac{a_6}{3} + a_{10}\right) t \\ &\quad + \left(\frac{a_2}{2} + \frac{a_8}{4} + \frac{a_9}{2}\right) t^2 + \frac{a_5}{3} t^3 \\ &\quad + a_7 \int_0^1 \int_0^t c^2 \, dudv - a_{10} \int_0^1 C(t, v)dv \end{aligned} \tag{3.1}$$

for each copula  $C$  with its density  $c$ .

Next, we will use the density of a Clayton copula (see Example 2.2) which is in the form

$$c_\theta(u, v) = (1 + \theta)(uv)^{-(\theta-1)}(u^{-\theta} + v^{-\theta} - 1)^{-(1+\theta)/\theta}.$$

Since two mappings  $\int_0^1 \int_0^t c_\theta^2 \, dudv$  and  $\int_0^1 C_\theta(t, v)dv$  contain two independent parameters  $\theta$  and  $t$ , they are linearly independent to  $t$ ,  $t^2$ , and  $t^3$ . Thus,  $a_1 + \frac{a_3}{2} + a_4 + \frac{a_6}{3} + a_{10} = 1$ ,  $2a_2 + a_8 + 2a_9 = 0$ ,  $a_5 = 0$ , and

$$0 = a_7 \int_0^1 \int_0^t c^2 \, dudv - a_{10} \int_0^1 C(t, v)dv.$$

To show  $a_7 = 0 = a_{10}$ , we use the Clayton copula  $C_\theta$  again. Now fixed  $t = 1$ , we obtain that

$$\int_0^1 C_\theta(t, v)dv = \int_0^1 vdv = \frac{1}{2} \quad \text{and} \quad \int_0^1 \int_0^1 c_\theta^2(u, v)dudv = (1 + \theta)^2 G(\theta)$$

where  $G(\theta) := \int_0^1 \int_0^1 (uv)^{-2(\theta-1)}(u^{-\theta} + v^{-\theta} - 1)^{-2(1+\theta)/\theta} dudv$  is a real-valued function containing only the variable  $\theta$ . So  $\int_0^1 C_\theta(t, v)dv$  and  $\int_0^1 \int_0^1 c_\theta^2(t, v)dudv$  are independent, and hence,  $a_7 = 0 = a_{10} = a_5$ . Thus,  $a_1 + \frac{a_3}{2} + a_4 + \frac{a_6}{3} = 1$  and  $2a_2 + a_8 + 2a_9 = 0$ .

Finally, to show that  $a_6 = 0 = a_9$ , we may set the other boundary condition that  $u = 1$  and  $v = s$  for any  $s \in [0, 1]$ . Similar to the previous argument, we obtain that

$$a_6 = 0 = a_9, 2a_3 + a_8 = 0, \text{ and } 2a_1 + a_2 + 2a_4 = 2.$$

Therefore,  $a_2 = a_3$ . □

**Remark.** Following the above lemma, we get

$$T(c) = a_1 + a_2u + a_3v + a_4c + a_8uv \tag{3.2}$$

where  $a_2 = a_3 = 2 - 2a_1 - 2a_4$  and  $a_8 = 4a_1 + 4a_4 - 4$ .

**Theorem 3.3.** *The mapping  $T$  is a CD-transformation if and only if*

$$T(c) = a_1 + a_2(u + v) + a_4c + a_8uv$$

where  $a_1 \geq 0, a_4 \geq 0, 2 \geq a_1 + 2a_4, a_2 = 2 - 2a_1 - 2a_4,$  and  $a_8 = 4a_1 + 4a_4 - 4$ .

*Proof.* Assume that  $T$  is a CD-transformation. Since any copula density is non-negative,  $T(c) \geq 0$  for each copula density  $c$ , i.e., for each  $(u, v) \in I^2$  and a copula density  $c$ ,

$$a_1 + a_2(u + v) + a_4c + a_8uv = T(c) \geq 0. \tag{3.3}$$

In the case that  $u = 0$  and  $v = 0$ , we get  $a_1 \geq 0$ . Following a fact of the Clayton copula (see Example 2.2), we obtain a density  $c$  of a copula whose gives an arbitrary large value at an arbitrary point  $(u, v) \in (0, 1)^2$ . Thus, the coefficient  $a_4$  can not be negative, otherwise  $T(c)$  may be negative.

To obtain  $2 \geq a_1 + 2a_4$ , we will construct a density  $c$  of copula with zero value at an arbitrary point  $(u, v) \in (0, 1)^2$ . Following the copula in Example 2.3, the density  $c$  of  $C$  gives zero at an arbitrary point  $(s, t) \in (0, 1)^2$ , and hence, we may ignore the term of density in the inequality (3.3). Following Lemma 3.2 (2) and (3),

$$0 \leq (2u + 2v - 4uv) + (1 - 2u - 2v + 4uv)a_1 + (-2u - 2v + 4uv)a_4.$$

Let  $h(u, v) = 2u + 2v - 4uv$ . Hence,

$$h(u, v) + (1 - h(u, v))a_1 - h(u, v)a_4 \geq 0.$$

It is not hard to see that  $0 \leq h(u, v) \leq 2$ , and hence,  $a_1$  and  $a_4$  satisfy the inequality  $2 - a_1 - 2a_4 \geq 0$ .

Conversely, assume that

$$T(c)(u, v) = a_1 + a_2(u + v) + a_4c + a_8uv$$

with  $a_1 \geq 0, a_4 \geq 0, 2 \geq a_1 + 2a_4, a_2 = 2 - 2a_1 - 2a_4,$  and  $a_8 = 4a_1 + 4a_4 - 4$ . We can see that  $T(c) \geq 0$  for each density  $c$ . This implies that the volume of  $\int_0^u \int_0^v T(c)(t, s) dt ds$  is non-negative. Therefore,  $\int_0^u \int_0^v T(c)(t, s) dt ds$  is a copula with the density  $T(c)$ . □

### 3.2 Transformations of Probability Density

At the begin of this section, we will introduce some probability densities with special properties as below.

1. A distribution density in  $\mathfrak{F}_{\alpha,\beta}$  which gives a large number at an arbitrary point in  $(0, 1)^2$ .
2. A distribution density in  $\mathfrak{F}_{\alpha,\beta}$  which is zero at an arbitrary point in  $I^2$ .

**Example 3.4.** Given  $(x, y) \in (0, 1)^2$  be such that  $x \neq \alpha$  and  $y \neq \beta$ . We will construct a mapping  $f : I^2 \rightarrow \mathbb{R}$  as following.

First, we construct a distribution density  $g : I \rightarrow \mathbb{R}$  with the expectation  $\alpha$ . For each  $x \in (0, 1)$  with  $x \neq \alpha$ , its reflection  $x'$  is denoted by

$$x' = \begin{cases} 1 - \frac{x(1-\alpha)}{\alpha}, & \text{if } x < \alpha \\ \frac{(1-x)\alpha}{1-\alpha}, & \text{if } x > \alpha \end{cases}$$

Note that  $x = x''$ , and either  $x < \alpha < x'$  or  $x' < \alpha < x$ . Without loss of generality, we may assume that  $x < \alpha$ . Fix a parameter  $r \in (0, 1)$ , let  $k = \frac{1-\alpha}{rx}$ , and  $h = \frac{\alpha}{r(1-x')}$ . Hence, we obtain  $krx + hr(1-x') = 1$  and two closed intervals  $A = [x - rx, x]$  and  $B = [x', x' + r(1-x')]$  which are subsets of  $I$ . Consequently, we define the mapping  $g$  by

$$g(t) = k \cdot \mathbf{1}_A(t) + h \cdot \mathbf{1}_B(t) \text{ for each } t \in I.$$

Thus,  $g$  is a distribution density since

$$\int_I g d\lambda = k \int_A d\lambda + h \int_B d\lambda = krx + hr(1-x') = 1$$

for each  $r, t \in (0, 1)$ . Additionally, we have that

$$\begin{aligned} \int_I tg(t)d\lambda &= k \int_{x-rx}^x t dt + h \int_{x'}^{x'+r(1-x')} t dt \\ &= \frac{k}{2} (2rx^2 - r^2x^2) + \frac{h}{2} (2rx'(1-x') + r^2(1-x')^2) \\ &= \frac{1-\alpha}{2} (2-r)x + \frac{\alpha}{2} [2x' + r(1-x')] = \alpha. \end{aligned}$$

Therefore,  $g : [0, 1] \rightarrow \mathbb{R}$  is a distribution density with the expectation  $\alpha$ .

Finally, for each  $(\alpha, \beta) \in (0, 1)^2$ , we obtain two distribution densities  $g_\alpha$  and  $g_\beta$  contained expectations  $\alpha$  and  $\beta$ , respectively, from above. Now we define the mapping  $f : I^2 \rightarrow (0, 1)$  by

$$f(s, t) = g_x(s) \cdot g_y(t) \text{ for each } (s, t) \in I^2.$$

Therefore,  $f \in \mathfrak{F}_{\alpha,\beta}$ .

To obtain the property (1) from above, we observe that



$$f(x, y) \in \left\{ \frac{(1-\alpha)(1-\beta)}{rx \quad ry}, \frac{(1-\alpha)\beta}{rx \quad r(1-y)}, \frac{\alpha(1-\beta)}{r(1-x) \quad ry}, \frac{\alpha\beta}{r(1-x) \quad r(1-y)} \right\}.$$

If  $r$  tends to zero, then  $f(x, y)$  tends to infinity. That is, we will obtain a distribution density in  $\mathfrak{F}_{\alpha, \beta}$  with a large number at an arbitrary point in  $(0, 1)^2$ .

To obtain the property (2), consider  $(\hat{x}, \hat{y}) \in I^2$  and  $(x, y) \in (0, 1)^2$  with  $\hat{x} \neq x$ ,  $x \neq \alpha$ , and  $y \neq \beta$ . We may construct the mapping  $f$  from  $(x, y)$  with  $r = \frac{1}{2} \min \left\{ \frac{|x-\hat{x}|}{x}, \frac{|x'-\hat{x}|}{1-x'} \right\}$ . Consequently,  $\hat{x} \notin A \cup B$  which implies  $f(\hat{x}, \hat{y}) = 0$ . Therefore, we obtain a distribution density in  $\mathfrak{F}_{\alpha, \beta}$  which is zero at an arbitrary point in  $I^2$ .

Given  $(\alpha, \beta) \in (0, 1)^2$  and a quadratic polynomial  $P$  of three variables, we always obtain a mapping  $T : \mathfrak{D}_{\alpha, \beta} \rightarrow \mathfrak{F}$  defined by

$$T(f)(x, y) = P(x, y, f(x, y))$$

for all  $(x, y) \in I^2$  and  $f \in \mathfrak{D}_{\alpha, \beta}$ .

**Definition 3.5.** The mapping  $T$  is said to be a **density transformation of distributions (shortly, DD-transformation)** if  $T(f)$  is a distribution density whenever  $f$  is a distribution density.

**Lemma 3.6.** *If  $g(x, y) = a_1 + a_2x + a_3y + a_5x^2 + a_6y^2 + a_8xy$  is nonnegative on  $I^2$ , then the following conditions are true.*

(A)  $a_1 \geq 0, a_1 + a_3 + a_6 \geq 0, a_1 + a_2 + a_5 \geq 0,$  and  $a_1 + a_2 + a_3 + a_5 + a_6 + a_8 \geq 0.$

(B1)  $4a_1a_5 - a_2^2 \geq 0$  when  $a_5 > 0$  and  $\frac{-a_2}{2a_5} \in I.$

(B2)  $4(a_1 + a_3 + a_6)a_5 - (a_2 + a_8)^2 \geq 0$  when  $a_5 > 0$  and  $\frac{-a_2 - a_8}{2a_5} \in I.$

(B3)  $4a_1a_6 - a_3^2 \geq 0$  when  $a_6 > 0$  and  $\frac{-a_3}{2a_6} \in I.$

(B4)  $4(a_1 + a_2 + a_5)a_6 - (a_3 + a_8)^2 \geq 0$  when  $a_6 > 0$  and  $\frac{-a_3 - a_8}{2a_6} \in I.$

*Proof.* Since  $g(0, 0), g(0, 1), g(1, 0)$  and  $g(1, 1)$  are nonnegative, we obtain the condition (A). Next, we will show (B1). When  $a_5 > 0$ , the restriction of  $g$  on  $\mathbb{R} \times \{0\}$  is an upward parabola and its minimum point is  $\left(\frac{-a_2}{2a_5}, 0\right)$ . Thus,  $a_1 - \frac{a_2^2}{4a_5} = g\left(\frac{-a_2}{2a_5}, 0\right) \geq 0$  when  $g$  is nonnegative on  $I^2$  and  $\left(\frac{-a_2}{2a_5}, 0\right) \in I^2$ . The other conditions (B2)-(B4) are the assumptions over the other line segments on the boundary of  $I^2$ . □

**Lemma 3.7.** *The function*

$$g(x, y) = a_1 + a_2x + a_3y + a_5x^2 + a_6y^2 + a_8xy$$

*is nonnegative on  $I^2$  if and only if conditions*

(A), (B1), (B2), (B3) (B4), and (C) hold

where the condition (C) is the following:

$$a_1(4a_5a_6 - a_8^2) + a_2a_3a_8 - a_2^2a_6 - a_5a_3^2 \geq 0$$

when  $a_5 > 0$ ,  $\det \begin{vmatrix} 2a_5 & a_8 \\ a_8 & 2a_6 \end{vmatrix} > 0$ , and  $\left( \frac{-2a_2a_6 + a_3a_8}{4a_5a_6 - a_8^2}, \frac{-2a_3a_5 + a_2a_8}{4a_5a_6 - a_8^2} \right) \in I^2$ .

*Proof.* To show sufficiency, we will show only the condition (C) and suppose  $g$  is nonnegative. Since  $a_5 > 0$  and  $\det \begin{vmatrix} 2a_5 & a_8 \\ a_8 & 2a_6 \end{vmatrix} > 0$ , the function  $g$  is an upward paraboloid and its vertex point is

$$(x_0, y_0) = \left( \frac{-2a_2a_6 + a_3a_8}{4a_5a_6 - a_8^2}, \frac{-2a_3a_5 + a_2a_8}{4a_5a_6 - a_8^2} \right).$$

Hence,  $a_1 + \frac{a_2a_3a_8 - a_2^2a_6 - a_5a_3^2}{4a_5a_6 - a_8^2} = g(x_0, y_0) \geq 0$  when  $(x_0, y_0) \in I^2$ . Therefore, we obtain the condition (C).

Conversely, we consider the following cases.

Case 1.  $g$  is an upward paraboloid and its vertex point is in  $I^2$ . This situation is equivalent to the assumption of (C). Thus,  $g$  is nonnegative.

Case 2.  $g$  is not an upward paraboloid or its vertex point is not in  $I^2$ . Since  $g$  is a quadratic polynomial, the minimum point  $(a, b)$  of the restriction of  $g$  on the box  $I^2$  must belong in the boundary of  $I^2$ . The condition (A) guarantees that the values of  $g$  at the four corners of  $I^2$  are nonnegative. So, we obtain the case that the minimum point  $(a, b)$  is a corner of  $I^2$ . Now suppose that the minimum point  $(a, b)$  is not a corners of  $I^2$  but is in a line segment around  $I^2$ . Consequently, the line segment is an upward paraboloid with its vertex point is in  $I^2$ . This implies an assumption of (B1)-(B4). Therefore,  $g$  is nonnegative.  $\square$

**Theorem 3.8.** *The mapping  $T$  is a DD-transformation, if and only if,  $T(f)$  is in the form*

$$T(f)(x, y) = a_1 + a_2x + a_3y + a_4f + a_5x^2 + a_6y^2 + a_8xy + a_9xf + a_{10}yf$$

and satisfies following conditions.

$$(M1) \quad 1 = a_1 + \frac{a_2}{2} + \frac{a_3}{2} + a_4 + \frac{a_5}{3} + \frac{a_6}{3} + \frac{a_8}{4} + a_9\alpha + a_{10}\beta,$$

$$(M2) \quad a_4 \geq 0, a_4 + a_9 \geq 0, a_4 + a_{10} \geq 0, \text{ and } a_4 + a_9 + a_{10} \geq 0,$$

$$(M3) \quad (A), (B1), (B2), (B3) (B4), \text{ and } (C) \text{ hold.}$$

*Proof.* Suppose  $T$  is a DD-transformation. Then

$$\begin{aligned} 1 &= \int_{I^2} T(f)d\lambda = \int_{I^2} (a_1 + a_2x + a_3y + a_4f + a_5x^2 + a_6y^2 + a_7f^2 + a_8xy \\ &\quad + a_9xf + a_{10}yf)(x, y)d\lambda \\ &= a_1 + \frac{a_2}{2} + \frac{a_3}{2} + a_4 + \frac{a_5}{3} + \frac{a_6}{3} + a_7 \int_{I^2} f^2(x, y)d\lambda + \frac{a_8}{4} \\ &\quad + a_9\alpha + a_{10}\beta. \end{aligned}$$

We get that  $a_7 = 0$  because  $\int_{I^2} f^2(x, y)d\lambda$  is varied with  $f$ . Now we get

$$T(f)(x, y) = a_1 + a_2x + a_3y + a_4f(x, y) + a_5x^2 + a_6y^2 + a_8xy + a_9xf(x, y) + a_{10}yf(x, y)$$

where  $1 = a_1 + \frac{a_2}{2} + \frac{a_3}{2} + a_4 + \frac{a_5}{3} + \frac{a_6}{3} + \frac{a_8}{4} + a_9\alpha + a_{10}\beta$ .

Following  $T(f) \geq 0$  for each  $f \in \mathfrak{F}_{\alpha, \beta}$ , we have that for all  $x, y \in [0, 1]$ ,

$$a_1 + a_2x + a_3y + a_4f(x, y) + a_5x^2 + a_6y^2 + a_8xy + a_9xf(x, y) + a_{10}yf(x, y) \geq 0.$$

Since some density  $f$  (see Example 3.4) may give a large positive number at an arbitrary point  $(x, y) \in (0, 1)^2$ , we must get  $(a_4 + a_9x + a_{10}y)f(x, y) \geq 0$ , and hence,  $a_4 + a_9x + a_{10}y \geq 0$ . We get the followings.

- $a_4 \geq 0$  when  $(x, y)$  converges to  $(0, 0)$ .
- $a_4 + a_9 \geq 0$  when  $(x, y)$  converges to  $(1, 0)$ .
- $a_4 + a_{10} \geq 0$  when  $(x, y)$  converges to  $(0, 1)$ .
- $a_4 + a_9 + a_{10} \geq 0$  when  $(x, y)$  converges to  $(1, 1)$ .

Additionally, we obtain other distribution densities  $f$  (see Example 3.4) that  $f(x, y) = 0$  at an arbitrary point  $(x, y) \in I^2$ . So

$$a_1 + a_2x + a_3y + a_5x^2 + a_6y^2 + a_8xy \geq 0$$

for each  $(x, y) \in I^2$ . Denote  $g(x, y) := a_1 + a_2x + a_3y + a_5x^2 + a_6y^2 + a_8xy$ . From Lemma 3.7, we obtain (A), (B1), (B2), (B3) (B4), and (C).

Conversely, we assume that the conditions hold. We know that

$$\int_{I^2} T(f)d\lambda = a_1 + \frac{a_2}{2} + \frac{a_3}{2} + a_4 + \frac{a_5}{3} + \frac{a_6}{3} + \frac{a_8}{4} + a_9\alpha + a_{10}\beta = 1.$$

To show that  $T(f) \geq 0$ , we will show only the function  $g(x, y) \geq 0$ , since  $(a_4 + a_9x + a_{10}y)f \geq 0$ . Following Lemma 3.7, the proof is done.  $\square$

We see that DD-transformations depend on eight coefficients. So we will show some example of DD-transformations.

**Example 3.9.** For each  $\alpha, \beta \in (0, 1)$ , we may obtain a DD-transformation  $T$  in the form

$$T(f)(x, y) = 1 + x - y - xy + \left(\frac{1}{4} - \alpha a_9 + a_9 x - \beta a_{10} + a_{10} y\right) f$$

where two coefficients  $a_9$  and  $a_{10}$  satisfy the following

1.  $\alpha a_9 + \beta a_{10} \geq \frac{1}{4}$ ,
2.  $(1 - \alpha)a_9 + (1 - \beta)a_{10} \geq -\frac{1}{4}$ ,
3.  $(1 - \alpha)a_9 - \beta a_{10} \geq -\frac{1}{4}$ ,
4.  $-\alpha a_9 + (1 - \beta)a_{10} \geq -\frac{1}{4}$ .

**Example 3.10.** For each  $\alpha, \beta \in (0, 1)$  satisfied  $\beta - \alpha = \frac{1}{2}$ , we obtain a DD-transformation  $T$  in the form

$$T(f)(x, y) = a_1 + a_2 x + a_5 x^2 + (1 + x - y)f$$

where the coefficients  $a_1, a_2$  and  $a_5$  satisfy the following

1.  $a_1 = \frac{1}{2} - \frac{a_2}{2} - \frac{a_5}{3}$ ,
2.  $\frac{1}{2} \geq \frac{a_2}{2} + \frac{a_5}{3}$ , and  $\frac{1}{2} \geq -\frac{a_2}{2} - \frac{2a_5}{3}$ ,
3.  $0 < -a_2 < 2a_5$ ,
4.  $4\left(\frac{1}{2} - \frac{a_2}{2} - \frac{a_5}{3}\right)a_5 - a_2^2 \geq 0$ .

Moreover, the condition 4. implies that the graph of the set of  $(a_2, a_5)$  is in an oval, see the figure below.

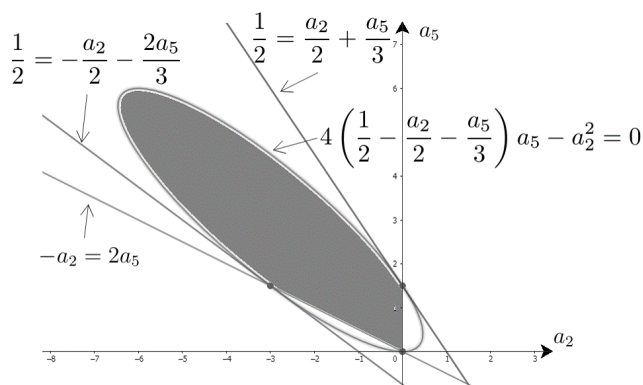


Fig. 1.

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