Thai Journal of Mathematics
Volume 5(2007) Number 2 : 253–259
www.math.science.cmu.ac.th/thaijournal

Application of Fixed Point Theory in Metric Spaces

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Abstract

Many problems in pure and applied mathematics have as their solutions the fixed point of some mapping F. Therefore a number of procedures in numerical analysis and approximations theory amount to obtaining successive approximations to the fixed point of an approximate mapping. Our object in this paper to discuss about fixed point theory and its applications in metric spaces, also we established some fixed point theorems in complete metric spaces, which generalized many results of great mathematicians.

Keywords : *Fixed-point theory, Metric space, Complete Metric space, continuous function.*

1. Introduction

The well known Banach [1] contraction principal states that "If X is complete metric space and f is a contraction mapping on X into itself, then f has unique fixed point in X". Many mathematicians worked on this principal. Kanan[4] proved that "If T is self mapping of a complete metric space X into itself satisfying:

$$d(Tx, Ty) \leq [d(Tx, x) + d(Ty, y)] \ \text{ for all } \ x, y \in X, \quad \text{where} \ \alpha \in \left[0, \frac{1}{2}\right].$$

Then T has unique fixed point in x.

Fisher [3] proved the result with

$$d(Tx, Ty) \leq [d(Ty, x) + d(Tx, y)], \quad \text{for all} \ \ x, y \in X, \quad \text{where} \ \alpha \in \left[0, \frac{1}{2}\right].$$

A similar conclusion was also obtained by Chaterjee [2].

In 1977, the mathematician Jaggi [6] introduced the rational expression first time as:

$$d(Tx,Ty) \leq \alpha d(x,y)+\beta\frac{d(x,Tx)d(y,Ty)}{d(x,y)} \text{ for all } x,y\in X,\ x\neq y, 0\leq \alpha+\beta \prec 1.$$

In 1980 the mathematicians Jaggi and Das [7] obtained some fixed point theorems with the mapping satisfying:

$$d(Tx,Ty) \leq \alpha d(x,y)+\beta\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \text{ for all } x,y\in X,\ \alpha+\beta \prec 1.$$

In the present paper we are also finding a new rational expression, using complete metric spaces, which satisfy the many results of great mathematicians.

2. Main Result

Let T be a continues self map, defined on a complete metric spaces x. Further T satisfies the following conditions:

$$\begin{aligned} d(Tx,Ty) \quad \leq \quad & \frac{d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)}{d(x,y)} + \beta\frac{d(x,Ty)[d(x,Tx)+d(y,Ty)]}{d(x,y)+d(y,Ty)+d(y,Tx)} \\ & + \frac{d(x,T_x)d(y,Tx)+d(y,Ty)d(x,Ty)}{d(x,Tx)+d(y,T_x)+d(y,Ty)+d(x,Ty)} \\ & +\delta[d(x,Tx)+d(y,Ty)]+\eta[d(y,Tx)+d(x,Ty)]+\mu d(x,y) \end{aligned}$$

For all $x,y\in X, x\neq y$ and for $\alpha,\beta,\gamma,\delta,\eta,\mu\in[0,1)$, and $2\alpha+\gamma+4\delta+4\eta+2\mu\prec 2$, *then* T *has unique fixed* - point in T.

Proof : Let x_o be an arbitrary point in X, and we define a sequence $\{x_n\}$ by means of iterates of T. By setting, $T^n x_o = x_n$, where n is a positive integers. If $x_n = x_{n+1}$, for some n, then we have $Tx_n = x_n$, then x_n is a fixed point of T taking $x_n \neq x_{n+1}$ for all n.

Now $d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1})$

$$\begin{aligned} & d(Tx_n,Tx_{n-1}) \leq \alpha\frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})+d(x_n,Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_n,x_{n-1})} \\ & +\beta\frac{d(x_{n-1_n},Tx_n)[d(x_{n-1},Tx_{n-1_n})+d(x_n,Tx_n)]}{d(x_n,x_{n-1_n})+d(x_{n-1},Tx_{n-1})+d(x_{n-1},Tx_n)} \\ & +\gamma\frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})+d(x_{n-1},Tx_{n-1})d(x_n,Tx_{n-1})}{d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})+d(x_{n-1},Tx_{n-1})+d(x_n,Tx_{n-1})} \\ & +\delta[(dx_n,Tx_n)+d(x_{n-1},Tx_{n-1})]+\eta[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})]+\mu d(x_n,x_{n-1}) \\ & \leq \left(\alpha+\frac{\gamma}{3}+\delta+\eta\right)d(x_n,x_{n+1})+(\delta+\eta+\mu)d(x_{n-1},x_n) \end{aligned}$$

i.e.

$$d(x_{n+1}, x_n) \leq \frac{\delta+\eta+\mu}{1-\left(\alpha+\frac{\gamma}{2}+\delta+\eta\right)} d(x_{n-1}, x_n).$$

On applying the same process, we get

$$d(x_{n+1}, x_n) \leq \frac{\delta+\eta+\mu}{1-\left(\alpha+\frac{\gamma}{2}+\delta+\eta\right)} d(x_0, x_1).$$

By the triangular inequality, we have for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x'_{n+1}, x_{n+2}) + \cdots + d(x'_{m-1} x_m) \\ &\leq (s^n + s^{n+1} + \cdots | s^{m-1}) d(x_0, Tx_0) \end{aligned}$$

where

$$s = \frac{\delta+\eta+\mu}{1-\left(\alpha+\frac{\gamma}{2}+\delta+\eta\right)} < 1.$$

Therefore,

$$d(x_n, x_m) \leq \frac{s^n}{1-s} d(x_0, Tx_0) \to 0 \quad \text{as} \quad m, n \to \infty.$$

So, $\{x_n\}$ is Cauchy sequence in x, so by completeness of X, there is a point, $u \in x$ such that $x_n \to u$ as $n \to \infty$. Further, the continuity of T in X implies

$$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim n \to \infty x_{n+1} = u.$$

Therefore, u is a fixed point of T in x.

3. Uniqueness

Suppose if there is any other $v \neq u$ in X such that $T(v) = v$, then $d(u, v) = d(T_u, T_v)$

$$\begin{aligned} d(u,v) \leq\ & \alpha \frac{d(u,Tu)d(v,Tv) + d(u,Tv)d(v,Tu)}{d(u,v)} + \frac{d(u,Tv)[(d(u,Tu) + d(v,Tv)]}{d(u,v) + d(v,Tv) + d(v,Tu)} \\ & +\gamma \frac{d(u,Tu) + d(v,Tu) + d(v,Tv)d(u,Tv)}{d(u,Tu)d(v,Tu)d(v,Tv)d(u,Tv)} \\ & +\delta[d(u,Tu) + d(v,Tv)] + \eta[d(u,Tv) + d(v,Tu)] + \mu d(u,v) \end{aligned}$$

i.e.

$$d(u,v) \leq [\alpha + 2\eta + \mu]d(u,v).$$

This is a contradiction because, $2\alpha + \gamma + 4\delta + 4\eta + 2\mu \prec 2$. Hence u is the unique fixed point.

Now we are proving an interesting result in which T is not necessarily continuous in X, but T^p is continuous for some positive integer P, T^p is continuous, Then T has a unique fixed point.

Theorem 2 : Let T be a self map defined on a complete metric space (X, d) such that (1:1) holds. If for some positive integer P, T^p is continuous, then T has a unique fixed point.

Proof : We define a sequence $\{x_n\}$ as in theorem 1, clearly it converges to some point u in X. Therefore there is a subsequence $\{X_{nk}\}$ of $\{X_n\}, (n_k = K_P)$ also converges to u. Also

$$\begin{aligned} T_u^p &= T^p(\lim_{k\to\infty} x_{nk}) \\ &= \lim_{k\to\infty}(T^p x_{nk}) \\ &= \lim_{k\to\infty}(X_{nk+1}) \\ &= u. \end{aligned}$$

Therefore, u is a fixed point of T_p.

Now, we show that, $T_u = u$.

Let m be the smallest positive integer such that $T_u^m = u$ but $T_u^q \neq u$. For $q = 1, 2, 3, \cdots, m-1$. If $m > 1$,

$$\begin{aligned} d(T_u, u) &= d(T_u, T_u^m) \\ &= d[(T_u, T(T_u^{m-1})] \end{aligned}$$

$$\begin{aligned} d(Tu,u) \leq\ & \alpha\frac{d(u,Tu)d(T^{m-1}u,T^{m}u)+d(u,T^{m}u)d(T^{m-1}u,Tu)}{d(u,T^{m-1}u)} \\ & +\beta\frac{d(u,T^{m}u)[d(u,T^{m}u)+d(T^{m-1}u,T^{m}u)}{d(u,T^{m-1}u)+d(T^{m-1}u,T^{m}u)+d(T^{m-1}u,Tu)} \\ & \gamma\frac{d(u,Tu)d(T^{m-1}u,Tu)+d(T^{m-1}u,T^{m}u)d(u,T^{m}u)}{d(u,Tu)+d(T^{m-1}u,Tu)+d(T^{m-1}u,T^{m}u)+d(u,T^{m}u)} \\ & +\delta[d(u,Tu)+d(T^{m-1}u,T^{m}u)]+\eta[d(T^{m-1}u,Tu)+d(u,T^{m}u)]+\mu d(u,T^{m-1},u) \end{aligned}$$

i.e.

$$d(u,Tu) \leq \frac{\delta+\eta+\mu}{1-\left(\alpha+\frac{\gamma}{2}+\delta+\eta\right)}d(u,T^{m-1}u)$$

i.e.

$$d(u,Tu) \leq s^{m}d(u,T^[m-1,u).$$

Thus we can write that

$$d(u,T_u) \leq s^{m}d(u,T_u)$$

Since, $s^m < 1$. Therefore $d(u,T_u) < d(u,T_u)$. This is contradiction. Hence $T_u = u$ i.e. u is a fixed point of T.

The uniqueness of u follows as in theorem 1.

We further generalize the result of theorem 1, in which T is neither continuous nor satisfies condition (1.1). In what follows T^m, for some positive integer m, satisfying the same rational expression and continuous, still T has unique fixed point.

Theorem 3 : Let T be a continues self map, defined on a complete metric space (X,d), such that for some positive integer m, T satisfies the following conditions:

$$\begin{aligned} d(T^{m}x,T^{m}y) \leq\ & \alpha\frac{d(x,T^{m}x)d(y,T^{m}y)+d(x,T^{m}y)d(y,T^{m}x)}{d(x,y)} \\ & +\beta\frac{d(x,T^{m}y)[d(x,T^{m}x)+d(y,T^{m}y)]}{d(x,y)+d(y,t^{m}y)+d(x,T^{m}y)+d(y,T^{m}x)} \\ & +\gamma\frac{d(x,T^{m}x)d(y,T^{m}x)+d(y,T^{m}y)d(x,T^{m}y)}{d(x,T^{m}x)+d(y,T^{m}x)+d(y,t^{m}y)+d(x,T^{m}y)} \\ & +\delta[d(x,T^{m}x)+d(y,T^{m}y)]+\eta[d(y,T^{m}x)+d(x,T^{m}y)]+\mu d(x,y) \end{aligned}$$

for all $x,y \in X, x \neq y$ and $\alpha,\beta,\gamma,\delta,\eta,\mu \geq 0$, with $2\alpha+\gamma+4\delta+4\eta+2\mu \prec 2$. If T^m is continuous then T has unique fixed - point.

Proof : By Theorem 2, we assume that T^m has unique fixed point. Also

$$Tu = T(T^{m}u) = T^{m}(Tu),$$

which implies $Tu = u$, further since fixed point of T is a fixed point of T^m, and T^m has a unique fixed point u, it follows that u is the unique fixed point of T.

Example : Let $X = [0, 1]$ with the usual metric and $T : X \to X$ be defined by

$$\begin{aligned} Tx &= \{0, \text{ when } 0 \leq x \leq \frac{1}{2} \\ &= \{\frac{1}{2} \text{ when } \frac{1}{2} \prec x \leq 1. \end{aligned}$$

It is clear that T is discontinuous and does not satisfy (1.1) for any $\alpha, \beta, \gamma, \delta, \eta, \mu \in [0, 1)$ with $2\alpha + \gamma + 4\delta + 4\eta + \mu \prec 2$, when $x = \frac{1}{2}, y = 1$. But it can be easily seen that T^2 is continuous and satisfies the condition of theorem 3, and 0 is unique fixed point of T^2 and so of T.

Remarks :

1. If $\alpha = \beta = \gamma = \delta = \eta = 0$, then theorem 1 reduce to Banach [1].
2. If $\alpha = \beta = \gamma = \eta = \mu = 0$, then theorem 1 reduce to Kannan [4].
3. If $\alpha = \beta = \gamma = \eta = 0$, then theorem 1 reduce to Chatterjee [2].
4. If $\alpha = \beta = \gamma = \delta = 0$, then theorem 1 reduce to Fisher [3].
5. If $\alpha = \beta = \gamma = 0$, then theorem 1 reduce to Reich [5].

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(Received 20 August 2007)

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