Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 527–538
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On the New Form of the Option Price of the Foreign Currency Related to Black-Scholes Formula

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Abstract : In this paper, we studied the option price of the Foreign Currency in the new form. Such new form of the option price can be related to the Black-Scholes Formula. Moreover we also studied the kerbel of the option price and found the interesting properties. However we hope that the results of this paper may be useful in the research area of Financial Mathematics.

Keywords : Black-Scholes equation; option price.
2010 Mathematics Subject Classification : 35K05; 91G20.

1 Introduction

We know that the Black-Scholes Formula is very useful in computing ther option price of the asset price and is accepted as the solution of the Black-Scholes Equation, see [[1],pp. 637-659]. In fact the Black-Scholes formula is rather complicated formula and is very difficult to be derived from such Black-Scholes Equation. So many people just only use the verification method to show that the Black-Scholes Formula is the solution of the Black-Scholes Equation. Fortunately in this

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paper, we succeeded in developed the option price which is the new form of the option price of the Foreign Currency and can be related to cover the Black-Scholes formula. The option price of the Foreign Currency can be obtain from the solution of the Black-Scholes Equation which is given by

$$\frac{\partial}{\partial t}C(s,t)+(r_D-r_F)s\frac{\partial}{\partial s}C(s,t)+\frac{1}{2}\sigma^2s^2\frac{\partial^2}{\partial s^2}C(s,t)-r_DC(s,t)=0 \tag{1.1}$$

with the call payof

$$C(S_T,T)=(S_T-p)^+\equiv \max(s_T-p,0), \tag{1.2}$$

see [[2], pp.237-238] where

1. s is the currency price(domestic unit for one foreign unit).
2. $C(s,t)$ is the price of foreign exchange call option.
3. T is the expiration time.
4. s_T is the currency price at time T.
5. r_D is the domestic riskless rate.
6. r_F is the foreign riskless rate.
7. σ is the volatility of the currency price.
8. p is the strike price (domestic unit for one foreign unit).

Now the solution of (1.1) sartisfies (1.2) is given by

$$C(s,t)=e^{-r_F(T-t)}sN(d_1)-e^{-r_D(T-t)}pN(d_2) \tag{1.3}$$

where

$$d_1=\frac{\log\left(\frac{s}{p}\right)+(r_D-r_F+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{1.4}$$

$$d_2=\frac{\log\left(\frac{s}{p}\right)+(r_D-r_F-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{1.5}$$

and denote $N(x)=\frac{1}{sqrt2\pi}\int_{-\infty}^{x}e^{-\frac{y^2}{2}}dy$.

Actually (1.3) is called the Black-Scholes Formula and we see that (1.3), (1.4) and (1.5) are complicated formula as mention before. Now in our work we obtained

$$C(s,t)=\frac{e^{-r_D(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\left[-\frac{(\log s+(r_D-r_F-\frac{\sigma^2}{2})(T-t)-y)^2}{2\sigma^2(T-t)}\right]f(y)dy \tag{1.6}$$

as the solution of (1.1) where $f(y) = e^y - p$. The equation (1.6) can also be computed as the new form of the option price

$$C(S,t) = se^{-r_F(T-t)} - e^{-r_D(T-t)}p \tag{1.7}$$

Moreover by choosing $\log p \leq p < \infty$ in (1.6). Then (1.6) can be related to cover the Black-Scholes formula in (1.3) and we also obtain the kernel of (1.1) which having thr interesting properties.

2 Preliminaries

The following Definitions and Lemmas are needed

Definition 2.1. Let f be integrable function the Fourier transform of f is defined by

$$\mathscr{F} f(x) = \widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{2.1}$$

and the inverse Fourier transform of $\widehat{f}(\omega)$ is also given by

$$f(x) = \mathscr{F}^{-1}\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega. \tag{2.2}$$

Definition 2.2. (The Dirac-delta distribution)
The Dirac-delta distribution or the impulse function is denoted by δ and is defined by

$$< f(x), \varphi(x) >\equiv \delta(\varphi(x)) = \int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0)$$

where $\varphi(x)$ is the testing function of indefinitely differentiable function with compact support. Actually
$varphi(x) \in \mathscr{D}$ and $\delta(x) \in \mathscr{D}'$ where $\mathscr{D}$ is the space of distribution and $\mathscr{D}'$ is the dual of $\mathscr{D}$, see [[3], pp. 6-14].

Lemma 2.3. *Given the equation*

$$\frac{\partial}{\partial\tau}v(R,\tau) - (r_D - r_F - \frac{\sigma^2}{2})\frac{\partial}{\partial R}v(R,\tau) - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}v(R,\tau) + r_D v(R,\tau) = 0 \tag{2.3}$$

which is obtained from (1.1) *by changing the variable* $R = \log s,\ \tau = T - t$ *and write* $C(s,t) = v(R,\tau)$ *with the call payoff or the initial condition*

$$v(R,0) = (e^R - p)^+ = f(R) \tag{2.4}$$

where f is continuous function of R then

$$v(R,\tau) = K(R,\tau) * f(R) \tag{2.5}$$

is the solution of (2.1) *in the convolution form where*

$$K(R,\tau) = \frac{e^{-r_D T}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(R + (r_D - r_F - \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}\right] \tag{2.6}$$

is the kernel of (2.3).

Proof. Take the Fourier transform defined by (2.1) with respect to R to both sides of (2.3). Then

$$\frac{\partial}{\partial\tau}\widehat{v}(\omega,\tau) - (r_D - r_F - \frac{\sigma^2}{2})i\omega\widehat{v}(\omega,\tau) + \frac{1}{2}\sigma^2\omega^2\widehat{v}(\omega,\tau) + r_D\widehat{v}(\omega,\tau) = 0$$

whose solution is

$$\widehat{v}(\omega,\tau) = C(\omega)\exp\left[(-\frac{1}{2}\sigma^2\omega^2 + i\omega(r_D - r_F - \frac{\sigma^2}{2}) - r_D)\tau\right].$$

Now from (2.4) by taking the Fourier transform

$$\widehat{v}(\omega,0) = \widehat{f}(\omega)$$

thus $\widehat{v}(\omega,0) = C(\omega) = \widehat{f}(\omega)$. It follows that

$$\widehat{v}(\omega,\tau) = \widehat{f}(\omega)\exp\left(-\frac{1}{2}\sigma^2\omega^2 + i\omega(r_D - r_F - \frac{\sigma^2}{2}) - r_D)\tau\right]$$

since from (2.2)

$$\begin{aligned} v(R,\tau) &= \mathscr{F}^{-}1\widehat{v}(\omega,\tau) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{v}(\omega,\tau)d\omega. \end{aligned}$$

Hence

$$v(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{f}(\omega)\exp\left(-\frac{1}{2}\sigma^2\omega^2 + i\omega(r_D - r_F - \frac{\sigma^2}{2}) - r_D)\tau\right] d\omega$$

Let $\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega y} f(y)dy$ thus

$$v(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega R}e^{-i\omega y}\exp\left[(-\frac{1}{2}\sigma^2\omega^2 + i\omega(r_D - r_F - \frac{\sigma^2}{2}) - r_D)\tau\right] f(y)dyd\omega$$

$$= \frac{e^{-r_D\tau}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\tau(\omega^2 - 2\frac{i\omega}{\sigma^2\tau}(r_D - r_F - \frac{\sigma^2}{2})\tau + R - y)\right] f(y)dyd\omega$$

$$= \frac{e^{-r_D\tau}}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\tau(\omega - i\frac{(r_D - r_F - \frac{\sigma^2}{\tau})\tau + R - y}{\sigma^2\tau})^2\right] d\omega \times$$

$$\int_{-\infty}^{\infty}\exp\left[-\frac{((r_D - r_F - \frac{\sigma^2}{2})\tau + R - y)^2}{2\sigma^2 y}\right] f(y)dy.$$

Put $u - \sigma\sqrt{\frac{\tau}{2}}(\omega - i\frac{(r_D - r_F - \frac{\sigma^2}{2})\tau + R - y}{\sigma^2\tau})$ then $d\omega = \frac{1}{\sigma}\sqrt{\frac{2}{\tau}}du$ thus

$$v(R,\tau) = \frac{e^{-r_D\tau}}{2\pi}\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}\int_{-\infty}^{\infty} e^{-u^2}du\int_{-\infty}^{\infty}\exp\left[-\frac{(R + (r_d - r_F - \frac{\sigma^2}{2})\tau - y)^2}{2\sigma^2\tau}\right] f(y)dy$$

$$= \frac{e^{-r_D\tau}}{2\pi}\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}\sqrt{\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{(R + (r_d - r_F - \frac{\sigma^2}{2})\tau - y)^2}{2\sigma^2\tau}\right] f(y)dy$$

$$= \frac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}\exp\left[-\frac{(R + (r_d - r_F - \frac{\sigma^2}{2})\tau - y)^2}{2\sigma^2\tau}\right] f(y)dy \qquad (2.7)$$

(Note that $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$). Now (2.7) can be written in the convolution form

$$v(R,\tau) = K(R,\tau) * f(R).$$

Thus we obtain (2.5) and (2.6) as required. Moreover it can shown that

$$\lim_{\tau\to 0} K(r,\tau) = \delta(R)$$

where $\delta(R)$ is the Dirac-delta distribution defined by definition (2.2), see [[4], pp. 36-37] thus

$$v(R,0) = \delta(R) * f(R) = \int_{-\infty}^{\infty}\delta(y)f(R - y)dy = f(R - 0) = f(R)$$

by definition (2.2). It follows that (2.4) holds. $\square$

Lemma 2.4. *The equation* (2.3) *with the call payoff* (2.4) *has the solution*

$$v(R,\tau) = e^{R-r_F\tau} - e^{-r_D\tau}p \tag{2.8}$$

as the new form of the option price of the Foreign Currency and moreover it has the solution that covers the Black-Scholes formula given by (1.3).

Proof. Form (2.7),

$$v(R,\tau) = \frac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{\infty} \exp\left[-\frac{(R+(r_D-r_F-\frac{\sigma^2}{2})\tau-y)^2}{2\sigma^2\tau}\right] e^y dy \right.$$
$$\left. - p\int_{-\infty}^{\infty} \exp\left[-\frac{(R+(r_D-r_F-\frac{\sigma^2}{2})\tau-y)^2}{2\sigma^2\tau}\right] dy\right)$$

where $f(y) = e^y - p$, put $u = \frac{1}{2\sigma^2\tau}(y-(r_D-r_F-\frac{\sigma^2}{2})\tau - R)$ then $dy- = \sqrt{2\sigma^2\tau}du$ and $y = \sqrt{2\sigma^2\tau}u + (r_D - r_F - \frac{\sigma^2}{2})\tau + R$ thus

$$v(R,\tau) = \frac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{\infty} \sqrt{2\sigma 62\tau}e^{-u^2} \exp\left[\sqrt{2\sigma^2\tau}u + (r_D-r_F-\frac{\sigma^2}{2})\tau + R\right] du \right.$$
$$\left. \sqrt{2\sigma^2\tau}\int_{-\infty}^{\infty} e^{-u^2}du\right)$$
$$= \frac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}}\sqrt{2\sigma^2\tau}e^R e^{r_D\tau}e^{-r_F\tau}e^{-\frac{\sigma^2}{2}\tau}\int_{-\infty}^{\infty} e^{-(u-\frac{\sqrt{2\sigma^2\tau}}{2})^2}du - \frac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}}\sqrt{2\sigma^2\tau}\sqrt{\pi}p$$
$$= \frac{\sqrt{2\sigma^2}\tau\sqrt{\pi}}{\sqrt{2\sigma^2}\tau}e^R - E^{-r_F\tau} - e^{r_F\tau}p$$
$$= e^{R-r_F\tau} - e^{r_D\tau}p$$

(Note that $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$). Thus $v(R,\tau) = e^{R-r_F\tau} - e^{-r_D\tau}p$. Thus we obtainb (2.8) as required. Next show that equation (2.3) has the on solution that covers the Black-Scholes Formula. Since from (2.7)

$$v(R,\tau) = \frac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{\infty} \exp\left[-\frac{(R+(r_D-r_F-\frac{\sigma^2}{2})\tau-y)^2}{2\sigma^2\tau}\right] e^y dy \right.$$
$$\left. - p\int_{-\infty}^{\infty} \exp\left[-\frac{(R+(r_D-r_F-\frac{\sigma^2}{2})\tau-y)^2}{2\sigma^2\tau}\right] dy\right)$$

where $f(y) = (e^y - p)^+$. Let $A = \exp\left[\dfrac{-(R + (r_D - r_F - \frac{\sigma^2}{2})\tau - y)^2}{2\sigma^2\tau}\right] e^y$ and

$B = \exp\left[\dfrac{-(R + (r_D - r_F - \frac{\sigma^2}{2})\tau - y)^2}{2\sigma^2\tau}\right] e^y$.

Hence $v(R,\tau) = \dfrac{e^{-r_D\tau}}{\sqrt{2\pi\sigma^2\tau}} \left(\int_{-\infty}^{\infty} A dy - p \int_{-\infty}^{\infty} B dy\right)$
consider the integral $\int_{-\infty}^{\infty} B dy$. Choose $y \geq \log p$ and put

$$\frac{u}{\sqrt{2}} = \frac{R + (r_D - r_F - \frac{\sigma^2}{2})\tau - y}{\sqrt{2\sigma^2\tau}}$$

thus $-\infty < u \leq \dfrac{R + (r_D - r_F - \frac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}}$ where $-y \leq -\log p$ and we have $d(-y) = dy = \sqrt{\sigma^2\tau} du$. Now let $a = \dfrac{R + (r_D - r_F - \frac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}}$ thus $\int_{-\infty} B dy = \sqrt{\sigma^2\tau} \int_{-\infty}^{a} e^{-\frac{u^2}{2}} du$.

Next consider the integral $\int_{-\infty}^{\infty} A dy$ Let $\theta = \dfrac{1}{\sqrt{2\sigma^2\tau}}(R + (r_D - r_F - \dfrac{\sigma^2}{2})\tau - y)$ then $d(-y) = dy = sqrt2\sigma^2\tau d\theta$. Since we choose $y \geq \log p$ hence $-\infty < \theta \leq \dfrac{R + (r_D - r_F - \frac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}}$. Let $b = \dfrac{R + (r_D - r_F - \frac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}}$ then

$$\begin{aligned}
\int_{-\infty}^{\infty} A dy &= \sqrt{2\sigma^2\tau} \int_{-\infty}^{b} e^{-\theta^2} \exp\left[R + (r_D - r_F - \frac{\sigma^2}{2}\tau - \sqrt{2\sigma^2\tau}\theta\right] d\theta \\
&= \sqrt{2\sigma^2\tau} e^{\frac{1}{2}\sigma^2\tau} \exp\left[R + (r_D - r_F - \frac{1}{2}\sigma^2)\tau\right] \int_{-\infty}^{b} \exp\left[-(\theta + \frac{1}{\sqrt{2}}\sqrt{\sigma^2\tau})^2\right] d\theta \\
&= \sqrt{2\sigma^2\tau} e^R e^{r_D - r_F}\tau \int_{-\infty}^{b} \exp\left[-(\theta + \frac{1}{2}\sqrt{\sigma^2\tau})^2\right] d\theta.
\end{aligned}$$

Put $\dfrac{\alpha}{\sqrt{2}} = \theta + \dfrac{1}{2}\sqrt{\sigma^2\tau}$ then $d\theta = \dfrac{1}{2} d\alpha$ and $\alpha = \sqrt{2}\theta + \sqrt{\sigma^2\tau}$. Since

$$-\infty < \theta \leq \frac{R + (r_D - r_F - \frac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}}$$

hence

$$-\infty < \theta \leq \frac{R + (r_D - r_F - \dfrac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}} + \sqrt{\sigma^2\tau}$$
$$-\infty < \theta \leq \frac{R + (r_D - r_F - \dfrac{\sigma^2}{2})\tau - \log p}{\sqrt{2\sigma^2\tau}}.$$

Let $C = dfracR + (r_D - r_F - \dfrac{\sigma^2}{2})\tau - \log p\sqrt{2\sigma^2\tau}$ then

$$\int_{-\infty}^{\infty} A dy = \sqrt{\sigma^2\tau} e^R e^{r_D - r_F}\tau \int_{-\infty}^{C} e^{-\frac{\alpha^2}{2}} d\alpha.$$

So we have

$$\begin{aligned} v(R,\tau) &= \frac{e^{r_D\tau}}{\sqrt{2\pi\sigma^2\tau}} \left(\sqrt{\sigma^2}\tau e^R e^{r_D - r_F}\tau \int_{-\infty}^{C} e^{-\frac{\alpha^2}{2}} d\alpha - p\sqrt{\sigma^2\tau} \int_{-\infty}^{a} e^{-\frac{u^2}{2}} du \right) \\ &= e^{R - r_F\tau} \left(e^{-\frac{\alpha^2}{2}} d\alpha \right) - e^{r_D\tau} p \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{u^2}{2}} du \right) \\ &= e^{R - r_D\tau} N(d_1) - e^{-r_D\tau} p N(d_2). \end{aligned}$$

It follows that

$$v(R,\tau) = e^{R - r_F\tau} N(d_1) - e^{-r_D\tau} p N(d_2) \tag{2.9}$$

is the Black-Scholes Formula where $d_1 = C$ and $d_2 = a$ and denote $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$ □

3 Main Results

Theorem 3.1. *Recall the equation* (1.1)

$$\frac{\partial}{\partial t} C(s,t) + (r_D - r_F) s \frac{\partial}{\partial s} C(s,t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} C(s,t) - r_D C(s,t) = 0 \tag{3.1}$$

with the call payoff

$$c(s_T, T) = (s_T - p)^+ \tag{3.2}$$

then (3.1) *has*

$$C(s,t) = s e^{r_F(T-t)} - e^{r_D(T-t)} p \tag{3.3}$$

as the solution of the new form of the option price and moreover (3.1) *has the solution that cover the Black-Scholes formula given by* (1.3), (1.4) *and* (1.5).

Proof. By (2.8) of Lemma (2.4),

$$v(R,\tau) = e^{R-r_F\tau} - e^{r_D\tau}p.$$

Since from Lemma (2.3), we write $C(s,t) = v(R,\tau)$ where $R = \log s$ and $\tau = T-t$ then we have

$$\begin{aligned} C(s,t) = v(\log s, T-t) &= e^{\log s}e^{-r_F(T-t)} - e^{r_D(T-t)} \\ &= se^{-r_F(T-t)} - e^{-r_D(T-t)}p. \end{aligned}$$

Thus we obtain (3.3). Next from (2.9) of Lemma (2.4)

$$v(R,\tau) = e^{R-r_F\tau}N(d_1) - e^{r_D\tau}pN(d_2)$$

since

$$u(s,t) = v(R,\tau) = v(\log s, T-t) = se^{-r_F(T-t)}N(d_1) - e^{-r_D(T-t)}pN(d_2).$$

Hence $u(s,t) = se^{-r_F(T-t)}N(d_1) - e^{-r_D(T-t)}pN(d_2)$ is the Black-Scholes formula where

$$d_1 = \frac{\log\left(\frac{s}{p}\right) + (r_D - r_F + \frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}}$$

$$d_1 = \frac{\log\left(\frac{s}{p}\right) + (r_D - r_F - \frac{\sigma^2}{2})(T-t)}{\sqrt{T-t}}.$$

Thus the solution of (3.1) covers teh Black-Scholes Formula given by (1.3), (1.4) and (1.5) as required. □

Theorem 3.2. *(The property of kernel $K(\log s, T-t)$)*
From (2.6),

$$K(\log s, T-t) = \frac{e^{-r_D(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left[\frac{-(\log s + (r_D - r_F - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right]$$

where $R = \log s$ and $\tau = T-t$. The kernel $K(\log s, T-t)$ have the following properties

(i) $K(\log s, T-t)$ *satisfies equation* (3.1)

(ii) $K(\log s, T-t) > 0$ *for* $0 < t \le T$

(iii) $K(\log s, T-t)$ *is the tempered distribution, that is* $K(\log s, T-t) \in S'(\mathbb{R})$ *where* $S'(\mathbb{R})$ *is the space of tempered distribution on the set of real number* $\mathbb{R}$.

(iv) $e^{r_D(T-t)} \int_{-\infty}^{\infty} K(\log s, T-t) d(\log s) = 1$

(v) $\lim_{t \to T} K(\log s, T-t) = \delta(\log s)$

(vi) $K(\log s, T-t)$ *is a Gaussian or Normal distribution with mean* $e^{r-D(T-t)}(\frac{1}{2}\sigma^2 - (r_D - r_F))(T-t)$ *and variance* $e^{-2r_D(T-t)}\sigma^2(T-t)$.

Proof. (i) Since $K(\log s, T-t)$ is the kernel of (3.1) which is the solution of (3.1) and we can also compute directly to show that $K(\log s, T-t)$ satisfies (3.1).

(ii) $K(\log s, T-t) > 0$ for $0 < t \leq T$ is obvious.

(iii) $K(\log s, T-t) \in S'(\mathbb{R})$, see [[5], pp.135-136].

(iv) Since

$$e^{r_D(T-t)} \int_{-\infty}^{\infty} K(\log s, T-t) d(\ln s)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} \exp\left[-\frac{(\log s + (r_D - r_F - \frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}\right] d(\log s)$$

put $u = \frac{1}{2\sigma^2(T-t)}(\log s + (r_D - r_F - \frac{\sigma^2}{2})(T-t))$ then

$$d(\log s) = \sqrt{2\sigma^2(T-t)}du.$$

Hence

$$e^{r_D(T-t)} \int_{-\infty}^{\infty} K(\log s, T-t) d(\log s) = \frac{\sqrt{2\sigma^2(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{\sqrt{2\sigma^2(T-t)}\sqrt{\pi}}{\sqrt{2\pi\sigma^2(T-t)}} = 1.$$

(v) $\lim_{t \to T} K(\log s, T-t) = \delta(\log s)$, see[[4], pp.36-37].

(vi) Since $K(\log s, T-t)$ is a Gaussian function or Normal distribution, hence

$$\text{mean } = E(K(\log s, T-t))$$

$$= e^{r_D(T-t)} E\left(\frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left[-\frac{(\log s - (\frac{\sigma^2}{2} - (r_D - r_F))(T-t))}{2\sigma^2(T-t)}\right]\right)$$

$$= e^{r_D(T-t)}(\frac{1}{2}\sigma^2 - (r_D - r_F))(T-t)$$

where E is expectation.

$$\begin{aligned}
&\text{variance } = V(K(\log s, T-t)) \\
&= e^{-2r_D(T-t)} V\left(\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left[-\frac{(\log s - (\frac{\sigma^2}{2} - (r_D - r_F))(T-t))}{2\sigma^2(T-t)}\right]\right) \\
&= e^{-2r_D(T-t)}\sigma^2(T-t)
\end{aligned}$$

where V denote variance.

□

4 Conclusion

The foreign Currency option is the kind of investing by buying or selling money with of foreign exchange in the option condition which is similar to then option of stock prices. For trading the foreign currency option, the Black-Scholes Formula is needed for computing such option. Actually the Black-Scholes Equation. In this paper we have introduced another kind of solutions whcich is the new result and name the form of the option price. Fortunately such new form of the option price is simple formula and can be related to cover such Black-Scholes Formula.

We hope that the new results of our work may be useful in the research of Financial Mathematics.

Acknowledgement : The author would like to thank the research Administration Center, Chiang Mai University for Financial supports.

References

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, Journal of political economy 81 (3) (1973) 637-654.

[2] M. Garman, S. Kohlhagon, Foreign Currency option values, Journal of International Money and Finance 2 (1983) 231-238.

[3] A.H. Zemanian, Distribution Theory and Transform Analysis: An Introduction to Generalized Functions with Applications, Courier Corporation, 1965.

[4] I.M. Gellfand, G.E. Shilov, Generalized Functions. Vol. 1, Properties and Operations, Academic Press, 1969.

[5] W.F. Donoghue, Distributions and Fourier Transforms, Academic Press, 1965.

(Received 25 September 2018)
(Accepted 1 October 2018)