



# Fixed Point Theorems for Some Generalized Multi-Valued Nonexpansive Mappings in Hadamard Spaces

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**Abstract :** We show that a condition on mappings introduced by Bunlue and Suantai is weaker than the notion of nonexpansive mappings and stronger than the notion of quasi-nonexpansive mappings. We also obtain the demiclosed principle as well as a fixed point theorem for the class of mappings satisfying this condition. A convergence theorem of the Ishikawa iteration for discontinuous quasi-nonexpansive mappings is also given.

**Keywords :** fixed point; multi-valued mapping; Ishikawa iteration; Hadamard space.

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## 1 Introduction

Fixed point theory is an important tool for finding solutions of various nonlinear equations and can be applied to many important problems such as minimiza-

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tion problems and equilibrium problems. One of the fundamental and celebrated results in metric fixed point theory is the Banach contraction principle [1] which states that every single-valued contractive mapping on a complete metric space always has a unique fixed point. However, many problems in science and technology can be arranged in the form of multi-valued mappings. It is natural to study the extension of the known fixed point results for single-valued mappings to the setting of multi-valued mappings. One of the most important fixed point theorems for multi-valued mappings was proved by Nadler [2] in 1969. He proved that every multi-valued contractive mapping on a complete metric space has a fixed point. Since then many results regarding the existence of fixed points for several kinds of multi-valued contractive mappings have been developed and many papers have appeared (see e.g., [3-14]). Among other things, Berinde and Berinde [7] proved the following theorem.

**Theorem 1.1.** *Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow CB(X)$  be a multi-valued mapping. Suppose there exist  $\theta \in [0, 1)$  and  $L \geq 0$  such that*

$$H(T(x), T(y)) \leq \theta \cdot d(x, y) + L \cdot \text{dist}(y, T(x)) \text{ for all } x, y \in X. \quad (1.1)$$

*Then  $T$  has a fixed point.*

Recently, inspired by (1.1), Bunlue and Suantai [15] introduced the concept of Berinde nonexpansive mappings and proved the existence of fixed points as well as the demiclosed principle for such kind of mappings in Banach spaces satisfying the Opial's condition.

A geodesic space  $X$  is said to be a Hadamard space if it is a complete metric space and every geodesic triangle in  $X$  is thinner than its comparison triangle in the Euclidean plane. The precise definition is given in section 3. Fixed point theory for nonexpansive mappings in Hadamard spaces was first studied by Kirk [16] in 2003. He showed that every single-valued nonexpansive mapping on a bounded closed convex subset of a Hadamard space always has a fixed point. Notice that complete  $\mathbb{R}$ -trees are outstanding examples of nonlinear Hadamard spaces and fixed point theory in complete  $\mathbb{R}$ -trees has many applications in graph theory (see e.g., [17, 18]).

In this paper, motivated by the above results, we prove the existence of fixed points and the demiclosed principle for the class of Berinde nonexpansive mappings in Hadamard spaces. We also obtain a convergence theorem of the Ishikawa iteration for discontinuous quasi-nonexpansive mappings in this setting. Our results extend and improve the results of [7, 15, 16] and many results in the literature.

## 2 Multi-Valued Mappings

Throughout this paper,  $\mathbb{N}$  stands for the set of natural numbers and  $\mathbb{R}$  stands for the set of real numbers. Let  $(X, d)$  be a metric space,  $x \in X$  and  $\emptyset \neq E \subseteq X$ . The *distance* from  $x$  to  $E$  is defined by

$$\text{dist}(x, E) := \inf\{d(x, y) : y \in E\}.$$

We denote by  $CB(E)$  the family of nonempty closed bounded subsets of  $E$  and by  $K(E)$  the family of nonempty compact subsets of  $E$ . The *Pompeiu-Hausdorff distance* on  $CB(E)$  is defined by

$$H(A, B) := \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\} \text{ for all } A, B \in CB(E).$$

**Definition 2.1.** Let  $E$  be a nonempty subset of a metric space  $(X, d)$  and  $T : E \rightarrow CB(X)$  be a multi-valued mapping. A point  $x \in E$  is said to be a *fixed point* of  $T$  if  $x \in T(x)$ . We denote by  $Fix(T)$  the *fixed point set* of  $T$ , that is,  $Fix(T) := \{x \in E : x \in T(x)\}$ .

**Definition 2.2.** A multi-valued mapping  $T : E \rightarrow CB(X)$  is said to be  
 (i) *contractive* if there exists  $\lambda \in [0, 1)$  such that

$$H(T(x), T(y)) \leq \lambda d(x, y) \text{ for all } x, y \in E;$$

(ii) *nonexpansive* if  $H(T(x), T(y)) \leq d(x, y)$  for all  $x, y \in E$ ;

(iii) *quasi-nonexpansive* if  $Fix(T) \neq \emptyset$  and

$$H(T(x), T(p)) \leq d(x, p) \text{ for all } x \in E \text{ and } p \in Fix(T).$$

It is clear that every contractive mapping is nonexpansive and every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. Inspired by (3.14) of [7], Bunlue and Suantai [15] introduce the following class of multi-valued mappings.

**Definition 2.3.** A multi-valued mapping  $T : E \rightarrow CB(X)$  is said to be *Berinde-Berinde nonexpansive* ( $B^2$ -nonexpansive in short) if there exists  $\mu \geq 0$  such that

$$H(T(x), T(y)) \leq d(x, y) + \mu \cdot \text{dist}(x, T(x)) \text{ for all } x, y \in E. \tag{2.1}$$

**Proposition 2.4.** *The following statements hold.*

(1) *If  $T$  is nonexpansive, then  $T$  is  $B^2$ -nonexpansive.*

(2) *If  $T$  is  $B^2$ -nonexpansive and  $Fix(T) \neq \emptyset$ , then  $T$  is quasi-nonexpansive.*

The converse of (2) in Proposition 2.4 is not true as shown in the following example. Notice also that the converse of (1) is not true, see Example 4.3 below.

**Example 2.5.** ([19, Example 2]) Let  $E = [-1, 1]$  and  $T : E \rightarrow CB(E)$  be defined by

$$T(x) := \begin{cases} \left\{ \frac{x}{1+|x|} \sin\left(\frac{1}{x}\right) \right\} & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0. \end{cases}$$

It is easy to see that  $Fix(T) = \{0\}$ . For  $x \in E$ , we have

$$H(T(x), T(0)) = \left| \frac{x}{1+|x|} \sin\left(\frac{1}{x}\right) \right| \leq \frac{|x|}{1+|x|} \leq |x - 0|.$$

This implies that  $T$  is a quasi-nonexpansive mapping. Next, we show that  $T$  is not  $B^2$ -nonexpansive. For each  $n \in \mathbb{N}$ , we set  $x_n := \frac{1}{2\pi n + \pi/2}$  and  $y_n := \frac{1}{2\pi n}$ . Then

$$\begin{aligned} \frac{H(T(x_n), T(y_n)) - |x_n - y_n|}{\text{dist}(x_n, T(x_n))} &= \left[ \frac{x_n}{1 + x_n} - (y_n - x_n) \right] \left( \frac{1 + x_n}{x_n^2} \right) \\ &= \frac{1}{x_n} - \frac{(y_n - x_n)(1 + x_n)}{x_n^2} \\ &= (2\pi n + \pi/2) - \frac{(2\pi n + \pi/2 + 1)}{4n} \rightarrow \infty. \end{aligned}$$

This implies that  $T$  is not  $B^2$ -nonexpansive.

### 3 Hadamard Spaces

Let  $[0, l]$  be a closed interval in  $\mathbb{R}$  and  $x, y$  be two points in a metric space  $(X, d)$ . A *geodesic* joining  $x$  to  $y$  is a map  $c : [0, l] \rightarrow X$  such that  $c(0) = x, c(l) = y$ , and  $d(c(s), c(t)) = |s - t|$  for all  $s, t \in [0, l]$ . The image of  $c$  is called a *geodesic segment* joining  $x$  and  $y$  which when unique is denoted by  $[x, y]$ . The space  $(X, d)$  is said to be a *geodesic space* if every two points in  $X$  are joined by a geodesic, and  $X$  is said to be *uniquely geodesic* if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ . A subset  $E$  of  $X$  is said to be *convex* if every pair of points  $x$  and  $y$  in  $E$  can be joined by a geodesic in  $X$  and the image of every such geodesic is contained in  $E$ .

A *geodesic triangle*  $\Delta(p, q, r)$  in a geodesic space  $(X, d)$  consists of three points  $p, q, r$  in  $X$  and a choice of three geodesic segments  $[p, q], [q, r], [r, p]$  joining them. A *comparison triangle* for geodesic triangle  $\Delta(p, q, r)$  in  $X$  is a triangle  $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$  in the Euclidean plane  $\mathbb{R}^2$  such that  $d_{\mathbb{R}^2}(\bar{p}, \bar{q}) = d(p, q), d_{\mathbb{R}^2}(\bar{q}, \bar{r}) = d(q, r)$ , and  $d_{\mathbb{R}^2}(\bar{r}, \bar{p}) = d(r, p)$ . A point  $\bar{u} \in [\bar{p}, \bar{q}]$  is called a *comparison point* of  $u \in [p, q]$  if  $d(p, u) = d_{\mathbb{R}^2}(\bar{p}, \bar{u})$ . Comparison points on  $[\bar{q}, \bar{r}]$  and  $[\bar{r}, \bar{p}]$  are defined in the same way.

**Definition 3.1.** A geodesic triangle  $\Delta(p, q, r)$  in  $(X, d)$  is said to satisfy the *CAT(0) inequality* if for any  $u, v \in \Delta(p, q, r)$  and for their comparison points  $\bar{u}, \bar{v} \in \bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$ , one has

$$d(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

A geodesic space  $(X, d)$  is said to be a *Hadamard space* if it is a complete metric space and all of its geodesic triangles satisfy the CAT(0) inequality. It is well-known that every Hadamard space is uniquely geodesic. Notice also that Hilbert spaces and complete  $\mathbb{R}$ -trees are examples of Hadamard spaces (see, e.g., [20]).

Let  $(X, d)$  be a Hadamard space,  $x, y \in X$  and  $t \in [0, 1]$ . By Lemma 2.1 of [21], there exists a unique point  $z \in [x, y]$  such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(y, z) = td(x, y). \quad (3.1)$$

We denote by  $tx \oplus (1 - t)y$  the unique point  $z$  satisfying (3.1). For a nonempty subset  $E$  of  $X$ , we set

$$tx \oplus (1 - t)E := \{tx \oplus (1 - t)y : y \in E\}.$$

Now, we collect some elementary facts about Hadamard spaces.

**Lemma 3.2.** ([22]) *Let  $x, y, z$  be points in a Hadamard space  $(X, d)$  and let  $t \in [0, 1]$ . Then the following inequalities hold:*

$$d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z), \tag{3.2}$$

$$d(tx \oplus (1 - t)y, tx \oplus (1 - t)z) \leq (1 - t)d(y, z). \tag{3.3}$$

If  $x, y_1, y_2$  are points in a Hadamard space and if  $y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$ , then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [23].

The following lemma is a generalization of the (CN) inequality which can be found in [21].

**Lemma 3.3.** *Let  $(X, d)$  be a Hadamard space. Then*

$$d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y), \tag{3.4}$$

for all  $x, y, z \in X$  and  $t \in [0, 1]$ .

Let  $\{x_n\}$  be a bounded sequence in a Hadamard space  $(X, d)$ . For  $x \in X$ , we set

$$r(x, \{x_n\}) := \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius  $r(\{x_n\})$  of  $\{x_n\}$  is given by

$$r(\{x_n\}) := \inf \{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center  $A(\{x_n\})$  of  $\{x_n\}$  is the set

$$A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [24] that in a Hadamard space,  $A(\{x_n\})$  consists of exactly one point. A sequence  $\{x_n\}$  in  $X$  is said to  $\Delta$ -converge to a point  $x$  in  $X$  if  $A(\{x_{n_k}\}) = \{x\}$  for every subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$ . In this case, we write  $\Delta - \lim_{n \rightarrow \infty} x_n = x$  and call  $x$  the  $\Delta$ -limit of  $\{x_n\}$ .

The following lemmas are also needed.

**Lemma 3.4.** ([25]) *Every bounded sequence in a Hadamard space always has a  $\Delta$ -convergent subsequence.*

**Lemma 3.5.** ([26]) *If  $E$  is a closed convex subset of a Hadamard space and if  $\{x_n\}$  is a bounded sequence in  $E$ , then the asymptotic center of  $\{x_n\}$  is in  $E$ .*

## 4 Fixed Point Theorems

This section is begun by proving the demiclosed principle for  $B^2$ -nonexpansive mappings in Hadamard spaces.

**Theorem 4.1.** *Let  $E$  be a nonempty closed convex subset of a Hadamard space  $(X, d)$  and  $T : E \rightarrow K(X)$  be a  $B^2$ -nonexpansive mapping. If  $\{x_n\}$  is a sequence in  $E$  and  $x \in X$ , then the conditions  $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} \text{dist}(x_n, T(x_n)) = 0$  imply  $x \in T(x)$ .*

*Proof.* By Lemma 3.5,  $x \in E$ . For each  $n \in \mathbb{N}$ , we can choose  $y_n \in T(x_n)$  and  $z_n \in T(x)$  such that

$$d(x_n, y_n) = \text{dist}(x_n, T(x_n)) \text{ and } d(y_n, z_n) = \text{dist}(y_n, T(x)).$$

Since  $T(x)$  is compact, there exists a subsequence  $\{z_{n_k}\}$  of  $\{z_n\}$  such that  $\lim_{n \rightarrow \infty} z_{n_k} = w$  for some  $w \in T(x)$ . By (2.1), we have

$$\begin{aligned} d(x_{n_k}, w) &\leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, w) \\ &\leq d(x_{n_k}, y_{n_k}) + H(T(x_{n_k}), T(x)) + d(z_{n_k}, w) \\ &\leq (1 + \mu)\text{dist}(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, x) + d(z_{n_k}, w). \end{aligned}$$

This implies that  $\limsup_{k \rightarrow \infty} d(x_{n_k}, w) \leq \limsup_{k \rightarrow \infty} d(x_{n_k}, x)$ . Therefore,  $w \in A(\{x_{n_k}\}) = \{x\}$  and hence  $x = w \in T(x)$ .  $\square$

Now, we prove a fixed point theorem which is an analog of Theorem 6 of [15].

**Theorem 4.2.** *Let  $E$  be a nonempty bounded closed convex subset of a Hadamard space  $(X, d)$  and  $T : E \rightarrow K(E)$  be a  $B^2$ -nonexpansive mapping. Suppose there exist  $u \in E$  and  $L \geq 0$  such that*

$$H(T(x), T(y)) \leq d(x, y) + L \cdot \text{dist}(y, \alpha u \oplus (1 - \alpha)T(x)), \quad (4.1)$$

for all  $x, y \in E$  and  $\alpha \in [0, 1]$ . Then  $T$  has a fixed point in  $E$ .

*Proof.* For each  $n \in \mathbb{N}$ , we define  $T_n : E \rightarrow K(E)$  by

$$T_n(x) := \frac{1}{n}u \oplus \left(1 - \frac{1}{n}\right)T(x), \text{ for all } x \in E.$$

It follows from (3.3) and (4.1) that

$$\begin{aligned} H(T_n(x), T_n(y)) &\leq \left(1 - \frac{1}{n}\right)H(T(x), T(y)) \\ &\leq \left(1 - \frac{1}{n}\right)d(x, y) + \left(1 - \frac{1}{n}\right)L \cdot \text{dist}(y, T_n(x)). \end{aligned}$$

Applying Theorem 1.1 we can conclude that  $T_n$  has a fixed point, say  $x_n$ . For each  $n \in \mathbb{N}$ , there exists  $z_n \in T(x_n)$  such that

$$x_n = \frac{1}{n}u \oplus (1 - \frac{1}{n})z_n.$$

By Lemmas 3.4 and 3.5, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\Delta - \lim_{k \rightarrow \infty} x_{n_k} = x$  for some  $x \in E$ . Notice also that

$$\text{dist}(x_{n_k}, T(x_{n_k})) \leq d(x_{n_k}, z_{n_k}) = \frac{1}{n_k}d(u, z_{n_k}) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

By Theorem 4.1,  $x$  is a fixed point of  $T$  and hence the proof is complete. □

One may observe that the condition (4.1) in Theorem 4.2 is strong. However, the following example shows that it is a necessary condition.

**Example 4.3.** Let  $E = [0, 1]$ ,  $X = (\mathbb{R}, |\cdot|)$  and  $T : E \rightarrow K(E)$  be defined by

$$T(x) := \begin{cases} \{1\} & \text{if } x \in [0, \frac{1}{2}); \\ \{0\} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Obviously,  $T$  is not nonexpansive and does not have a fixed point. Next, we show that  $T$  is a  $B^2$ -nonexpansive mapping. Let  $\mu = 2$  and  $x, y \in [0, 1]$ .

**Case 1.** If  $x \in [0, \frac{1}{2}]$  and  $y \in (\frac{1}{2}, 1]$ , then  $T(x) = \{1\}$  and  $T(y) = \{0\}$ . This implies that  $\text{dist}(x, T(x)) \geq \frac{1}{2}$  and hence

$$H(T(x), T(y)) = |1 - 0| = 1 \leq |x - y| + \mu \cdot \text{dist}(x, T(x)).$$

**Case 2.** If  $x \in (\frac{1}{2}, 1]$  and  $y \in [0, \frac{1}{2}]$ , then  $T(x) = \{0\}$  and  $T(y) = \{1\}$ . This implies that  $\text{dist}(x, T(x)) \geq \frac{1}{2}$  and hence

$$H(T(x), T(y)) = |0 - 1| = 1 \leq |x - y| + \mu \cdot \text{dist}(x, T(x)).$$

Next, we show that  $T$  does not satisfy (4.1). Given  $u \in E$  and  $L \geq 0$ . Choose  $x = \frac{1}{2}$ ,  $y = 1$  and  $\alpha = 0$ . Then  $T(x) = \{1\}$ ,  $T(y) = \{0\}$  and

$$\text{dist}(y, \alpha u \oplus (1 - \alpha)T(x)) = \text{dist}(y, T(x)) = 0.$$

This implies that

$$H(T(x), T(y)) = |1 - 0| = 1 > \frac{1}{2} = |x - y| + L \cdot \text{dist}(y, \alpha u \oplus (1 - \alpha)T(x)).$$

## 5 Convergence Theorems

In 2009, Shahzad and Zegeye [27] defined an Ishikawa iteration for multi-valued mappings in Banach spaces and proved, under some appropriate conditions, that the proposed iteration converges to a fixed point of a quasi-nonexpansive mapping. In 2010, Puttasontiphot [28] extended the idea of Shahzad and Zegeye to the setting of Hadamard spaces and defined the sequence of Ishikawa iteration in the following manner:

**Definition 5.1.** Let  $E$  be a nonempty convex subset of a Hadamard space  $X$ ,  $\{\alpha_n\}$ ,  $\{\beta_n\}$  be sequences in  $[0, 1]$ , and  $T : E \rightarrow CB(E)$  be a multi-valued mapping. The sequence of Ishikawa iteration is defined by  $x_1 \in E$ ,

$$y_n = \beta_n z_n \oplus (1 - \beta_n)x_n, \quad n \in \mathbb{N},$$

where  $z_n \in T(x_n)$ , and

$$x_{n+1} = \alpha_n z'_n \oplus (1 - \alpha_n)x_n, \quad n \in \mathbb{N}, \quad (5.1)$$

where  $z'_n \in T(y_n)$ .

**Definition 5.2.** A multi-valued mapping  $T : E \rightarrow CB(E)$  is said to be

- (i) *continuous* if for any sequence  $\{x_n\}$  in  $E$  such that  $\lim_{n \rightarrow \infty} x_n = x$ , we have  $\lim_{n \rightarrow \infty} H(T(x_n), T(x)) = 0$ ;
- (ii) *hemicompact* if for any sequence  $\{x_n\}$  in  $E$  such that  $\lim_{n \rightarrow \infty} \text{dist}(x_n, T(x_n)) = 0$ , there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and  $q \in E$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = q$ ;
- (iii) *Berinde nonexpansive* ( $B$ -nonexpansive in short) if there exists  $L \geq 0$  such that

$$H(T(x), T(y)) \leq d(x, y) + L \cdot \text{dist}(y, T(x)) \text{ for all } x, y \in E.$$

The following results can be found in [28].

**Lemma 5.3.** Let  $E$  be a nonempty closed convex subset of a Hadamard space  $(X, d)$  and  $T : E \rightarrow CB(E)$  be a quasi-nonexpansive mapping with  $T(p) = \{p\}$  for each  $p \in \text{Fix}(T)$ . Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (5.1). Then  $\lim_{n \rightarrow \infty} d(x_n, p)$  exists for each  $p \in \text{Fix}(T)$ .

**Theorem 5.4.** Let  $E$  be a nonempty closed convex subset of a Hadamard space  $(X, d)$  and  $T : E \rightarrow CB(E)$  be a quasi-nonexpansive mapping with  $T(p) = \{p\}$  for each  $p \in \text{Fix}(T)$ . Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (5.1). Assume that  $T$  is hemicompact and continuous, and (i)  $0 \leq \alpha_n, \beta_n < 1$ ; (ii)  $\beta_n \rightarrow 0$ ; (iii)  $\sum \alpha_n \beta_n = \infty$ . Then  $\{x_n\}$  converges to a fixed point of  $T$ .

Next, we show that the continuity of  $T$  in Theorem 5.4 can be replaced by the  $B$ -nonexpansiveness of  $T$ . For this we will make use of the following lemma.



**Lemma 5.5.** ([29]) *Let  $\{\alpha_n\}, \{\beta_n\}$  be two real sequences such that*

- (i)  $0 \leq \alpha_n, \beta_n < 1$ ;
- (ii)  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ ;
- (iii)  $\sum \alpha_n \beta_n = \infty$ .

*Let  $\{\gamma_n\}$  be a nonnegative real sequence such that  $\sum \alpha_n \beta_n (1 - \beta_n) \gamma_n < \infty$ . Then  $\{\gamma_n\}$  has a subsequence which converges to zero.*

**Theorem 5.6.** *Let  $E$  be a nonempty closed convex subset of a Hadamard space  $(X, d)$  and  $T : E \rightarrow CB(E)$  be a quasi-nonexpansive mapping with  $T(p) = \{p\}$  for each  $p \in \text{Fix}(T)$ . Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (5.1). Assume that  $T$  is hemicompact and  $B$ -nonexpansive, and (i)  $0 \leq \alpha_n, \beta_n < 1$ ; (ii)  $\beta_n \rightarrow 0$ ; (iii)  $\sum \alpha_n \beta_n = \infty$ . Then  $\{x_n\}$  converges to a fixed point of  $T$ .*

*Proof.* Let  $p \in \text{Fix}(T)$ . It follows from (3.4) that

$$\begin{aligned}
 d^2(x_{n+1}, p) &= d^2(\alpha_n z'_n \oplus (1 - \alpha_n)x_n, p) \\
 &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(z'_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, z'_n) \\
 &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n H^2(T(y_n), T(p)) - \alpha_n(1 - \alpha_n)d^2(x_n, z'_n) \\
 &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, z'_n) \\
 &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p)
 \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
 d^2(y_n, p) &= d^2(\beta_n z_n \oplus (1 - \beta_n)x_n, p) \\
 &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) - \beta_n(1 - \beta_n)d^2(x_n, z_n) \\
 &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n H^2(T(x_n), T(p)) - \beta_n(1 - \beta_n)d^2(x_n, z_n) \\
 &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - \beta_n(1 - \beta_n)d^2(x_n, z_n) \\
 &\leq d^2(x_n, p) - \beta_n(1 - \beta_n)d^2(x_n, z_n).
 \end{aligned} \tag{5.3}$$

By (5.2) and (5.3), we have

$$d^2(x_{n+1}, p) \leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) - \alpha_n \beta_n (1 - \beta_n) d^2(x_n, z_n).$$

This implies

$$\alpha_n \beta_n (1 - \beta_n) d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p).$$

Thus

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) d^2(x_n, z_n) < \infty.$$

By Lemma 5.5, there exist subsequences  $\{x_{n_k}\}$  and  $\{z_{n_k}\}$  of  $\{x_n\}$  and  $\{z_n\}$  respectively, such that  $\lim_{k \rightarrow \infty} d(x_{n_k}, z_{n_k}) = 0$ . Hence

$$\lim_{k \rightarrow \infty} \text{dist}(x_{n_k}, T(x_{n_k})) \leq \lim_{k \rightarrow \infty} d(x_{n_k}, z_{n_k}) = 0.$$

Since  $T$  is hemicompact, by passing to a subsequence, we may assume that  $\lim_{k \rightarrow \infty} x_{n_k} = q$  for some  $q \in E$ . Since  $T$  is  $B$ -nonexpansive, there exists  $L \geq 0$  such that

$$H(T(x_{n_k}), T(q)) \leq d(x_{n_k}, q) + L \cdot \text{dist}(q, T(x_{n_k})) \text{ for all } k \in \mathbb{N}.$$

This implies that

$$\begin{aligned} \text{dist}(q, T(q)) &\leq d(q, x_{n_k}) + \text{dist}(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \\ &\leq 2d(x_{n_k}, q) + \text{dist}(x_{n_k}, T(x_{n_k})) + L \cdot \text{dist}(q, T(x_{n_k})) \\ &\leq (2 + L)d(x_{n_k}, q) + (1 + L)\text{dist}(x_{n_k}, T(x_{n_k})) \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

That is  $q \in \text{Fix}(T)$ . By Lemma 5.3,  $\lim_{n \rightarrow \infty} d(x_n, q)$  exists, it follows that  $\{x_n\}$  converges to  $q$ . Therefore, the proof is complete.  $\square$

Finally, we finish the paper by providing an example supporting Theorem 5.6.

**Example 5.7.** Let  $E = [0, 1]$ ,  $X = (\mathbb{R}, |\cdot|)$  and  $T : E \rightarrow CB(E)$  be defined by

$$T(x) := \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0, 1); \\ \{0\} & \text{if } x = 1. \end{cases}$$

It is clear that  $\text{Fix}(T) = \{\frac{1}{2}\}$ . Since  $E$  is compact,  $T$  is hemicompact. Now, we show that  $T$  is a quasi-nonexpansive mapping. Let  $x \in [0, 1)$ . Then  $T(x) = \{\frac{1}{2}\}$ , which implies that

$$H(T(x), T(\frac{1}{2})) = 0 \leq \left| x - \frac{1}{2} \right|.$$

On the other hand, if  $x = 1$  then

$$H(T(x), T(\frac{1}{2})) = \left| 0 - \frac{1}{2} \right| = \frac{1}{2} = \left| x - \frac{1}{2} \right|.$$

This shows that  $T$  is a quasi-nonexpansive mapping.

Next, we show that  $T$  is a  $B$ -nonexpansive mapping. Let  $L = 1$  and  $x, y \in [0, 1]$ .

**Case 1.** If  $x, y \in [0, 1)$ , then

$$H(T(x), T(y)) = 0 \leq |x - y| + L \cdot \text{dist}(y, T(x)).$$

**Case 2.** If  $x \in [0, 1)$  and  $y = 1$ , then  $T(x) = \{\frac{1}{2}\}$  and  $T(y) = \{0\}$ . This implies that  $\text{dist}(y, T(x)) = \frac{1}{2}$  and hence

$$H(T(x), T(y)) = \frac{1}{2} \leq |x - y| + L \cdot \text{dist}(y, T(x)).$$

**Case 3.** If  $x = 1$  and  $y \in [0, \frac{1}{2})$ , then  $T(x) = \{0\}$  and  $T(y) = \{\frac{1}{2}\}$ . This implies that  $|x - y| \geq \frac{1}{2}$  and hence

$$H(T(x), T(y)) = \frac{1}{2} \leq |x - y| + L \cdot \text{dist}(y, T(x)).$$

**Case 4.** If  $x = 1$  and  $y \in [\frac{1}{2}, 1)$ , then  $T(x) = \{0\}$  and  $T(y) = \{\frac{1}{2}\}$ . This implies that  $\text{dist}(y, T(x)) \geq \frac{1}{2}$  and hence

$$H(T(x), T(y)) = \frac{1}{2} \leq |x - y| + L \cdot \text{dist}(y, T(x)).$$

This shows that  $T$  is a  $B$ -nonexpansive mapping. By Theorem 5.6, for any starting point  $x_1$  in  $E$ , the sequence of Ishikawa iteration defined by (5.1) converges to  $\frac{1}{2}$ . However, in this situation, Theorem 5.4 cannot be applied since  $T$  is not continuous.

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