Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 515–525
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

# A Local Discontinuous Galerkin Method for the Reduced Burgers-Poisson Equation

**Nattapol Ploymaklam**

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand

Research Center in Mathematics and Applied Mathematics
Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand

Centre of Excellence in Mathematics, CHE, Bangkok, 10400, Thailand
e-mail : nattapol.p@cmu.ac.th

**Abstract :** In this work, we discuss a numerical approximation of the solution of the reduced Burgers-Poisson equation using the local discontinuous Galerkin method (LDG). The reduced Burgers-Poisson equation comes from rewriting the system of Burger-Poisson equations into a single equation. The equation is then rewritten into a system of first-order partial differential equation before the discontinuous Galerkin framework is applied. Numerical tests show that optimal order of convergence can be achieved when using polynomials of even degree in the approximation. The result agrees with the behavior of the numerical solution of the system of Burgers-Poisson equations using LDG method.

**Keywords :** local discontinuous Galerkin; non-linear PDE; optimal convergence.
**2010 Mathematics Subject Classification :** 65M60.

## 1 Introduction

Partial differential equations (PDE) have been used as a mathematical tool to describe various natural phenomena by means of simplified models. One of the

most widely-used PDE's is the advection equation

$$\partial_t u + \partial_x u + = 0 \tag{1.1}$$

which describes the transportation for bulk of substances, such as pollutants or small particles in the air or river. One main property of the advection equation is the conservation of quantity in a bulk motion. A generalized form of (1.1)

$$\partial_t u + \partial_x [f(u)] = 0 \tag{1.2}$$

can be used to described more complicated models, enabling us to describe a wider variety of natural phenomena. One of many interesting models is shallow water wave model. In 1974, Whitham [1] proposed a model of the form

$$\partial_t u + u\partial_x u + \partial_x [G * u] = 0 \tag{1.3}$$

When the kernel $G$ is $G(x) = \frac{1}{2}e^{-|x|}$, equation (1.3) reduces to the system of Burgers-Poisson equations

$$u_t + uu_x = \phi_x \tag{1.4}$$

$$\phi_{xx} - \phi = u. \tag{1.5}$$

The Burgers-Poisson system can be rewritten into a single equation

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx} \tag{1.6}$$

by applying $1 - \partial_x^2$ to (1.4), then replacing the right-hand side by -$\partial_x$ of (1.5). In [2], the authors note that (1.6) has similar form to the Camassa-Holn equation

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

In this paper, we apply a local discontinuous Galerkin method (LDG method) to numerically approximate the solution of the reduced Burgers-Poisson equation (1.6). The LDG method adapts from the discontinuous Galerkin method (DG method) which was used for the first time by Reed and Hill in 1973 to solve the neutron transport equations [3]. Since then, it has been made popular by Shu and Cockburn to solve the first-order hyperbolic conservative equation. See [4] for general framework. When a PDE of interest is of higher order, can rewrite it into a system of first-order PDE and apply the DG framework to the resulting system. See [5] for a comprehensive introductory to LDG method.

In Chapter 2, we will formulate an LDG methods for the reduced Burgers-Poisson equation (1.6). In Chapter 3, we will show numerical results on various test problems. We will then make a conclusion in Chapter 4.

# 2 LDG Method

In this section, we discuss the formulation of a local discontinuous Galekin method for approximating solutions of the reduced Burgers-Poisson equation (1.6)

subject to initial data $u_0(x)$. The problem is defined on the domain $I = [0, L]$ with periodic boundary conditions in spatial direction.

The interval $I$ is partitioned into $N$ sub-interval with the partition $0 = x_{1/2}$, $x_{3/2}, \ldots, x_{N+1/2} = L$. Let $I_j = [x_{j-1/2}, x_{j+1/2}]$ with mesh size $h_j = x_{j+1/2} - x_{j-1/2}$ for $j = 1, 2, ...N$. Denote the center of cell by $x_j = \frac{1}{2}(x_{j-1/2} + x_{j+1/2})$.

Let $V_h^k$ be defined as the space of piecewise polynomials of degree up to $k$ in each cell $I_j$, that is,

$$V_h^k = \{v_h : v_h|_{I_j} \in P^k(I_j), j = 1, 2, ....N\}.$$

Since functions in $V_h^k$ are allowed to have discontinuities across the cell interfaces, then for $v_h \in V_h^k$, $v_h$ may have two different values on the interface: we use $(v_h)^-_{j+1/2}$ and $(v_h)^+_{j+1/2}$ to denote the limit values of $v_h$ at $x_{j+1/2}$ from the left and right directions respectively. The jump and average across the cell interface are denoted by $[v_h] := v_h^+ - v_h^-$ and $\{v_h\} := \frac{v_h^+ + v_h^-}{2}$ respectively.

To formulate the LDG scheme, we first rewrite (1.6) by introducing auxiliary variables $r, w, z, q$ to obtain

$$r - u_x = 0, \tag{2.1}$$

$$w - \left(\frac{u^2}{2}\right)_x = 0, \tag{2.2}$$

$$z - w_x = 0, \tag{2.3}$$

$$q - u + r_x = 0, \tag{2.4}$$

$$q_t + \left(\frac{u^2}{2}\right)_x - z_x + u_x = 0. \tag{2.5}$$

Now, our proposed scheme is to seek $u_h, r_h, w_h, z_h, q_h \in V_h^k$ such that, for all $j$, the followings hold for any $\alpha, \beta, \gamma, \rho, \phi \in V_h^k$

$$\int_{I_j} r_h \alpha \, dx + \int_{I_j} u_h \alpha_x \, dx - \hat{u}_h \alpha|_{\partial I_j} = 0, \tag{2.6}$$

$$\int_{I_j} w_h \beta \, dx + \int_{I_j} \left(\frac{u_h^2}{2}\right) \beta_x \, dx - \widehat{\frac{u_h^2}{2}} \beta|_{\partial I_j} = 0, \tag{2.7}$$

$$\int_{I_j} z_h \gamma \, dx + \int_{I_j} w_h \gamma_x \, dx - \hat{w}_h \gamma|_{\partial I_j} = 0, \tag{2.8}$$

$$\int_{I_j} q_h \rho \, dx - \int_{I_j} u_h \rho \, dx - \int_{I_j} r_h \rho_x \, dx + \hat{r}_h \rho|_{\partial I_j} = 0, \tag{2.9}$$

$$\int_{I_j} (q_h)_t \phi \, dx - \int_{I_j} \left(\frac{u_h^2}{2}\right) \phi_x \, dx + \widehat{\frac{u_h^2}{2}} \phi|_{\partial I_j}$$
$$+ \int_{I_j} z_h \phi_x \, dx - \hat{z}_h \phi|_{\partial I_j} - \int_{I_j} u_h \phi_x \, dx + \hat{u}_h \phi|_{\partial I_j} = 0, \tag{2.10}$$

where the numerical fluxes $\widehat{u_h^2}, \hat{u}_h, \hat{r}_h, \hat{w}_h$ are given by

$$\widehat{u_h^2} = \frac{1}{3}\left((u_h^+)^2 + u_h^+ u_h^- + (u_h^-)^2\right), \tag{2.11}$$

$$\hat{u}_h = \theta u_h^+ + (1-\theta)u_h^-, \tag{2.12}$$

$$\hat{r}_h = (1-\theta)r_h^+ + \theta r_h^-, \tag{2.13}$$

$$\hat{w}_h = \eta w_h^+ + (1-\eta)w_h^-, \tag{2.14}$$

$$\hat{z}_h = (1-\eta)z_h^+ + \eta z_h^-, \tag{2.15}$$

for $\theta, \eta \in [0,1]$. Note that because of the periodic boundary conditions on $u$ and its derivatives, we have that $U_{1/2}^- := U_{N+1/2}^-$ and $U_{N+1/2}^+ := U_{1/2}^+$, where $U$ represents $u_h, r_h, w_h$, $z_h$, or $q_h$.

Let $\vec{U}$ be a vector consisting of unknown coefficients for $u_h$. Similarly, let $\vec{R}, \vec{W}, \vec{Z}$, and $\vec{Q}$ be coefficient vectors for $r_h, w_h, z_h$, and $q_h$ respectively. The scheme (2.6)-(2.10) can be written as matrix-vector equations. Equations involving the unknown $\vec{W}, \vec{Z}$, and $\vec{Q}$ can be solved and written in terms of $\vec{U}$. This is a characteristic of the local discontinuous Galerkin method. Finally, the scheme is reduced to a system of ODE to solve for the unknown $\vec{U}$

$$\frac{d}{dt}\vec{U} = \mathcal{L}\vec{U}, \tag{2.16}$$

for some operator $\mathcal{L}$.

To approximate the solution of the ODE system (2.16), the third order TVD Runge-Kutta scheme [6] is used:

$$\begin{aligned}
\vec{a}_1 &= \vec{a}(t) + \Delta t \mathcal{L}(\vec{a}(t), t) \\
\vec{a}_2 &= \frac{3}{4}\vec{a}(t) + \frac{1}{4}\vec{a}_1 + \frac{1}{4}\Delta t \mathcal{L}(\vec{a}_1, t + \Delta t) \\
\vec{a}(t+\Delta t) &= \frac{1}{3}\vec{a}(t) + \frac{2}{3}\vec{a}_2 + \frac{2}{3}\Delta t \mathcal{L}(\vec{a}_2, t + \frac{\Delta t}{2}).
\end{aligned}$$

# 3 Numerical Results

**Example 3.1.** Test on smooth problem.

For preliminary test, we perform the numerical simulation on

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx} + f(t,x), \quad t > 0, x \in [0, L], \tag{3.1}$$

with periodic boundary conditions in spatial direction. Given the initial condition $u(0,x) = \sin(x), x \in [0, 2\pi]$ and the right-hand side $f(t,x) = 5\sin(x-t)\cos(x-t) - \cos(x-t)$, we have that the exact solution is given by $u(t,x) = \sin(x-t)$.

We run the simulation at $t_{\max} = 1$ using $\Delta t = 0.001$. The values of $\theta$ and $\eta$ are fixed at $1/2$. The errors and orders of accuracy are shown in Table 1.

| $k$ | $N$ | $\theta = 1/2, \eta = 1/2$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\|u - u_h\|_1$ | order | $\|u - u_h\|_2$ | order | $\|w - w_h\|_\infty$ | order |
| 1 | 5 | 8.6661e-01 | | 4.2031e-01 | | 4.6340e-01 | |
| | 10 | 3.7600e-01 | 1.2046 | 2.0017e-01 | 1.0703 | 2.7131e-01 | 0.7723 |
| | 20 | 1.8139e-01 | 1.0517 | 9.7873e-02 | 1.0322 | 1.3801e-01 | 0.9751 |
| | 40 | 8.9612e-02 | 1.0173 | 4.8620e-02 | 1.0094 | 7.5370e-02 | 0.8727 |
| 2 | 5 | 3.4386e-01 | | 1.6822e-01 | | 2.7631e-01 | |
| | 10 | 3.5604e-03 | 6.5936 | 1.9376e-03 | 6.4399 | 4.0843e-03 | 6.0801 |
| | 20 | 3.5137e-04 | 3.3410 | 1.8256e-04 | 3.4079 | 2.9395e-04 | 3.7965 |
| | 40 | 4.2357e-05 | 3.0523 | 2.1833e-05 | 3.0638 | 3.1790e-05 | 3.2089 |
| 3 | 5 | 8.8794e-03 | | 4.8249e-03 | | 5.7403e-03 | |
| | 10 | 1.1972e-03 | 2.8908 | 7.0118e-04 | 2.7826 | 1.7062e-03 | 1.7503 |
| | 20 | 1.4708e-04 | 3.0250 | 9.0346e-05 | 2.9563 | 2.4536e-04 | 2.7978 |
| | 40 | 1.8845e-05 | 2.9643 | 1.1478e-05 | 2.9765 | 4.2798e-05 | 2.5193 |
| 4 | 5 | 4.3616e-03 | | 2.3620e-03 | | 4.4082e-03 | |
| | 10 | 9.0941e-06 | 8.9057 | 4.9966e-06 | 8.8849 | 1.1177e-05 | 8.6235 |
| | 20 | 1.1445e-07 | 6.3122 | 6.0476e-08 | 6.3684 | 1.5722e-07 | 6.1516 |
| | 40 | 3.4958e-09 | 5.0329 | 1.7270e-09 | 5.1300 | 3.0889e-09 | 5.6696 |

Table 1: Errors and orders of accuracy for the test problem.

**Example 3.2.** Test on problem with exact solution.

We also test the proposed scheme on the exact solution of the Burgers-Poisson system. It has been shown in [2] that the periodic solution of the system (1.4)-(1.5) is given by

$$u(t,x) = U(x - u_0 t) + u_0, \quad \text{where}$$
$$U(x) = \frac{4}{3}\left(\frac{\cosh\left(\frac{x}{2}\right)}{\cosh\left(\frac{p}{2}\right)} - 1\right), \quad x \in [-p, p].$$

The parameters used in the simulation are: $u_0 = 0$, $p = 2$, $\Delta t = 0.001$, $t_{\max} = 1$, $\theta = 1/2$, and $\eta = 1/2$. The errors and orders of accuracy are shown in Table 2. It is shown in Figure 1 that the errors behave according to their respective orders. For example, when $k = 1$ the graphs of errors are linear, when $k = 2, 3$ the graphs have parabola shape, and when $k = 4$, the graphs exhibit the behavior of a polynomial of higher degree. We also plot the exact solution and the numerical solution with $k = 4$ in Figure 2.

| $k$ | $N$ | $\theta = 1/2, \eta = 1/2$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\|u - u_h\|_1$ | order | $\|u - u_h\|_2$ | order | $\|u - u_h\|_\infty$ | order |
| 1 | 5 | 5.9583e-02 | | 3.5475e-02 | | 4.7052e-02 | |
| | 10 | 2.9768e-02 | 1.0011 | 1.7505e-02 | 1.0190 | 2.3282e-02 | 1.0150 |
| | 20 | 1.4904e-02 | 0.9981 | 8.7248e-03 | 1.0045 | 1.1611e-02 | 1.0037 |
| | 40 | 7.4700e-03 | 0.9965 | 4.3589e-03 | 1.0012 | 5.7909e-03 | 1.0037 |
| 2 | 5 | 1.0530e-03 | | 7.4286e-04 | | 1.7019e-03 | |
| | 10 | 9.6554e-04 | 0.1251 | 5.7714e-04 | 0.3642 | 1.0886e-03 | 0.6447 |
| | 20 | 1.2174e-04 | 2.9876 | 7.2933e-05 | 2.9843 | 1.4418e-04 | 2.9165 |
| | 40 | 1.5239e-05 | 2.9979 | 9.1399e-06 | 2.9963 | 1.8626e-05 | 2.9525 |
| 3 | 5 | 1.3366e-03 | | 8.1221e-04 | | 1.7354e-03 | |
| | 10 | 1.6740e-04 | 2.9972 | 1.0340e-04 | 2.9736 | 2.5114e-04 | 2.7887 |
| | 20 | 2.0917e-05 | 3.0005 | 1.2987e-05 | 2.9931 | 3.5173e-05 | 2.8360 |
| | 40 | 2.6106e-06 | 3.0023 | 1.6254e-06 | 2.9982 | 4.7803e-06 | 2.8793 |
| 4 | 5 | 1.5839e-06 | | 9.6701e-07 | | 2.4079e-06 | |
| | 10 | 8.8775e-07 | 0.8353 | 5.8124e-07 | 0.7344 | 1.8080e-06 | 0.4134 |
| | 20 | 2.7775e-08 | 4.9983 | 1.8359e-08 | 4.9846 | 6.5268e-08 | 4.7919 |
| | 40 | 8.6647e-10 | 5.0025 | 5.7511e-10 | 4.9965 | 2.2649e-09 | 4.8488 |

Table 2: Errors and orders of accuracy for the problem with exact solution.

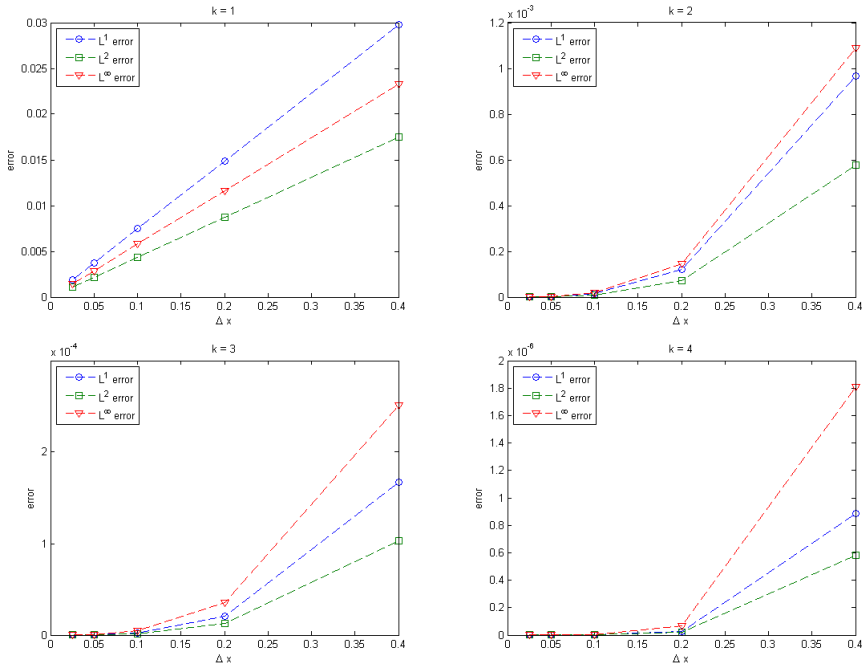


Figure 1: Errors of the problem with exact solution plotted against $\Delta x$ for when $k = 1, 2, 3, 4$.

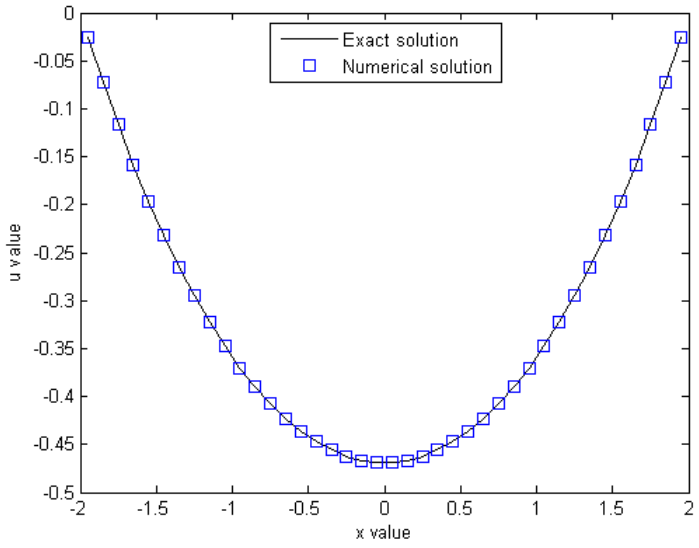


Figure 2: Comparison between the exact solution and numerical solution at $t = 1$ using $k = 4, \Delta t = 0.001, \theta = 1/2, \eta = 1/2$.

From the result in Table 2, we can see that the optimal order of convergence can be achieved when the order of the polynomials used for the approximation is even. This agrees with the behavior of the numerical solutions of the viscous and inviscous Burgers systems obtained from the LDG framework. See [7] and [8].

**Example 3.3.** Test on problem with a corner point.

We test the scheme on the exact solution [2] of the form

$$u(t,x) = V(x - u_0 t) + u_0, \quad \text{where}$$
$$V(x) = \frac{4}{3}\left(e^{-|x|/2} - 1\right), \quad x \in [-\infty, \infty].$$

In the numerical simulation, we impose periodic boundary condition on the restricted domain $[-20, 20]$. The parameters used in the simulation are: $u_0 = 1$, $\Delta t = 0.001$, $t_{\max} = 1$, $\theta = 1/2$, and $\eta = 1/2$. We can see from Table 3 that the order of convergence is not optimal. This is to be expected because the solution has a kink at the symmetry axis.

We also run a long-time simulation to test the stability-preserving ability of the scheme. To make a comparison, we replace the flux in (2.11) with the Lax-Friedrich flux [9]

$$\widehat{u_h^2} = \frac{1}{2}\left[(u_h^-)^2 + (u_h^+)^2 - \sigma(u_h^+ - u_h^-)\right], \quad \sigma = 2\max|u| \tag{3.2}$$

Using $N = 160$, $k = 4$, $u_0 = 1$, $\Delta t = 0.001$, $\theta = 1/2$, and $\eta = 1/2$, we get the numerical approximations at $t_{\max} = 40$ and 400. In Figure 3, we see that the two schemes hardly make any difference in the short time simulation at $t_{\max} = 40$. However, when $t_{\max} = 400$, we can see that the presented scheme performs better because the numerical solution obtained from our scheme does not shift from the exact solution as much as the other numerical solution.

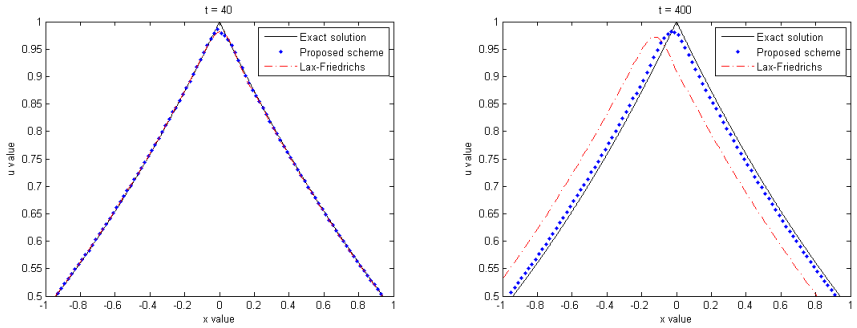


Figure 3: Comparison between the proposed scheme and the scheme with Lax-Friedrichs flux in a long time simulation.

| $k$ | $N$ | $\theta = 1/2, \eta = 1/2$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\|u - u_h\|_1$ | order | $\|u - u_h\|_2$ | order | $\|u - u_h\|_\infty$ | order |
| 1 | 40 | 2.2790e-01 | | 1.2262e-01 | | 1.9844e-01 | |
| | 80 | 1.2383e-01 | 0.8801 | 6.7529e-02 | 0.8607 | 1.0553e-01 | 0.9110 |
| | 160 | 6.5519e-02 | 0.9184 | 3.7553e-02 | 0.8466 | 5.8564e-02 | 0.8496 |
| | 320 | 3.4390e-02 | 0.9299 | 2.1566e-02 | 0.8001 | 4.9671e-02 | 0.2376 |
| 2 | 40 | 9.1294e-02 | | 5.7252e-02 | | 1.0684e-01 | |
| | 80 | 4.8582e-02 | 0.9101 | 3.9776e-02 | 0.5254 | 1.0822e-01 | -0.0185 |
| | 160 | 2.6871e-02 | 0.8543 | 2.3354e-02 | 0.7682 | 8.4802e-02 | 0.3518 |
| | 320 | 1.0320e-02 | 1.3806 | 9.2260e-03 | 1.3399 | 3.2238e-02 | 1.3953 |
| 3 | 40 | 6.3505e-02 | | 4.9491e-02 | | 1.4760e-01 | |
| | 80 | 3.2563e-02 | 0.9636 | 2.7606e-02 | 0.8422 | 8.9635e-02 | 0.7195 |
| | 160 | 9.9228e-03 | 1.7144 | 9.3720e-03 | 1.5586 | 4.6763e-02 | 0.9387 |
| | 320 | 2.6627e-03 | 1.8979 | 2.9899e-03 | 1.6482 | 2.1931e-02 | 1.0924 |
| 4 | 40 | 5.0777e-02 | | 4.1715e-02 | | 1.4126e-01 | |
| | 80 | 1.2886e-02 | 1.9783 | 1.0732e-02 | 1.9586 | 3.9485e-02 | 1.8390 |
| | 160 | 3.2848e-03 | 1.9720 | 2.6463e-03 | 2.0199 | 8.1308e-03 | 2.2798 |
| | 320 | 1.0379e-03 | 1.6621 | 9.0915e-04 | 1.5414 | 3.3071e-03 | 1.2978 |

Table 3: Errors and orders of accuracy for the problem with kink.

**Example 3.4.** Variable tests.

Finally, we run the simulation on different values of $\theta$ and $\gamma$. In Tables 4 - 6, we show the $L^2$ errors from simulation using $\theta, \eta \in \{0, 1/2, 1\}$. Here, we use the same setting as in Example 3.2.

| $k$ | $N$ | $\theta = 0, \eta = 0$ | | $\theta = 0, \eta = 1/2$ | | $\theta = 0, \eta = 1$ | |
|---|---|---|---|---|---|---|---|
| | | $\|u - u_h\|_2$ | order | $\|u - u_h\|_2$ | order | $\|u - u_h\|_2$ | order |
| 2 | 5 | 9.3632e-04 | | 3.6803e-04 | | 3.0285e-03 | |
| | 10 | 9.6907e-05 | 3.2723 | 3.3928e-05 | 3.4392 | 3.8756e-04 | 2.9661 |
| | 20 | 9.5045e-06 | 3.3499 | 3.6181e-06 | 3.2292 | 4.1047e-05 | 3.2391 |
| | 40 | 9.2381e-07 | 3.3629 | 4.3720e-07 | 3.0488 | 3.9615e-06 | 3.3731 |
| 3 | 5 | 7.2904e-05 | | 3.2481e-05 | | 1.8785e-04 | |
| | 10 | 7.8427e-06 | 3.2166 | 1.9832e-06 | 4.0337 | 1.1400e-05 | 4.0424 |
| | 20 | 9.2159e-07 | 3.0891 | 1.1143e-07 | 4.1536 | 7.7874e-07 | 3.8718 |
| | 40 | 1.1284e-07 | 3.0298 | 6.8896e-09 | 4.0156 | 7.4330e-08 | 3.3891 |

Table 4: Errors and orders of accuracy for when $\theta = 0$.

| $k$ | $N$ | $\theta = 1/2, \eta = 0$ | | $\theta = 1/2, \eta = 1/2$ | | $\theta = 1/2, \eta = 1$ | |
|---|---|---|---|---|---|---|---|
| | | $\|u - u_h\|_2$ | order | $\|u - u_h\|_2$ | order | $\|u - u_h\|_2$ | order |
| 2 | 5 | 1.6900e-03 | | 7.4286e-04 | | 1.3618e-03 | |
| | 10 | 1.5765e-04 | 3.4223 | 5.7714e-04 | 0.3642 | 1.3910e-04 | 3.2913 |
| | 20 | 1.5805e-05 | 3.3183 | 7.2933e-05 | 2.9843 | 2.0043e-05 | 2.7949 |
| | 40 | 1.1966e-06 | 3.7233 | 9.1399e-06 | 2.9963 | 1.6207e-06 | 3.6285 |
| 3 | 5 | 1.5677e-02 | | 8.1221e-04 | | 2.0261e-04 | |
| | 10 | NaN | NaN | 1.0340e-04 | 2.9736 | 2.7043e-05 | 2.9053 |
| | 20 | NaN | NaN | 1.2987e-05 | 2.9931 | 5.2243e-06 | 2.3720 |
| | 40 | NaN | NaN | 1.6254e-06 | 2.9982 | NaN | NaN |

Table 5: Errors and orders of accuracy for when $\theta = 1/2$.

From the results in Tables 4 - 6, we can see that the scheme gives optimal result when $\theta = 0$ or when $\eta = 1/2$. It should be noted that in some cases, the optimal order of convergence occurs only when the order of polynomial is even.

| $k$ | $N$ | $\theta = 1, \eta = 0$ | | $\theta = 1, \eta = 1/2$ | | $\theta = 1, \eta = 1$ | |
|---|---|---|---|---|---|---|---|
| | | $\|u - u_h\|_2$ | order | $\|u - u_h\|_2$ | order | $\|u - u_h\|_2$ | order |
| 2 | 5 | 1.7732e-02 | | 4.3296e-04 | | 8.9090e-04 | |
| | 10 | 4.6448e-03 | 1.9327 | 3.3900e-05 | 3.6749 | 9.4656e-05 | 3.2345 |
| | 20 | 1.2931e-02 | -1.4771 | 3.6405e-06 | 3.2191 | 9.4029e-06 | 3.3315 |
| | 40 | 5.9441e-01 | -5.5226 | 4.3825e-07 | 3.0543 | 9.1944e-07 | 3.3543 |
| 3 | 5 | 2.8080e-02 | | 3.3181e-05 | | 7.1123e-05 | |
| | 10 | 7.0719e-02 | -1.3326 | 2.1566e-06 | 3.9435 | 7.6028e-06 | 3.2257 |
| | 20 | NaN | NaN | 1.2129e-07 | 4.1523 | 9.0346e-07 | 3.0730 |
| | 40 | NaN | NaN | 7.2524e-09 | 4.0638 | 1.1163e-07 | 3.0167 |

Table 6: Errors and orders of accuracy for when $\theta = 1$.

# 4 Conclusion

In this work, we design a of local discontinuous Galerkin method for approximating the solution of the reduced Burgers-Poission equation. From numerical evidences, it is shown that we can achieve an optimal order of accuracy when using even-order polynomials for the approximation as long as the parameters are chosen appropriately. This behavior is similar to the previous results on related equations. The proposed scheme also performs better than the scheme with Lax-Friedrichs flux in a long-time simulation.

**Acknowledgements :** This research was supported by the Institute for the Promotion of Teaching Science and Technology (IPST) under the research Fund for DPST Graduate with First Placement [Grant no. 036/2558], by Chiang Mai University, and by the Centre of Excellence in Mathematics, The Commission on Higher Education, Thailand.

# References

[1] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.

[2] K. Fellner, C. Schmeiser, Burgers-Poisson: a nonlinear dispersive model equation, SIAM J. Appl. Math. 64 (5) (2004) 1509-1525.

[3] W.H. Reed, T.R. Hill, Triangular Mesh Methods for the Neutron Transport Equation, Los Alamos, Report, LA-UR-73-479, 1973.

[4] B. Cockburn, C.-W. Shu, TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework, Math. Comput. 52 (1989) 411-435.

[5] B. Cockburn, C.-W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, SIAM J. Numer. Anal. 35 (6) (199) 2440-2463.

[6] S. Gottlieb, C.-W. Shu, Total variation diminishing RungeKutta schemes, Math. Comput 67 (221) (1998) 73-85.

[7] H. Liu, N. Ploymaklam, A local discontinuous Galerkin method for the Burgers-Poisson equation, Numer. Math. 129 (2015) 321-351.

[8] N. Ploymaklam, P.M. Kumbhar, A.K. Pani, A Priori error analysis of the local discontinuous Galerkin method for the viscous Burgers-Poisson system, Int. J. Numer. Anal. Model. 14 (4-5) (2017) 784-807.

[9] P.D. Lax, Weak solutions of nonlinear hyperbolic equations and their numerical computation, Comm. Pure. Appl. Math. 7 (1) (1954) 159-193.

(Received 30 August 2018)
(Accepted 17 June 2019)