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# An Elementary Proof of the Brouwer Fixed Point Theorem

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**Abstract :** A new proof of the Brouwer fixed point theorem is given. The proof does not depend on deep known results and it is self-contained.

**Keywords :** fixed point; the Brouwer fixed point theorem.
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## 1 Introduction

Effort on discovering simple proofs of the Brouwer fixed point theorem had been made for decades. Recent results on such effort can be found in [1, 2]. This paper appears to be more elementary since no knowledge beyond underagraduate level is required.

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# 2 The Brouwer Fixed Point Theorem (BFT)

**Theorem 2.1.** *For the closed unit ball $B_d(0,1)$ of the Euclidean space $(R^d, \|\cdot\|)$, every continuous mapping $T : B_d(0,1) \to B_d(0,1)$ has a fixed point, i.e., a point $x \in B_d(0,1)$ with $T(x) = x$.*

## Preliminaries

Let $-s = n = (0, \dots, 0, 1) \in S^{d-1}$. For each $\alpha \in [-1, 1]$, let

$$D_\alpha = \{(x_1, \dots, x_{d-1}, \alpha) \in B_d(0,1)\},$$

and $S_\alpha$ its sphere (with center at $c_\alpha := (0, \dots, 0, \alpha)$ and radius $\sqrt{1-\alpha^2}$). In what follows, we will apply several projections:

(a) A metric projection $P_D$ onto a set $D$.

(b) $P_\alpha : D_\alpha \backslash c_\alpha \to S_\alpha, z \mapsto \sqrt{1-\alpha^2}\frac{z-c_\alpha}{\|z-c_\alpha\|}$.

Note that for $D = S^{d-1}, P_D = P_\alpha$ where $\alpha = 0$ in (b) and $D_0 \backslash c_0$ is replaced by $R^d \backslash 0$.

Clearly, $(NR) \Rightarrow (BFT)$, where $(NR)$ states that:

*The unit sphere is not a retract of the unit ball.* (NR)

Recall that a subset $A$ of a metric space $M$ is a retract of $M$ if the identity mapping on $A$ can be extended to a continuous mapping $r : M \to A$. Call such mapping $r$ a retraction.

(Actually, (BFT) and (NR) are equivalent.)

## Proof of Theorem 2.1

Clearly, $(NR)$ as well as $(BFT)$ are valid for $d = 1$. Suppose $U : B_d(0,1) \to S^{d-1}$ is a retraction for some $d > 1$, assuming that $S^{d-2}$ is not a retract of $B_{d-1}(0,1)$. Set $S_+^{d-1} = \{x = (x_1, \dots, x_d) \in S^{d-1} : x_d \geq 0\}$ then write $x' = (x_1, \dots, x_{d-1}, -x_d)$ for $x = (x_1, \dots, x_{d-1}, x_d) \in S_+^{d-1}$ and write $S_-^{d-1}$ for the collection of all those points $x'$. For $\alpha \in [0,1]$, let $S^\alpha = \{x_\alpha := (1-\alpha)x + \alpha x' : x \in S_+^{d-1}\}$ and write $U_\alpha$ for the restriction $U|_{S^\alpha}$.

Take $\delta > 0$ having the property that $\|U(x) - U(y)\| \leq 1/8$ if $\|x - y\| < \delta$. Now consider the deformation of (the image of) $U_\alpha$ as $\alpha$ moves from 0 to 1. The plan is to block the image not to pass through the sets $S_\gamma$ for all $\gamma \in [-1, .25]$. If for some $\alpha < 1, V_\alpha$ is the new image on $S^{d-1}$ after modifying the image of $U_\alpha$ as stated above with an additional property that $V_\alpha$ does not produce any new image on $S_+^{d-1}$, take any $\beta \in (\alpha, 1)$ with $\beta - \alpha < \delta/2$. For $x \in S_+^{d-1}$, define $W_\beta(x_\beta) = P_{S^{d-1}}\big(V_\alpha(x_\alpha) + (U_\beta(x_\beta) - U_\alpha(x_\alpha))\big)$.

It is noted that

$(i)$ $\|W_\beta(x_\beta) - V_\alpha(x_\alpha)\| < 1/8$.

$(ii)$ If $V_\alpha(x_\alpha) = U_\alpha(x_\alpha)$, then $W_\beta(x_\beta) = U_\beta(x_\beta)$.

$(iii)$ If $V_\alpha(x_\alpha) \neq U_\alpha(x_\alpha)$, we must have $U_\alpha(x_\alpha) \in S_-^{d-1}$. Consider two subcases:

$(iii.1)$ If $W_\beta(x_\beta) \in S_+^{d-1}$, put $V_\beta(x_\beta) := P_0 \circ P_{D_0}(W_\beta(x_\beta))$ (which can be done by $(i)$).

$(iii.2)$ If $W_\beta(x_\beta) \in S_-^{d-1}$ but $W_\beta(x_\beta) \notin S_\gamma$ for all $\gamma \in [-1, -.25]$, put $V_\beta(x_\beta) := W_\beta(x_\beta)$. For the case $W_\beta(x_\beta) \in S_\gamma$ for some $\gamma \in [-1, -.25]$, we apply the projections $P_{D_{-.25}}$ and $P_{c_{-.25}}$ to obtain $P_{c_{-.25}} \circ P_{D_{-.25}}(W_\beta(x_\beta)) =: V_\beta(x_\beta)$. Note that, by $(i)$ $W_\beta(x_\beta) \neq s$.

Now if $A$ is a collection of all $\alpha$ for which a deformed $V_\alpha$ exists, then $A$ is obviously nonempty. Moreover, the construction of $V_\beta$ as above assures that $sup(A) = 1$. Finally, modify $V_\alpha$ so that all the image in $S_-^{d-1}$ are mapped into the boundary $S_0$ of $S^{1/2}(= D_0)$ which is easily done by the composition $P_0 \circ P_{D_0}$.

If $V_\alpha(S^\alpha) \not\supset S_+^{d-1}$ for some $\alpha$, we claim that the sphere $S^{d-2}$ is a retract of $B_{d-1}(0,1)$ contradicting to the above assumption. To see this, suppose $V_\alpha(S^\alpha)$ does not contain $u$ for some $u \in S_+^{d-1}$. For simplicity, suppose $u = n$. Denote $x^0 = (x_1, \ldots, x_{d-1}, 0)$ for $x = (x_1, \ldots, x_{d-1}, x_d) \in S_+^{d-1}$. Thus the map

$$x^0 \mapsto x_\alpha \mapsto V_\alpha(x_\alpha) \mapsto (V_\alpha(x_\alpha))^0 \mapsto P_0((V_\alpha(x_\alpha))^0)$$

is clearly a retraction from $S^{1/2}$ onto its boundary $S_0$. This contradiction proves our claim. Thus, $V_\alpha(S^\alpha) \supset S_+^{(d-1)}$for all $\alpha$ which is impossible when $\alpha = 1$. The proof is now complete.

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