Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 495–514

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

The Viscosity Implicit Midpoint Rule for Finding Common Fixed Points of Two Asymptotically Nonexpansive Mappings with Applications

Shrijana Dhakal and Wutiphol Sintunavarat[1]

Department of Mathematics and Statistics, Faculty of Science
and Technology, Thammasat University Rangsit Center
Pathumthani 12121, Thailand
e-mail: shrijana.usc@gmail.com (S. Dhakal)
wutiphol@mathstat.sci.tu.ac.th (W. Sintunavarat)

Abstract : The purpose of this work is to introduce the viscosity implicit midpoint rule for finding the common fixed points of two asymptotically nonexpansive mappings in real Hilbert spaces. The strong convergence theorem of this method is proved under some favorable conditions imposed on the control parameters. Furthermore, we provide applications of the main result which are related to variational inequality problems, constrained convex minimization problems, Fredholm integral equations and nonlinear evolution equations.

Keywords : viscosity; implicit midpoint rule; asymptotically nonexpansive mapping; variational inequality; fixed point.
2010 Mathematics Subject Classification : 47J25; 47N20; 34G20; 65J15.

1 Introduction

In 1996, Atouch [1] first introduced the viscosity method for solving minimization problems.

[1]Corresponding author email.

From the idea of Atouch [1], Moudafi [2] proposed the viscosity approximation method for solving fixed point problems of nonexpansive mappings in real Hilbert spaces. Also, Moudafi [2] gave the strong convergence results of the viscosity explicit method and the viscosity implicit method (in short, VEM and VIM, respectively) for searching fixed points of a nonexpansive self-mapping on a closed convex nonempty subset of a real Hilbert space.

In the recent years, the implicit midpoint rule (in short, IMR) have been studied by many authors [3–7] and references therein. Here, we give some research in this direction which is the main motivation of this paper. In 2015, Xu et al. [3] proposed the viscosity implicit midpoint rule (in short, VIMR) for finding fixed points of a nonexpansive self-mapping T on a nonempty closed convex subset C of a real Hilbert space H as follows:

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{1.1}$$

where $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$ and $f : C \to C$ is a given contraction mapping. The idea of contractions in this method is used to regularize a nonexpansive mappings in Hilbert spaces for selecting the particular fixed points. Xu et al. [3] also proved that the VIMR (1.1) converges strongly to a unique fixed point x^* of T under some appropriate assumption. Furthermore, this fixed point is also the unique solution of the following variational inequality (in short, VI):

$$\langle (I-f)x^*, x - x^* \rangle \quad \geq \quad 0, \qquad\qquad x \in Fix(T), \tag{1.2}$$

where $I : H \to H$ is the identity mapping and $Fix(T) := \{x \in H : T(x) = x\}$, that is, the set of all fixed points of T.

Nowadays, many authors generalized and extended the idea of viscosity implicit midpoint rule in different ways. Herein, we include some of them. In 2016, Zhao et al. [6] improved the idea of Xu et al. [3] by introducing the following iterative algorithm for finding fixed points of a asymptotically nonexpansive self-mapping T on a nonempty closed convex subset C of a real Hilbert space:

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T^n\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{1.3}$$

where $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$ and $f : C \to C$ is a given contraction mapping.

Recently, Naqvi and Khan [8] presented the following viscosity rule for finding the common fixed points of two nonexpansive mappings $S, T : C \to C$, where C is a nonempty closed convex subset of a real Hilbert space:

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n S\Big(\frac{x_n + x_{n+1}}{2}\Big) + \gamma_n T\Big(\frac{x_n + x_{n+1}}{2}\Big) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{1.4}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ for all $n \in \mathbb{N}$, $f : C \to C$ is a given contraction mapping. Under some specific conditions, they proved that the sequence defined by (1.4) converges strongly to a common fixed point of S and T, which is also solved the VI (1.2).

The purposed of this paper is to present the viscosity implicit midpoint rule for searching common fixed points of two asymptotically nonexpansive self-mappings S and T on a nonempty closed convex subset C of a real Hilbert space H, which is stated as follows:

$$\left.\begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n T^n\Big(\frac{x_n + x_{n+1}}{2}\Big) + \gamma_n S^n\Big(\frac{x_n + x_{n+1}}{2}\Big) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{1.5}$$

where $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$, $f : C \to C$ is a contraction mapping and $S, T : C \to C$ are two asymptotically nonexpansive mappings. Under some favorable conditions imposed on the control parameters, we will show that the sequence $\{x_n\}$ defined by (1.5) strongly converges to a point $z \in Fix(T) \cap Fix(S)$, which is also the unique solution of the VI (1.2). Moreover, we provide some applications which illustrate our result.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and the convergence lemmas which are necessary for proving our convergence theorem. In Section 3, we present the strong convergence theorem of common fixed points of two asymptotically nonexpansive mappings under some favorable conditions. In the last Section, we give the applications which are derived from the main results. These applications are related to the variational inequality problems, constrained minimization problems, Fredholm integral equations and nonlinear evolution equations.

2 Preliminaries

Throughout this paper, we assume that C is a nonempty closed and convex subset of a real Hilbert space H. In this section, we recall some basic definitions and convergence lemmas which are necessary for proving the convergence theorem.

Definition 2.1. A mapping $T : C \to C$ is said to be:

(i) *α-inverse strongly monotone* if there exists $\alpha > 0$ satisfying

$$\langle x - y, Tx - Ty\rangle \geq \alpha \| Ax - Ay \|^2$$

for all $x, y \in C$;

(ii) *L-Lipschitz continuous* if there exists $L \geq 0$ satisfying

$$\| Tx - Ty \| \leq L \| x - y \|$$

for all $x, y \in C$;

(iii) *nonexpansive* if

$$\| Tx - Ty \| \leq \| x - y \|$$

for all $x, y \in C$;

(iv) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|$$

for all $x, y \in C$ and for all $n \in \mathbb{N}$;

(v) *contraction* if there exists the contractive constant $\alpha \in [0, 1)$ such that

$$\| Tx - Ty \| \leq \alpha \| x - y \|$$

for all $x, y \in C$.

Remark 2.2. *If T is an α-inverse strongly monotone mapping, then T is a $\frac{1}{\alpha}$-Lipschitz mapping. Moreover, if T is a asymptotically nonexpansive mapping, then $Fix(T)$ is always closed and convex, if in addition, C is bounded then $Fix(T)$ is nonempty.*

Definition 2.3. An operator $P_C : H \to C$ that assigns every point $x \in H$ to its unique nearest point in C is called the *metric projection* onto C, defined by:

$$P_C(x) := \arg\min_{z\in C} \| x - z \|^2, \ x \in H.$$

It is denoted by P_C.

Remark 2.4. *Note that $P_C x$ is characterized as follows:*

$$P_C x \in C \ \text{and} \ \langle x - P_C x, z - P_C x\rangle \leq 0, \qquad \forall z \in C. \tag{2.1}$$

The next lemma is the demiclosedness principle of asymptotically nonexpansive mappings which is quite helpful in the proof of the convergence result of our algorithm for finding the common fixed point of two asymptotically nonexpansive mappings.

Lemma 2.5 (The demiclosedness principle [9])**.** *Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $T : C \to C$ be a asymptotically nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $\{x_n\}$ weakly converges to x and $\{(I - T)x_n\}$ converges strongly to 0, then $x = T(x)$.*

Next, we give the famous lemma for using in the proof of many convergence theorems.

Lemma 2.6 ([10])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \delta_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subseteq (0, 1)$ and $\{\delta_n\} \subseteq \mathbb{R}$ are two sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\lambda_n} \leq 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

In this section, we prove the strong convergence theorem of the viscosity implicit midpoint rule for two asymptotically nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset a real Hilbert space H, $T : C \to C$ and $S : C \to C$ be two asymptotically nonexpansive mappings with the same sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $U := Fix(T) \cap Fix(S) \neq \emptyset$ and $f : C \to C$ be a contraction mapping with the contractive constant $\alpha \in [0, 1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + \beta_n T^n\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n S^n\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ are sequences satisfying the following conditions:

(A1) $\alpha_n + \beta_n + \gamma_n = 1$;

(A2) $\lim_{n\to\infty} \frac{k_n^2 - 1}{\alpha_n} = 0$;

(A3) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A4) $\lim_{n\to\infty} \gamma_n = 1$ *and* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 1$.

Suppose that $(1 - \alpha_n)k_n \leq 2$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = \lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{3.2}$$

Then the sequence $\{x_n\}$ strongly converges to a common fixed point z of S and T, which is also the unique solution of the following variational inequality

$$\langle (I - f)x, y - x\rangle \geq 0 \quad \forall y \in U.$$

In other word, z is the unique fixed point of the contraction $P_U f$, that is, $P_U f(z) = z$.

Proof. First, we will show the existence of a sequence $\{x_n\}$ defined by (3.1). Consider the mapping $W_n : C \to C$ by

$$W_n x = \alpha_n f(w) + \beta_n T^n\left(\frac{w+x}{2}\right) + \gamma_n S^n\left(\frac{w+x}{2}\right)$$

for all $x \in C$. We want to show that W_n is a contraction mapping for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $x, y \in C$, we have

$$\begin{aligned}\|W_n x - W_n y\| &= \left\| \left[\beta_n T^n\left(\frac{w+x}{2}\right) + \gamma_n S^n\left(\frac{w+x}{2}\right)\right] \right.\\ &\qquad \left. - \left[\beta_n T^n\left(\frac{w+y}{2}\right) + \gamma_n S^n\left(\frac{w+y}{2}\right)\right] \right\| \\ &\le \beta_n \left\|T^n\left(\frac{w+x}{2}\right) - T^n\left(\frac{w+y}{2}\right)\right\| + \gamma_n \left\|S^n\left(\frac{w+x}{2}\right) - S^n\left(\frac{w+y}{2}\right)\right\| \\ &\le (\beta_n + \gamma_n)\frac{k_n}{2}\|x-y\| \\ &= (1-\alpha_n)\frac{k_n}{2}\|x-y\|.\end{aligned}$$

It follows from $(1-\alpha_n)k_n \le 2$ for all $n \in \mathbb{N}$ that W_n is a contraction mapping for all $n \in \mathbb{N}$. By using the Banach contraction principle, W_n has a unique fixed point for all $n \in \mathbb{N}$. This yields the existence of a sequence $\{x_n\}$ defined by (3.1).

Next, we will prove that the sequence $\{x_n\}$ defined by (3.1) converges to $y \in U \neq \emptyset$. We will divide the proof into six steps.

Step (I): We will show that $\{x_n\}$ is bounded. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_{n+1} - y\| &= \left\|\alpha_n f(x_n) + \beta_n T^n\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n S^n\left(\frac{x_n + x_{n+1}}{2}\right) - y\right\| \\ &\le \alpha_n \|f(x_n) - y\| + \beta_n \left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - y\right\| \\ &\quad + \gamma_n \left\|S^n\left(\frac{x_n + x_{n+1}}{2}\right) - y\right\| \\ &\le \alpha_n \|f(x_n) - f(y)\| + \alpha_n \|f(y) - y\| + \beta_n k_n \left\|\frac{x_n + x_{n+1}}{2} - y\right\| \\ &\quad + \gamma_n k_n \left\|\frac{x_n + x_{n+1}}{2} - y\right\| \\ &\le \alpha_n \alpha \|x_n - y\| + \alpha_n \|f(y) - y\| \\ &\quad + \frac{(1-\alpha_n)k_n}{2}\left(\|x_n - y\| + \|x_{n+1} - y\|\right).\end{aligned}$$

It implies that

$$\frac{2-(1-\alpha_n)k_n}{2}\|x_{n+1}-y\| \le \frac{2\alpha\alpha_n+(1-\alpha_n)k_n}{2}\|x_n-y\|+\alpha_n\|f(y)-y\| \tag{3.3}$$

for all $n \in \mathbb{N}$. From the condition (A2), for any given positive number ϵ with $0 < \epsilon < 1-\alpha$, there exists a sufficient large positive integer $N \in \mathbb{N}$ such that for any $n \ge N$, we have $k_n^2 - 1 \le 2\epsilon\alpha_n$ and so

$$k_n - 1 \;\le\; \frac{k_n+1}{2}(k_n-1) \;=\; \frac{k_n^2-1}{2} \;\le\; \epsilon\alpha_n. \tag{3.4}$$

Since $k_n \in [1,\infty)$, $k_n^2 - 1 \le 2\epsilon\alpha_n$ and $k_n - 1 \le \epsilon\alpha_n$ for all $n \ge N$, we have

$$\left.\begin{aligned}&2-(1-\alpha_n)k_n \ge 1-\epsilon\alpha_n+\alpha_n k_n = 1+(k_n-\epsilon)\alpha_n \ge 1+(1-\epsilon)\alpha_n,\\ &2\alpha\alpha_n+(1-\alpha_n)k_n \le 2\alpha\alpha_n+1+\epsilon\alpha_n-\alpha_n = 1-(1-\epsilon-2\alpha)\alpha_n.\end{aligned}\right\} \tag{3.5}$$

Substituting (3.5) into (3.3), we have

$$\begin{aligned}\|x_{n+1}-y\| &\le \frac{1-(1-\epsilon-2\alpha)\alpha_n}{1+(1-\epsilon)\alpha_n}\|x_n-y\|+\frac{2\alpha_n}{1+(1-\epsilon)\alpha_n}\|f(y)-y\|\\ &= 1-\frac{2(1-\epsilon-\alpha)\alpha_n}{1+(1-\epsilon)\alpha_n}\|x_n-y\|\\ &\quad+\frac{2(1-\epsilon-\alpha)\alpha_n}{1+(1-\epsilon)\alpha_n}\left(\frac{1}{1-\epsilon-\alpha}\|f(y)-y\|\right)\\ &\le \max\left\{\|x_n-y\|, \frac{1}{1-\epsilon-\alpha}\|f(y)-y\|\right\}\end{aligned}$$

for all $n \ge N$. By the mathematical induction, we obtain

$$\|x_{n+1}-y\| \le \max\left\{\|x_0-y\|, \frac{1}{1-\epsilon-\alpha}\|f(y)-y\|\right\}$$

for all $n \ge N$. Hence $\{x_n\}$ is bounded. Consequently, $\{f(x_n)\}$, $\{T^n x_n\}$, $\left\{T^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\}$, $\{S^n x_n\}$ and $\left\{S^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\}$ are also bounded.

Step (II): In this step, we will show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.6}$$

For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
= & \left\| \alpha_n f(x_n) + \beta_n T^n \left(\frac{x_n + x_{n+1}}{2} \right) + \gamma_n S^n \left(\frac{x_n + x_{n+1}}{2} \right) \right. \\
& \left. -(\alpha_n + \beta_n + \gamma_n) T^n x_n + T^n x_n - x_n \right\| \\
\leq & \left\| \alpha_n \left(f(x_n) - T^n x_n \right) + \beta_n \left(T^n \left(\frac{x_n + x_{n+1}}{2} \right) - T^n x_n \right) \right. \\
& \left. + \gamma_n \left(S^n \left(\frac{x_n + x_{n+1}}{2} \right) - S^n x_n \right) \right\| \\
& + \|\gamma_n S^n x_n - \gamma_n T^n x_n + T^n x_n - x_n\| \\
\leq & \alpha_n \|f(x_n) - T^n x_n\| + \beta_n \left\| T^n \left(\frac{x_n + x_{n+1}}{2} \right) - T^n x_n \right\| \\
& + \gamma_n \left\| S^n \left(\frac{x_n + x_{n+1}}{2} \right) - S^n x_n \right\| + \gamma_n \|S^n x_n - x_n\| \\
& + \|\gamma_n x_n - \gamma_n T^n x_n + T^n x_n - x_n\| \\
\leq & \alpha_n \|f(x_n) - T^n x_n\| + (\beta_n + \gamma_n) k_n \left\| \left(\frac{x_n + x_{n+1}}{2} \right) - x_n \right\| \\
& + \gamma_n \|S^n x_n - x_n\| + (1 - \gamma_n) \|T^n x_n - x_n\| \\
\leq & \alpha_n M_1 + \frac{(1 - \alpha_n) k_n}{2} \|x_{n+1} - x_n\| + \gamma_n \|S^n x_n - x_n\| \\
& + (1 - \gamma_n) \|T^n x_n - x_n\|,
\end{aligned}$$

where $M_1 := \sup_{n \in \mathbb{N}} \{\|f(x_n) - T^n(x_n)\|\} < \infty$. It implies that

$$\left(\frac{2 - (1 - \alpha_n) k_n}{2} \right) \|x_{n+1} - x_n\| \leq \alpha_n M_1 + \gamma_n \|S^n x_n - x_n\| + (1 - \gamma_n) \|T^n x_n - x_n\| \tag{3.7}$$

for all $n \in \mathbb{N}$. It follows from (3.5) that

$$\frac{1 + (1 - \epsilon)\alpha_n}{2} \|x_{n+1} - x_n\| \leq \alpha_n M_1 + \gamma_n \|S^n x_n - x_n\| + (1 - \gamma_n) \|T^n x_n - x_n\|$$

for all $n \geq N$ and so

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq & \frac{2\alpha_n M_1}{1 + (1 - \epsilon)\alpha_n} + \frac{2\gamma_n}{1 + (1 - \epsilon)\alpha_n} \|S^n x_n - x_n\| \\
& + \frac{2(1 - \gamma_n)}{1 + (1 - \epsilon)\alpha_n} \|T^n x_n - x_n\|
\end{aligned}$$

for all $n \geq N$. Using (A4) and (3.2), we obtain $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Step (III): We will show that $\lim_{n\to\infty}\|x_n - T(x_n)\| = 0$ and $\lim_{n\to\infty}\|x_n - S(x_n)\| = 0$.

For each $n \in \mathbb{N}$ with $n \geq 2$, we have

$$\begin{aligned}
&\|x_n - T^{n-1}(x_n)\| \\
= \; & \left\|\alpha_{n-1} f(x_{n-1}) + \beta_{n-1} T^{n-1}\left(\frac{x_{n-1}+x_n}{2}\right)\right. \\
& \left. + \gamma_{n-1} S^{n-1}\left(\frac{x_{n-1}+x_n}{2}\right) - T^{n-1}(x_n)\right\| \\
\leq \; & \alpha_{n-1}\left\|f(x_{n-1}) - T^{n-1}(x_{n-1})\right\| \\
& + \beta_{n-1}\left\|T^{n-1}\left(\frac{x_{n-1}+x_n}{2}\right) - T^{n-1}(x_{n-1})\right\| \\
& + \gamma_{n-1}\left\|S^{n-1}\left(\frac{x_{n-1}+x_n}{2} - S^{n-1}(x_{n-1})\right)\right\| \\
& + \left\|(\alpha_{n-1}+\beta_{n-1}) T^{n-1}(x_{n-1}) + \gamma_{n-1} S^{n-1}(x_{n-1}) - T^{n-1}(x_n)\right\| \\
\leq \; & \alpha_{n-1} M_1 + (\beta_{n-1}+\gamma_{n-1}) k_{n-1}\left\|\left(\frac{x_{n-1}+x_n}{2}\right) - x_{n-1}\right\| \\
& + \left\|(1-\gamma_{n-1}) T^{n-1}(x_{n-1}) + \gamma_{n-1} S^{n-1}(x_{n-1}) - T^{n-1}(x_n)\right\| \\
= \; & \alpha_{n-1} M_1 + \frac{(1-\alpha_{n-1}) k_{n-1}}{2}\|x_n - x_{n-1}\| \\
& + \left\|T^{n-1}(x_{n-1}) - T^{n-1}(x_n) + \gamma_{n-1}\left(S^{n-1}(x_{n-1}) - T^{n-1}(x_{n-1})\right)\right\| \\
\leq \; & \alpha_{n-1} M_1 + \frac{(1-\alpha_{n-1}) k_{n-1}}{2}\|x_n - x_{n-1}\| + k_{n-1}\|x_{n-1} - x_n\| \\
& + \gamma_{n-1}\left\|S^{n-1}(x_{n-1}) - T^{n-1}(x_{n-1})\right\| \\
= \; & \alpha_{n-1} M_1 + \frac{(3-\alpha_{n-1}) k_{n-1}}{2}\|x_n - x_{n-1}\| \\
& + \gamma_{n-1}\left\|S^{n-1}(x_{n-1}) - T^{n-1}(x_{n-1})\right\|.
\end{aligned}$$

Using (3.2), (3.6) and the condition (A4), we obtain

$$\lim_{n\to\infty}\|x_n - T^{n-1}(x_n)\| = 0. \tag{3.8}$$

Moreover, we have

$$\begin{aligned}
\|x_n - T(x_n)\| &\leq \|x_n - T^n(x_n)\| + \|T^n(x_n) - T(x_n)\| \\
&\leq \|x_n - T^n(x_n)\| + k_1\|T^{n-1}(x_n) - x_n\|
\end{aligned}$$

for all $n \in \mathbb{N}$.

By using (3.2) and (3.8), we conclude that

$$\lim_{n\to\infty}\|x_n - T(x_n)\| = 0. \tag{3.9}$$

Similarly, also we can obtain the following result.

$$\lim_{n\to\infty}\|x_n - S(x_n)\| = 0. \tag{3.10}$$

Step (IV): In this step, we will show that $\omega_w(x_n) \subseteq Fix(S) \cap Fix(T)$, where

$$\omega_w(x_n) := \{x \in H : \text{there exist a subsequence of } \{x_n\} \text{converges weakly to } x\}.$$

Suppose that $x \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$ as $i \to \infty$. From (3.9), we have

$$\lim_{i\to\infty} \|(I-T)x_{n_i}\| = \lim_{n\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0.$$

This implies that $\{(I-T)x_{n_i}\}$ converges strongly to 0. By using Lemma 2.5, we have $Tx = x$ and so $x \in Fix(T)$. From (3.10), we have

$$\lim_{i\to\infty} \|(I-S)x_{n_i}\| = \lim_{n\to\infty} \|x_{n_i} - Sx_{n_i}\| = 0.$$

This implies that $\{(I-S)x_{n_i}\}$ converges strongly to 0. By using Lemma 2.5, we have $Sx = x$ and so $x \in Fix(S)$. Therefore, we can conclude that $x \in Fix(S) \cap Fix(T)$ and then $\omega_w(x_n) \subseteq Fix(S) \cap Fix(T)$.

Step (V): In this step, we will show that

$$\limsup_{n\to\infty} \langle z - f(z), z - x_n\rangle \le 0, \tag{3.11}$$

where $z \in U$ is the unique fixed point of $P_U \circ f$, that is, $z = P_U(f(z))$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$ for some $\bar{x} \in H$ and

$$\limsup_{n\to\infty} \langle z - f(z), z - x_n\rangle = \lim_{i\to\infty} \langle z - f(z), z - x_{n_i}\rangle. \tag{3.12}$$

From Step (IV), we get $\bar{x} \in U$. By using (2.1), we obtain

$$\limsup_{n\to\infty} \langle z - f(z), z - x_n\rangle = \lim_{i\to\infty} \langle z - f(z), z - x_{n_i}\rangle = \langle z - f(z), z - \bar{x}\rangle \le 0.$$

Step (VI): In this step, we will show that $x_n \to z$ as $n \to \infty$, where z is the common fixed point of S and T. Suppose that $z \in Fix(S) \cap Fix(T)$ and then z is also the unique fixed point of the contraction mapping $P_U \circ f$ or in other words, $z = P_U f(z)$. For each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
& \|x_{n+1} - z\|^2 \\
= \;& \left\|\alpha_n f(x_n) + \beta_n T^n\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\|^2 \\
= \;& \left\|\alpha_n\left(f(x_n) - z\right) + \beta_n\left(T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right) + \gamma_n\left(S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right)\right\|^2 \\
\le \;& \alpha_n^2\|f(x_n) - z\|^2 + \beta_n^2\left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\|^2 + \gamma_n^2\left\|S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\|^2 \\
& + 2\alpha_n\beta_n\left\langle f(x_n) - z, T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\beta_n\gamma_n\left\langle T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\alpha_n\gamma_n\left\langle f(x_n) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
\le \;& \alpha_n^2\|f(x_n) - z\|^2 + \beta_n^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - z\right\|^2 + \gamma_n^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - z\right\|^2 \\
& + 2\alpha_n\beta_n\left\langle f(x_n) - f(z), T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\alpha_n\beta_n\left\langle f(z) - z, T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\beta_n\gamma_n\left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\|\left\|S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\| \\
& + 2\alpha_n\gamma_n\left\langle f(x_n) - f(z), S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\alpha_n\gamma_n\left\langle f(z) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
\le \;& \alpha_n^2\|f(x_n) - z\|^2 + \left(\beta_n^2 + \gamma_n^2\right)k_n^2\left\|\frac{x_n + x_{n+1}}{2} - z\right\|^2 \\
& + 2\beta_n\gamma_n k_n^2\left\|\frac{x_n + x_{n+1}}{2} - z\right\|^2 \\
& + 2\alpha_n\beta_n\|f(x_n) - f(z)\|\left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\| \\
& + 2\alpha_n\beta_n\left\langle f(z) - z, T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\alpha_n\gamma_n\|f(x_n) - f(z)\|\left\|S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\| \\
& + 2\alpha_n\gamma_n\left\langle f(z) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
\le \;& \alpha_n^2\|f(x_n) - z\|^2 + (1 - \alpha_n)^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - z\right\|^2 \\
& + 2\alpha_n\beta_n\alpha k_n\|x_n - z\|\left\|\frac{x_n + x_{n+1}}{2} - z\right\| \\
& + 2\alpha_n\beta_n\left\langle f(z) - z, T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\alpha_n\gamma_n\alpha k_n\|x_n - z\|\left\|\frac{x_n + x_{n+1}}{2} - z\right\| \\
& + 2\alpha_n\gamma_n\left\langle f(z) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
= \;& \alpha_n^2\|f(x_n) - z\|^2 + (1 - \alpha_n)^2 k_n^2\left\|\frac{x_n + x_{n+1}}{2} - z\right\|^2 \\
& + 2\alpha\alpha_n(1 - \alpha_n)k_n\|x_n - z\|\left\|\frac{x_n + x_{n+1}}{2} - z\right\| \\
& + 2\alpha_n\beta_n\left\langle f(z) - z, T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
& + 2\alpha_n\gamma_n\left\langle f(z) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\
\le \;& \frac{(1 - \alpha_n)^2 k_n^2}{4}\left(\|x_n - z\|^2 + 2\|x_n - z\|\|x_{n+1} - z\| + \|x_{n+1} - z\|^2\right) \\
& + \alpha\alpha_n(1 - \alpha_n)k_n\|x_n - z\|\left(\|x_n - z\| + \|x_{n+1} - z\|\right) + h_n, \qquad (3.13)
\end{aligned}
$$

where

$$\begin{aligned} h_n \quad := \quad & \alpha_n^2 \|f(x_n) - z\|^2 + 2\alpha_n\beta_n \left\langle f(z) - z, T^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle \\ & +2\alpha_n\gamma_n \left\langle f(z) - z, S^n\left(\frac{x_n + x_{n+1}}{2}\right) - z\right\rangle. \end{aligned} \tag{3.14}$$

From (3.13) and the fact that

$$2\|x_n - z\|\|x_{n+1} - z\| \leq \|x_n - z\|^2 + \|x_{n+1} - z\|^2$$

for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 \leq & \frac{(1-\alpha_n)^2 k_n^2}{4}\left(2\|x_n - z\|^2 + 2\|x_{n+1} - z\|^2\right) \\ & + \alpha\alpha_n(1-\alpha_n)k_n\left(\|x_n - z\|^2 + \|x_n - z\|\|x_{n+1} - z\|\right) + h_n \\ \leq & \frac{(1-\alpha_n)^2 k_n^2}{2}\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) \\ & + \alpha\alpha_n(1-\alpha_n)k_n\left(\|x_n - z\|^2 + \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2\right) + h_n \\ = & \frac{(1-\alpha_n)^2 k_n^2 + 3\alpha\alpha_n(1-\alpha_n)k_n}{2}\|x_n - z\|^2 \\ & + \frac{(1-\alpha_n)^2 k_n^2 + \alpha\alpha_n(1-\alpha_n)k_n}{2}\|x_{n+1} - z\|^2 + h_n. \end{aligned}$$

It implies that

$$\begin{aligned} & \frac{2 - (1-\alpha_n)^2 k_n^2 - \alpha\alpha_n(1-\alpha_n)k_n}{2}\|x_{n+1} - z\|^2 \\ & \leq \frac{(1-\alpha_n)^2 k_n^2 + 3\alpha\alpha_n(1-\alpha_n)k_n}{2}\|x_n - z\|^2 + h_n \end{aligned} \tag{3.15}$$

for all $n \in \mathbb{N}$. From (3.12), $k_n \in [1, \infty)$, $k_n^2 - 1 \leq 2\epsilon\alpha_n$ and $k_n - 1 \leq \epsilon\alpha_n$ for all $n \geq N$, we have

$$\left.\begin{aligned} 2 - (1-\alpha_n)^2 k_n^2 - \alpha\alpha_n(1-\alpha_n)k_n \geq 2 - (1-\alpha_n)^2(1+2\epsilon\alpha_n) - \alpha\alpha_n(1-\alpha_n)(1+\epsilon\alpha_n), \\ (1-\alpha_n)^2 k_n^2 + 3\alpha\alpha_n(1-\alpha_n)k_n \leq (1-\alpha_n)^2(1+2\epsilon\alpha_n) + 3\alpha\alpha_n(1-\alpha_n)(1+\epsilon\alpha_n) \end{aligned}\right\} \tag{3.16}$$

for all $n \geq N$. Substituting (3.16) into (3.15) then it turns out that

$$\begin{aligned} \|x_{n+1} - z\|^2 \leq & \frac{(1-\alpha_n)^2(1+2\epsilon\alpha_n) + 3\alpha\alpha_n(1-\alpha_n)(1+\epsilon\alpha_n)}{2 - (1-\alpha_n)^2(1+2\epsilon\alpha_n) - \alpha\alpha_n(1-\alpha_n)(1+\epsilon\alpha_n)}\|x_n - z\|^2 \\ & + \lambda_n \end{aligned} \tag{3.17}$$

for all $n \geq N$, where

$$\lambda_n := \frac{2h_n}{2 - (1-\alpha_n)^2(1+2\epsilon\alpha_n) - \alpha\alpha_n(1-\alpha_n)(1+\epsilon\alpha_n)}.$$

Let us consider the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(t) := \frac{1}{t}\left\{1 - \frac{(1-t)^2(1+2\epsilon t) + 3\alpha t(1-t)(1+\epsilon t)}{2-(1-t)^2(1+2\epsilon t) - \alpha t(1-t)(1+\epsilon t)}\right\}$$

for all $t > 0$. After certain manipulations and simplification, we obtain $\lim_{t\to 0} g(t) = 4(1-\alpha-\epsilon) > 0$. Let $\delta_0 > 0$. There exists $\epsilon_0 := 3(1-\alpha-\epsilon) > 0$ such that $g(t) > \epsilon_0,\ \ 0 < t < \delta_0$. Thus, we have

$$\frac{(1-t)^2(1+2\epsilon t) + 3\alpha t(1-t)(1+\epsilon t)}{2-(1-t)^2(1+2\epsilon t) - \alpha t(1-t)(1+\epsilon t)} < 1 - \epsilon_0 t, \quad 0 < t < \delta_0 \tag{3.18}$$

Since $\alpha_n \to 0$ as $n \to \infty$, there exists an integer $N_1 \in \mathbb{N}$ such that $\alpha_n < \delta_0$ for all $n \geq N_1$. From (3.17) and (3.18) we obtain,

$$\|x_{n+1} - z\|^2 \leq (1 - \epsilon_0\alpha_n)\|x_n - z\|^2 + \lambda_n. \tag{3.19}$$

From the definition h_n of (3.14) and (3.11), it turns out that $\limsup_{n\to\infty} \frac{h_n}{\alpha_n} \leq 0$, which implies that

$$\limsup_{n\to\infty} \frac{\lambda_n}{\alpha_n} \leq 0 \tag{3.20}$$

Finally, we use the conditions (A3), (A4) and (3.20) to apply Lemma 2.6 to the inequality (3.19), we conclude that $\lim_{n\to\infty} \|x_n - z\|^2 = 0$, this implies that $x_n \to z$ as $n \to \infty$. This completes the proof. □

The following result is derived from Theorem 3.1 immediately.

Theorem 3.2. *Let C be a nonempty closed convex subset a real Hilbert space H, $T : C \to C$ be a asymptotically nonexpansive mapping with $Fix(T) \neq \emptyset$ and $f : C \to C$ be a contraction mapping with the contractive constant $\alpha \in [0,1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{3.21}$$

where $\{\alpha_n\} \subseteq (0,1)$ is a sequence satisfying the following conditions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$;

(A2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Suppose that $(1-\alpha_n)k_n \leq 2$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. Then the sequence $\{x_n\}$ defined by (3.21) strongly converges to a fixed point z of T, which is also the unique solution of the following variational inequality

$$\langle (I-f)x, y-x\rangle \geq 0 \quad \forall y \in Fix(T).$$

In other word, z is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(z) = z$.

Proof. By taking $S = T$ and $\beta_n = 0$ and $\gamma_n = 1 - \alpha_n$ for all $n \in \mathbb{N}$ in Theorem 3.1, we get this result. □

Next, we give the another obtained result from Theorem 3.1 which is the main tool for applying various applications in the next section.

Theorem 3.3. *Let C be a nonempty closed convex subset a real Hilbert space H, $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $f : C \to C$ be a contraction mapping with the contractive constant $\alpha \in [0, 1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{3.22}$$

where $\{\alpha_n\} \subseteq (0, 1)$ is a sequence satisfying the following conditions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$;

(A2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ defined by (3.22) strongly converges to a fixed point z of T, which is also the unique solution of the following variational inequality

$$\langle (I-f)x, y-x\rangle \geq 0 \quad \forall y \in Fix(T).$$

In other word, z is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(z) = z$.

Proof. By taking $S = T$ and $\beta_n = 0$ and $\gamma_n = 1 - \alpha_n$ for all $n \in \mathbb{N}$ in Theorem 3.1, it is sufficient to prove that the following condition is satisfied:

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0.$$

We can use the same technique of the Theorem 3.1 for proving $\|x_{n+1} - x_n\| \to 0$

as $n \to \infty$ and $\{f(x_n)\}$, $\{T^n(x_n)\}$ are bounded. Therefore we have

$$\begin{aligned}
\|x_n - T^n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n x_n\| \\
&\le \|x_n - x_{n+1}\| + \left\|\alpha_n f(x_n) + (1-\alpha_n) T^n\left(\frac{x_n + x_{n+1}}{2}\right) - T^n x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - T^n(x_n)\| \\
&\quad + (1-\alpha_n)\left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - T^n x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - T^n(x_n)\| + \frac{(1-\alpha_n)}{2}\|x_{n+1} - x_n\| \\
&\le \frac{(3-\alpha_n)}{2}\|x_{n+1} - x_n\| + \alpha_n M,
\end{aligned}$$

where $M := \sup_{n\in\mathbb{N}} \{\|f(x_n) - T^n(x_n)\|\} < \infty$. By taking the limit as $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$. This completes the proof. $\square$

Now we give an open question whether Theorem 3.1 holds whenever we change some condition, that is, we have the following:

Open Question: Let C be a nonempty closed convex subset a real Hilbert space $H, T : C \to C$ and $S : C \to C$ be two asymptotically nonexpansive mappings with the same sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $U := Fix(T) \cap Fix(S) \neq \emptyset$ and $f : C \to C$ be a contraction mapping with the contractive constant $\alpha \in [0, 1)$. Define a sequence $\{x_n\}$ in C as follows:

$$\left.\begin{aligned}
&x_1 \in C, \\
&x_{n+1} = \alpha_n f(x_n) + \beta_n T^n\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n S^n\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N},
\end{aligned}\right\} \quad (3.23)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ are sequences satisfying the following conditions:

(A1) $\alpha_n + \beta_n + \gamma_n = 1$;

(A2) $\lim_{n\to\infty} \frac{k_n^2 - 1}{\alpha_n} = 0$;

(A3) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(A4) and $\lim_{n\to\infty} \gamma_n = 1$ and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$.

Suppose that $(1 - \alpha_n)k_n \le 2$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|T^n x_n - S^n x_n\| = 0$. Is it possible that the sequence $\{x_n\}$ defined by (3.23) converges strongly to a common fixed point of S and T?

4 Applications

In this section, we present the applications of the results in the previous section. These applications are related to variational inequality problems, constrained convex minimization problems, Fredholm integral equations and nonlinear evolution equations.

4.1 Variational Inequality Problems

Let C be a closed convex subset of a real Hilbert space H and $A : C \to H$ be a given operator. Let us consider the variational inequality problem (in short, VIP) as follows:

$$\text{finding } x^* \in C \quad \text{such that} \quad \langle Ax^*, x - x^* \rangle \geq 0 \qquad \forall x \in C. \tag{4.1}$$

Note that the variational inequality problem (4.1) is equivalent to the fixed point problem

$$Tx^* = x^*,$$

where $T := P_C(I - \lambda A)$ and $\lambda > 0$. If A is an L-Lipschitzian mapping and A is a strongly monotone mapping, for small enough $\lambda > 0$, T is a contraction mapping. So we can apply the Picard iteration for finding the unique fixed point of T which is also the unique solution of the VIP (4.1). However, if A is a θ-inverse strongly monotone mapping and A is an L-Lipschitzian mapping, then $P_C(I - \lambda A)$ is a nonexpansive mapping provided that $\lambda \in (0, 2\theta)$. Therefore, we can apply Theorem 3.3 for finding a solution of VIP (4.1) as follows.

Theorem 4.1. *Let C be a nonempty closed convex subset C of a real Hilbert space H, $A : C \to H$ be a θ-inverse strongly monotone maping for some $\theta > 0$, A is an L-Lipschitzian mapping and $\lambda \in (0, 2\theta)$. Suppose that $f : C \to C$ is a contraction mapping with the contractive constant $\alpha \in [0, 1)$. A sequence $\{x_n\}$ define by as follows:*

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[P_C(I - \lambda A)]^n \left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\}$$

where $\{\alpha_n\} \subseteq (0, 1)$ is a sequence satisfying the conditions (A1) and (A2) of Theorem 3.3. Then the sequence $\{x_n\}$ strongly convergence to a solution x^* of the VIP (4.1). Also, x^* solves the following variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0 \qquad \forall x \in A^{-1}(0).$$

4.2 Constrained Convex Minimization Problems

Consider the following constrained convex minimization problem:

$$\min_{x \in C} \phi(x), \tag{4.2}$$

where C is a closed convex subset of a real Hilbert space H and $\phi : C \to \mathbb{R}$ is a lower semicontinuous convex function and it is Frechet differentiable. Note that the constrained convex minimization problem (4.2) to the fixed point problem

$$Tx^* = x^*,$$

where $T := P_C(I - \lambda\nabla\phi)$ and $\lambda > 0$. Furthermore, $P_C(I - \lambda\nabla\phi)$ is a nonexpansive mapping provided that $\nabla\phi$ is an L-Lipschitzian mapping, $\nabla\phi$ is a θ-inverse strongly monotone mapping and $\lambda \in (0, 2\theta)$. Thus, we can apply Theorem 3.3 to find the solution of the constrained convex minimization problem (4.2) as follows.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $\phi : C \to \mathbb{R}$ is Frechet differentiable, $\nabla\phi$ is an L-Lipschitzian mapping, $\nabla\phi$ is a θ-inverse strongly monotone mapping and $\lambda \in (0, 2\theta)$. Suppose that $f : C \to C$ is a contraction mapping with the contractive constant $\alpha \in [0, 1)$. Define a sequence $\{x_n\}$ as follows:*

$$\left.\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[P_C(I - \lambda\nabla\phi)]^n \left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\}$$

where $\{\alpha_n\} \subseteq (0, 1)$ is a sequence satisfying the conditions (A1) and (A2) of Theorem 3.3. Then the sequence $\{x_n\}$ strongly converges to a solution x^* of constrained convex minimization problem (4.2). Also, x^* solves the following variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0 \qquad \forall x \in (\nabla\phi)^{-1}(0).$$

4.3 Fredholm Integral Equations

Consider the Fredholm integral equation

$$x(t) = g(t) + \int_0^1 F(t, s, x(s))ds, \qquad t \in [0, 1], \tag{4.3}$$

where $x \in L^2[0, 1]$ is an unknown function, g is a continuous function on $[0, 1]$ and $F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition:

$$|F(t, s, a) - F(t, s, b)| \leq |a - b|$$

for all $t, s \in [0, 1]$ and $a, b \in \mathbb{R}$. Define a mapping $T : L^2[0, 1] \to L^2[0, 1]$ by

$$(Tx)(t) = g(t) + \int_0^1 F(t, s, x(s))ds, \qquad t \in [0, 1]$$

for all $x \in L^2[0, 1]$. It is easy to see that T is a nonexpansive mapping. Furthermore, the solution of an integral equation (4.3) is equivalence to a fixed point of nonexpansive mapping T. Therefore, we can apply Theorem 3.3 for finding the solution of an integral equation (4.3) as follows.

Theorem 4.3. *Let us consider g, F, T and $L^2[0,1]$ define same as above. Suppose that $f : L^2[0,1] \to L^2[0,1]$ is a contraction mapping with the contractive constant $\alpha \in [0,1)$. Define a sequence $\{x_n\}$ in $L^2[0,1]$ as follows:*

$$\left.\begin{aligned} &x_1 \in L^2[0,1], \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\}$$

where $\{\alpha_n\} \subseteq (0,1)$ is a sequence satisfying the conditions (A1) and (A2) of Theorem 3.3. Then the sequence $\{x_n\}$ strongly converges to a solution x^* of the Fredholm integral equation (4.3). Also, x^* solves the following variational inequality

$$\langle (I-f)x^*, x - x^* \rangle \geq 0 \qquad \forall x \in Fix(T).$$

4.4 Nonlinear Evolution Equations

Let us consider the time-dependent nonlinear evolution equation in a real Hilbert space H as follows:

$$\frac{du}{dt} + A(t)u = f(t,u), \qquad t > 0, \tag{4.4}$$

where $A(t)$ is a family of closed linear operators in H and $f : \mathbb{R} \times H \to H$ is a given operator. The following existence result of a periodic solution of a nonlinear evolution equation (4.4) is proved by Browder [11] in 1965.

Theorem 4.4. *Consider the time-dependent nonlinear equation of the evolution (4.4). Assume that $A(t)$ and $f(t,u)$ are periodic in t with a common period $\xi > 0$ and the following assumptions hold.*

(i) For each t and each pair u and v in H, we get

$$\langle f(t,u) - f(t,v), u - v \rangle \leq 0.$$

(ii) For each t and u in $D(A(t))$, we have $\langle A(t)u, u \rangle \geq 0$.

(iii) There exists a mild solution u of an equation (4.4) on $\mathbb{R}^+$ with the initial value $v \in H$. Recall that u is a mild solution of an equation (4.4) on $\mathbb{R}^+$ with the initial value $u(0) = v$ if and only if for each $t > 0$, we have

$$u(t) = U(t,0)v + \int_0^t U(t,s)f(s,u(s))ds,$$

where $\{U(t,s)\}_{t \geq s \geq 0}$ is the evolution system for the homogeneous linear system

$$\frac{du}{dt} + A(t)u = 0, \qquad (t > s).$$

(iv) There exists $R > 0$ such that $\langle f(t,u), u \rangle < 0$ for $\| u \| = R$ and all t in $[0,\xi]$.

Then there exists an element v of H with $\| u \|< R$ such that the mild solution of nonlinear evolution equation (4.4) *with initial condition* $u(0) = v$*, is periodic of period* ξ*.*

Consider the time-dependent nonlinear evolution equation (4.4). If we define a mapping $T : H \to H$ by

$$T(v) = u(\xi) \text{ for all } v \in H,$$

where u is the solution of (4.4) which satisfy $u(0) = v$. From [11], T is a nonexpansive mapping. Also, if (iv) holds, then T is a self mapping on the closed ball $B := \{v \in H : \| v \| \leq R\}$. Consequently, T has a fixed point in B which we denote by v and the corresponding solution of u of (4.4) with $u(0) = v$ is the desired periodic solution of (4.4) with period ξ. In other word, to find a periodic solution of the time-dependent nonlinear evolution equation (4.4) is equivalent to find a fixed point of T. Therefore, we can use Theorem 3.22 for finding the periodic solution of a nonlinear evolution equation (4.4). Thus, the following algorithm:

$$\left.\begin{aligned} &v_1 \in B, \\ &v_{n+1} = \alpha_n f(v_n) + (1-\alpha_n)[P_C(I-\lambda A)]^n \left(\frac{v_n + v_{n+1}}{2}\right) \quad \forall n \in \mathbb{N}, \end{aligned}\right\} \tag{4.5}$$

where $\{\alpha_n\}$ is a sequence satisfying the conditions (A1) and (A2) of Theorem 3.3, converges weakly to a fixed point v of T and then the corresponding mild solution u of (4.4) with $u(0) = \xi$ is a periodic solution of (4.4).

Acknowledgements : The second author would like to thank the Thailand Research Fund and Office of the Higher Education Commission under grant no. MRG6180283 for financial support during the preparation of this manuscript.

References

[1] H. Attouch, Viscosity approximation methods for minimization problems, SIAM J. Optim. 6 (3) (1996) 769-806.

[2] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (1) (2000) 46-55.

[3] H.K. Xu, M.A. Alghamdi, N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 41 (2015).

[4] Y. Ke, C. Ma, The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces, Fixed Point Theory and Appl. 190 (2015).

[5] M.A. Alghamdi, N. Shahzad, H.K. Xu, The implicit midpoint rule for nonexpansive mappings, Fixed Point Theory and Appl. 96 (2014).

[6] L.-C. Zhao, S.-S. Chang, C.-F. Wen, Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mappings in Hilbert spaces, J. Nonlinear Sci. Appl. 9 (6) (2016) 4478-4488.

[7] S. He, Y. Mao, Z. Zhou, J.Q. Zhang, The generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces, Applied Mathematical Science 11 (12) (2017) 549-560.

[8] S.F.A. Naqvi, M.S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, Open J. Math. Sci. 1 (1) (2017) 111-125.

[9] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990).

[10] H.K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66 (2) (2002) 240-256.

[11] F.E. Browder, Existence of periodic solutions for nonlinear equations of evolution, Proc. Natl. Acad. Sci. USA 53 (5) (1965), 1100-1103.

(Received 26 August 2018)
(Accepted 14 March 2019)