Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 475–493
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Modified CQ Algorithm for Solving the Multiple-Sets Split Feasibility Problem and the Fixed Point Problem for Nonexpansive Mappings

Suparat Kesornprom, Nattawut Pholasa and Prasit Cholamjiak[1]

School of Science, University of Phayao, Phayao 56000, Thailand
e-mail : suparat.ke@gmail.com (S. Kesornprom)
nattawut_math@hotmail.com (N. Pholasa)
prasitch2008@yahoo.com (P. Cholamjiak)

Abstract : In this work, we propose a new relaxed CQ algorithm for solving the multiple-sets split feasibility problem (MSFP) and the fixed point problem for nonexpansive mappings. We obtain weak and strong convergence theorems of the proposed algorithm in Hilbert spaces. Finally, we provide numerical experiments to show the efficiency of our algorithm.

Keywords : multiple-sets split feasibility problem; relaxed CQ algorithm; Hilbert space; weak and strong convergence; fixed point problem.
2010 Mathematics Subject Classification : 65K05; 65K10; 49J52; 58C30.

1 Introduction

Censor et al. [1] introduced the multiple-sets feasibility problem (MSFP) which is formulated as the problem of finding a point x^* such that

$$x^* \in C := \bigcap_{i=1}^{t} C_i, \ Ax^* \in Q := \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

[1]Corresponding author.

Copyright © 2019 by the Mathematical Association of Thailand. All rights reserved.

where $t \geq 1$ and $r \geq 1$ are given integers, A is a given $M \times N$ real matrix, and $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ are closed convex subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. If $t = r = 1$, then (1.1) becomes the split feasibility problem (SFP) studied in [2].

In this work, we assume that the MSFP (1.1) is consistent *i.e.* its solution set, denoted by S, is nonempty. We know that the MSFP is equivalent to the following minimization problem:

$$\min \frac{1}{2}\|x - P_C(x)\|^2 + \frac{1}{2}\|Ax - P_Q(Ax)\|^2, \tag{1.2}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively. However, it should be noted that the projections onto the sets C and Q are usually difficult to be calculated in general.

In order to solve MSFP, Censor et al. [1] defined the following proximity function :

$$p(x) := \frac{1}{2}\sum_{i=1}^{t} l_i\|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r}\lambda_j\|Ax - P_{Q_j}(Ax)\|^2, \tag{1.3}$$

where $l_i (i = 1, ..., t)$ and $\lambda_j (j = 1, ..., r)$ are all positive constants such that $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r}\lambda_j = 1$. In this case, they obtained the following:

$$\nabla p(x) := \sum_{i=1}^{t} l_i(x - P_{C_i}(x)) + \sum_{j=1}^{r}\lambda_j A^*(I - P_{Q_j})Ax, \tag{1.4}$$

where $\nabla p(x)$ is a gradient of p at x. They considered the following problem:

$$\text{find } x^* \in \Omega \text{ such that } x^* \text{ solves (1.1)}, \tag{1.5}$$

where $\Omega \subseteq \mathbb{R}^N$ is a nonempty, closed and convex set such that $\Omega \bigcap S \neq \emptyset$. They proposed the following projection algorithm:

$$x_{n+1} = P_\Omega(x_n - s\nabla p(x_n)), \tag{1.6}$$

where s is a step size. It was proved that if $0 < s_L \leq s \leq s_U < \frac{2}{L}$, with L being the Lipschitz constant of ∇p, then the sequence (x_n) converges to a solution of (1.5). However, in general the Lipschitz constant L may be computed very hard.

Subsequently, MSFP and SFP are investigated in a more general setting (see [3]- [18]) for example, Zhang et al. [19] proposed a self-adaptive projection method for solving the MSFP in Hilbert spaces. Recently, López et al. [20] proposed the iterative scheme for the split feasibility problem without prior knowledge of operator norms.

Set $f_n(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2$ and $\nabla f_n(x) = A^*(I - P_{Q_n})Ax$. Define

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, 0 < \rho_n < 4. \tag{1.7}$$

Algorithm 1.1. Choose an arbitrary initial guess x_0. Assume x_n has been constructed. If $\nabla f_n(x_n) = 0$, then stop; otherwise, continue and construct x_{n+1} by the following manner:

$$x_{n+1} = P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{1.8}$$

where $C_n = \{x \in H : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}$, $\xi_n \in \partial c(x_n)$; $Q_n = \{y \in K : q(Ax_n) \leq \langle \zeta_n, Ax_n - y\rangle\}$, $\zeta_n \in \partial q(Ax_n)$. Let H and K be real Hilbert spaces and $A : H \to K$ a bounded linear operator and A^* denotes its adjoint.

López et al. [20] proved that the sequence (x_n) generated by Algorithm 1.1 converges weakly to a solution of the SFP under some certain conditions. We observe that the projections onto half-spaces C_n and Q_n have closed forms and τ_n is obtained adaptively via the formula (1.7). Hence the above relaxed CQ Algorithm 1.1 is implementable.

Recently, He et al. [21] introduced a new relaxed CQ algorithm for solving the MSFP (1.1), and proved the strong convergence by using the Halpern-type algorithm in real Hilbert spaces.

Algorithm 1.2. Let $u \in H$, and start an initial guess $x_0 \in H$ arbitrarily. Assume that the nth iterate (x_n) has been constructed. If $\nabla p_n(x_n) = 0$, then stop (x_n) is a approximate solution of MSFP(1.1). Otherwise continue and calculate the $(n+1)th$ iterate x_{n+1} by the following manner:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(x_n - \tau_n \nabla p_n(x_n)), \tag{1.9}$$

where $(\alpha_n) \subset (0, 1), \nabla p_n$ is given as (1.4), $\tau_n = \dfrac{\rho_n p_n(x_n)}{\|\nabla p_n(x_n)\|^2}, 0 < \rho_n < 4$.

In this paper, we introduce a new relaxed CQ algorithm for solving the multiple-sets feasibility problem and the fixed point problem in Hilbert spaces. We prove its weak and strong convergence theorems under some suitable conditions. Finally, we provide numerical experiments to show the efficiency of the proposed algorithm.

2 Preliminaries

Let H and K be real Hilbert spaces. In what follows, we will use the following notations:

- $\to$ denotes strong convergence.
- $\rightharpoonup$ denotes weak convergence.
- $\omega_w(x_n) = \{x | \exists (x_{n_k}) \subset (x_n)$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak $\omega - limit$ set of (x_n).

Recall that a mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H. \tag{2.1}$$

A mapping $T : H \to H$ is said to be firmly nonexpansive if, for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2. \tag{2.2}$$

A point $x \in H$ is said to be a fixed point of T if

$$T(x) = x. \tag{2.3}$$

We denote its solutions set by $F(T)$.
A mapping $f : H \to H$ is said to be a contraction on H if there exists a constant $a \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq a\|x - y\|, \ \forall x, y \in H. \tag{2.4}$$

Recall that a function $f : H \to \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \ \forall \lambda \in (0, 1), \forall x, y \in H. \tag{2.5}$$

A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x\rangle, \ \forall z \in H. \tag{2.6}$$

Recall that an element $g \in H$ is said to be a subgradient of $f : H \to \mathbb{R}$ at x if

$$f(z) \geq f(x) + \langle g, z - x\rangle, \ \forall z \in H. \tag{2.7}$$

This relation is called the subdifferentiable inequality.

A function $f : H \to \mathbb{R}$ is said to be subdifferentiable at x, if it has at least one subgradient at x. The set of subgradients of f at the point x is called the subdifferentiable of f at x, and it is denoted by $\partial f(x)$. A function f is called subdifferentiable, if it is subdifferentiable at all $x \in H$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n). \tag{2.8}$$

A mapping $T : H \to H$ is demiclosed (at y) if $T(x) = y$ whenever $(x_n) \subset H$ with $x_n \rightharpoonup x$ and $T(x_n) \to y$. It is well-known that if T is nonexpansive, then it is demiclosed in real Hilbert spaces.

We know that the orthogonal projection P_C from H onto a nonempty closed convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y\in C} \|x - y\|^2, \ x \in H. \tag{2.9}$$

We know that $P_C x$ satisfies the following inequality (for $x \in H$)

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall y \in C. \tag{2.10}$$

Lemma 2.1. [1] *Let* $\{C_i\}_{i=1}^t$ *and* $\{Q_j\}_{j=1}^r$ *be closed convex subsets of* H *and* K*, respectively and* $A : H \to K$ *a bounded linear operator. Let* $p(x)$ *be the function defined as in* (1.3)*. Then* $\nabla p(x)$ *is Lipschitz continuous with* $L := \sum_{i=1}^{t} l_i + \|A\|^2 \sum_{j=1}^{r} \lambda_j$ *as the Lipschitz constant.*

Lemma 2.2. [22] *Let* $T : H \to H$ *be an operator. The following statements are equivalent.*
(i) T *is firmly nonexpansive;*
(ii) $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle \ \forall x, y \in H$;
(iii) $I - T$ *is firmly nonexpansive.*

Lemma 2.3. [23,24] *Let* (a_n) *and* (c_n) *be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \ n \geq 1, \tag{2.11}$$

where (δ_n) *is a sequence in* $(0, 1)$ *and* (b_n) *is a real sequence. Assume* $\sum_{n=1}^{\infty} c_n < \infty$*. Then the following results hold:*
(i) If $b_n \leq \delta_n M$ *for some* $M \geq 0$*, then* (a_n) *is a bounded sequence.*
(ii) If $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\limsup_{n\to\infty} b_n/\delta_n \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [25] *Let* (s_n) *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* (s_{n_i}) *of* (s_n) *which satisfies* $s_{n_i} < s_{n_i+1}$ *for all* $i \in \mathbb{N}$*. Define the sequence* $(\psi(n))_{n\geq n_0}$ *of integers as follows:*

$$\psi(n) = \max\{k \leq n : s_k < s_{k+1}\}, \tag{2.12}$$

where $n_0 \in \mathbb{N}$ *such that* $\{k \leq n_0 : s_k < s_{k+1}\} \neq \emptyset$*. Then, the following hold:*
(i) $\psi(n_0) \leq \psi(n_0 + 1) \leq ...$ *and* $\psi(n) \to \infty$*;*
(ii) $s_{\psi(n)} \leq s_{\psi(n)+1}$ *and* $s_n \leq s_{\psi(n)+1}, \ \forall n \geq n_0$.

Recall that a sequence $(x_n) \subset H$ is said to be Fejér monotone with respect to a nonempty closed convex subset C in H if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \forall n \geq 1, \ \forall z \in C. \tag{2.13}$$

Lemma 2.5. [20] *Let* C *be a nonempty closed convex subset in* H*. If the sequence* (x_n) *is Féjer monotone with respect to* C*, then the following hold:*
(i) $x_n \rightharpoonup x^* \in C$ *if and only if* $\omega_w(x_n) \subset C$*;*
(ii) the sequence $(P_C x_n)$ *converges strongly;*
(iii) if $x_n \rightharpoonup x^* \in C$*, then* $x^* = \lim_{n\to\infty} P_C x_n$.

3 Main Results

3.1 Strong Convergence Theorem

In this section, we prove strong convergence theorem for the MSFP and the fixed point problem for nonexpansive mappings. Let $C_i(i = 1, \dots, t)$ and $Q_j(j = 1, \dots, r)$ be defined by

$$C_i = \{x \in H : c_i(x) \le 0\},\ Q_j = \{y \in K : q_j(y) \le 0\}, \tag{3.1}$$

where $c_i : H \to \mathbb{R}, i = 1, \dots, t$, and $q_j : K \to \mathbb{R}, j = 1, \dots, r$, are convex functions. We assume that $c_i(i = 1, \dots, t)$ and $q_j(j = 1, \dots, r)$ are subdifferentiable on H and K, respectively, and that $\partial c_i(i = 1, \dots, t)$ and $\partial q_j(j = 1, \dots, r)$ are bounded operators (*i.e.* bounded on bounded sets). By the way, we mention that every convex function defined on a finite-dimensional Hilbert space is subdifferentible and its subdifferential operator is a bounded operator (see [26]).

Set

$$C_i^n = \{x \in H : c_i(x_n) \le \langle \xi_i^n, x_n - x\rangle\}, \tag{3.2}$$

where $\xi_i^n \in \partial c_i(x_n)$ for $i = 1, \dots, t$, and

$$Q_j^n = \{y \in K : q_j(Ax_n) \le \langle \zeta_j^n, Ax_n - y\rangle\}, \tag{3.3}$$

where $\zeta_j^n \in \partial q_j(Ax_n)$ for $j = 1, \dots, r$.

We see that $C_i^n(i = 1, \dots, t)$ and $Q_j^n(j = 1, \dots, r)$ are half-spaces. We define the following function:

$$p_n(x) := \frac{1}{2}\sum_{i=1}^{t} l_i\|x - P_{C_i^n}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j\|Ax - P_{Q_j^n}Ax\|^2, \tag{3.4}$$

where $C_i^n(i = 1, \dots, t)$ and $Q_j^n(j = 1, \dots, r)$ are given as in (3.2) and (3.3), respectively. So we have

$$\nabla p_n(x) := \sum_{i=1}^{t} l_i\big(x - P_{C_i^n}(x)\big) + \sum_{j=1}^{r} \lambda_j A^*\big(I - P_{Q_j^n}\big)Ax, \tag{3.5}$$

where A^* is the adjoint operator of A.

Algorithm 3.1. Let $f : H \to H$ be a contraction, $T : H \to H$ be a nonexpansive mapping and start an initial guess $x_1 \in H$ arbitrarily. Assume that the nth iterate x_n has been constructed. If $\nabla p_n(x_n) = 0$ and $x_n = Tx_n$ then stop. Otherwise continue and calculate the $(n+1)th$ iterate x_{n+1} by the following manner:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - \tau_n \nabla p_n(x_n)),\\ x_{n+1} &= \beta_n y_n + (1-\beta_n)Ty_n,\ n \geq 1, \end{aligned} \tag{3.6}$$

where the sequences $(\alpha_n), (\beta_n) \subset (0,1)$,$\nabla p_n$ is given as (1.4),
$\tau_n = \dfrac{\rho_n p_n(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}$, $0 < \rho_n < 4$.

We are now ready to prove the strong convergence theorem.

Theorem 3.2. *Assume that $(\alpha_n), (\beta_n)$ and (ρ_n) satisfy the assumptions:*

(a1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(a2) $\inf_n \rho_n(4-\rho_n) > 0$;

(a3) $\inf_n \beta_n(1-\beta_n) > 0$.

Then the sequence (x_n) generated by Algorithm 3.1 converges strongly to $P_{S\cap F(T)}f(z)$.

Proof. We set $z = P_{S\cap F(T)}f(z)$. Then

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\|^2\\ &= \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2\\ &\quad -\beta_n(1-\beta_n)\|(y_n - z) - (Ty_n - z)\|^2\\ &= \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2\\ &= \|y_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2. \end{aligned} \tag{3.7}$$

Note that $I - P_{C_i^n}, (i = 1,\dots,t)$ and $I - P_{Q_j^n}, (j = 1,\dots,r)$ are firmly nonexpansive and $\nabla p_n(z) = 0$. So by Lemma 2.2 we have

$$\begin{aligned} \langle \nabla p_n(x_n), x_n - z\rangle &= \langle \sum_{i=1}^{t} l_i(x_n - P_{C_i^n}(x_n)) + \sum_{j=1}^{r} \lambda_j A^*(I - P_{Q_j^n})Ax_n, x_n - z\rangle\\ &= \sum_{i=1}^{t} l_i\langle (I - P_{C_i^n})x_n, x_n - z\rangle\\ &\quad + \sum_{j=1}^{r} \lambda_j\langle (I - P_{Q_j^n})Ax_n, Ax_n - Az\rangle\\ &\geq \sum_{i=1}^{t} \|(I - P_{C_i^n})x_n\|^2 + \sum_{j=1}^{r} \lambda_j\|(I - P_{Q_j^n})Ax_n\|^2\\ &= 2p_n(x_n), \end{aligned} \tag{3.8}$$

which gives

$$\begin{aligned}
& \|x_n - \tau_n \nabla p_n(x_n) - z\|^2 \\
= \;& \|x_n - z\|^2 + \|\tau_n \nabla p_n(x_n)\|^2 - 2\tau_n \langle \nabla p_n(x_n), x_n - z\rangle \\
\le \;& \|x_n - z\|^2 + \frac{\rho_n^2 p_n^2(x_n)}{(\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2)^2} \cdot \|\nabla p_n(x_n)\|^2 \\
& - \frac{4\rho_n p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \\
\le \;& \|x_n - z\|^2 + \frac{\rho_n^2 p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \\
& - \frac{4\rho_n p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \\
= \;& \|x_n - z\|^2 - \rho_n(4 - \rho_n) \frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}. \qquad (3.9)
\end{aligned}$$

Using (3.9), we have the following estimation:

$$\begin{aligned}
\|y_n - z\|^2 = \;& \|(\alpha_n f(x_n) + (1-\alpha_n)(x_n - \tau_n \nabla p_n(x_n))) - z\|^2 \\
= \;& \langle \alpha_n f(x_n) + (1-\alpha_n)(x_n - \tau_n \nabla p_n(x_n)) - z, y_n - z\rangle \\
= \;& \alpha_n \langle f(x_n) - f(z), y_n - z\rangle + \alpha_n \langle f(z) - z, y_n - z\rangle \\
& + (1-\alpha_n)\langle (x_n - \tau_n \nabla p_n(x_n) - z), y_n - z\rangle \\
\le \;& \alpha_n \|f(x_n) - f(z)\| \|y_n - z\| + \alpha_n \langle f(z) - z, y_n - z\rangle \\
& + (1-\alpha_n)\|x_n - \tau_n \nabla p_n(x_n) - z\| \|y_n - z\| \\
\le \;& \frac{1}{2}\alpha_n(\|f(x_n) - f(z)\|^2 + \|y_n - z\|^2) + \alpha_n \langle f(z) - z, y_n - z\rangle \\
& + \frac{1}{2}(1-\alpha_n)(\|x_n - \tau_n \nabla p_n(x_n) - z\|^2 + \|y_n - z\|^2) \\
= \;& \frac{1}{2}\alpha_n \|f(x_n) - f(z)\|^2 + \frac{1}{2}\alpha_n \|y_n - z\|^2 + \alpha_n \langle f(z) - z, y_n - z\rangle \\
& + \frac{1}{2}(1-\alpha_n)\|x_n - \tau_n \nabla p_n(x_n) - z\|^2 + \frac{1}{2}(1-\alpha_n)\|y_n - z\|^2 \\
\le \;& \frac{1}{2}\alpha_n a \|x_n - z\|^2 + \frac{1}{2}\alpha_n \|y_n - z\|^2 + \alpha_n \langle f(z) - z, y_n - z\rangle \\
& + \frac{1}{2}(1-\alpha_n)\left(\|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}\right) \\
& + \frac{1}{2}(1-\alpha_n)\|y_n - z\|^2 \\
= \;& \frac{1}{2}(1 - \alpha_n(1-a))\|x_n - z\|^2 + \frac{1}{2}\|y_n - z\|^2 + \alpha_n \langle f(z) - z, y_n - z\rangle \\
& - \frac{1}{2}(1-\alpha_n)\rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}. \qquad (3.10)
\end{aligned}$$

It follows that

$$\frac{1}{2}\|y_n - z\|^2 \leq \frac{1}{2}(1-\alpha_n(1-a))\|x_n - z\|^2 + \alpha_n\langle f(z) - z, y_n - z\rangle - \frac{1}{2}(1-\alpha_n)\rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}. \quad (3.11)$$

Hence

$$\|y_n - z\|^2 \leq (1-\alpha_n(1-a))\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, y_n - z\rangle - (1-\alpha_n)\rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}. \quad (3.12)$$

From (3.7) and (3.12), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 \quad \leq \quad & (1-\alpha_n(1-a))\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, y_n - z\rangle \\ & -(1-\alpha_n)\rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \\ & -\beta_n(1-\beta_n)\|y_n - Ty_n\|^2. \end{aligned} \quad (3.13)$$

Next, we will show that (x_n) is bounded. We see that

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n(y_n - z) + (1-\beta_n)(Ty_n - z)\| \\ &\leq \beta_n\|y_n - z\| + (1-\beta_n)\|Ty_n - z\| \\ &\leq \beta_n\|y_n - z\| + (1-\beta_n)\|y_n - z\| \\ &= \|y_n - z\| \\ &= \|\alpha_n f(x_n) + (1-\alpha_n)(x_n - \tau_n\nabla p_n(x_n)) - z\| \\ &\leq \alpha_n\|f(x_n) - z\| + (1-\alpha_n)\|x_n - \tau_n\nabla p_n(x_n) - z\| \\ &\leq \alpha_n(\|f(x_n) - f(z)\| + \|f(z) - z\|) + (1-\alpha_n)\|x_n - \tau_n\nabla p_n(x_n) - z\| \\ &\leq \alpha_n a\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\ &= (1-\alpha_n(1-a))\|x_n - z\| + \alpha_n(1-a)\cdot\frac{1}{1-a}\|f(z) - z\|. \end{aligned} \quad (3.14)$$

By induction, we can show that (x_n) is bounded. Using conditions (a1), (a2) and (a3), with no loss of generality, we can assume that there exist $\sigma, \gamma > 0$ such that $\rho_n(4-\rho_n)(1-\alpha_n) \geq \sigma$ and $\beta_n(1-\beta_n) \geq \gamma$ for all n. Setting $s_n = \|x_n - z\|^2$ by (3.13), we have

$$\begin{aligned} s_{n+1} \quad \leq \quad & (1-\alpha_n(1-a))s_n + 2\alpha_n\langle f(z) - z, y_n - z\rangle \\ & -\frac{\sigma p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} - \gamma\|y_n - Ty_n\|^2. \end{aligned} \quad (3.15)$$

We next consider the following two cases:

Case1 (x_n) is eventually decreasing, that is there exists $k \geq 0$ such that $s_n > s_{n+1}$ for all $n \geq k$. In this case, (s_n) must be convergent, from (3.15) and using condition (a1), we have

$$\begin{aligned}
&\frac{\sigma p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} + \gamma\|y_n - Ty_n\|^2 \\
\leq\ & 2\alpha_n\langle f(z) - z, y_n - z\rangle - s_{n+1} + (1 - \alpha_n(1-a))s_n \\
\leq\ & 2\alpha_n\langle f(z) - z, y_n - z\rangle - s_{n+1} + s_n. \qquad (3.16)
\end{aligned}$$

Since $\alpha_n \to 0$ and (s_n) is convergent, $\dfrac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \to 0$ and $\|y_n - Ty_n\|^2 \to 0$. To show that $p_n(x_n) \to 0$, it suffices to show that $(\|\nabla p(x_n)\|)$ is bounded. In fact, by Lemma 2.1, we see that

$$\|\nabla p_n(x_n)\| = \|\nabla p_n(x_n) - \nabla p_n(z)\| \leq L\|x_n - z\|, \qquad (3.17)$$

where $L = \sum_{i=1}^{t} l_i + \|A\|^2 \sum_{j=1}^{r} \lambda_j$. This implies that $(\|\nabla p_n(x_n)\|)$ is bounded and consequently $p_n(x_n) \to 0$. Hence $\|(I - P_{C_i^n})x_n\| \to 0 (i = 1, \ldots, t)$, and $\|(I - P_{Q_j^n})Ax_n\| \to 0 (j = 1, \ldots, r)$.

Next, we show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Consider

$$\begin{aligned}
\|x_n - y_n\| &= \|x_n - (\alpha_n f(x_n) + (1-\alpha_n)(x_n - \tau_n \nabla p_n(x_n)))\| \qquad (3.18)\\
&\leq \alpha_n\|x_n - f(x_n)\| + (1-\alpha_n)\left\|\frac{\rho_n p_n(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \cdot \nabla p_n(x_n)\right\| \\
&= \alpha_n\|x_n - f(x_n)\| + (1-\alpha_n)\rho_n p_n(x_n)\left(\frac{\|\nabla p_n(x_n)\|}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}\right).
\end{aligned}$$

Hence $\|x_n - y_n\| \to 0$ and we obtain

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\
&\leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|y_n - x_n\|. \qquad (3.19)
\end{aligned}$$

Thus $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Since $\partial_{q_j} (j = 1, \ldots, r)$ are bounded on bounded sets, there exists a constant $\eta > 0$ such that $\|\zeta_j^n\| \leq \eta (j = 1, \ldots, r)$ for all $n \geq 0$. From (3.3) and $P_{Q_j^n}(Ax_n) \in Q_j^n (j = 1, \ldots, r)$, it follows that

$$q_j(Ax_n) \leq \langle \zeta_j^n, Ax_n - P_{Q_j^n}(Ax_n)\rangle \leq \eta\|(I - P_{Q_j^n})Ax_n\| \to 0. \qquad (3.20)$$

If $x^* \in \omega_w(x_n)$, and (x_{n_k}) is a subsequence of (x_n) such that $x_{n_k} \rightharpoonup x^*$, then the $w - lsc$ of q_j and (3.20) implies that

$$q_j(Ax^*) \leq \liminf_{k\to\infty} q_j(Ax_{n_k}) \leq 0. \qquad (3.21)$$

This shows that $Ax^* \in Q_j (j = 1, \ldots, r)$. Next we prove that $x^* \in C_i (i = 1, \ldots, t)$. By the definition of $C_i^n (i = 1, \ldots, t)$, we have

$$c_i(x_n) \leq \langle \zeta_i^n, x_n - P_{C_i^n}(x_n) \rangle \leq \delta \| x_n - P_{C_i^n} x_n \| \to 0, (n \to \infty), \tag{3.22}$$

where δ is a constant such that $\|\zeta_i^n\| \leq \delta (i = 1, \ldots, t)$ for all $n \geq 0$. The $w - lsc$ of $c_i (i = 1, \ldots, t)$ also implies that

$$c_i(x^*) \leq \liminf_{k\to\infty} c_i(x_{n_k}) = 0. \tag{3.23}$$

So, $x^* \in C_i (i = 1, \ldots, t)$. By the demiclosedness principle, we can show that $\omega_w(x_n) \subset F(T)$. Hence $\omega_w(x_n) \subset S \cap F(T)$. Moreover, by (2.10), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle f(z) - z, y_n - z \rangle &= \limsup_{n\to\infty} \langle f(z) - z, x_n - z \rangle \\ &= \lim_{k\to\infty} \langle f(z) - z, x_{n_k} - z \rangle \\ &= \langle f(z) - P_{S\cap F(T)} f(z), x^* - P_{S\cap F(T)} f(z) \rangle \\ &\leq 0. \end{aligned} \tag{3.24}$$

From (3.15), we have

$$s_{n+1} \leq (1 - (\alpha_n (1 - a))) s_n + 2\alpha_n \langle f(z) - z, y_n - z \rangle. \tag{3.25}$$

By Lemma 2.3 (ii), (3.24) and (3.25), we conclude that $s_n \to 0$. Hence (x_n) converges strongly to z.

Case2 Suppose that there exists a subsequence (s_{n_i}) of the sequence (s_n) such that $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\psi : \mathbb{N} \to \mathbb{N}$ as in (2.12). Then, by Lemma 2.4, we have $s_{\psi(n)} \leq s_{\psi(n)+1}$. From (3.16), it follows that

$$\begin{aligned}&\frac{\sigma p^2_{\psi(n)}(x_{\psi(n)})}{\|\nabla p_{\psi(n)}(x_{\psi(n)})\|^2 + \|x_{\psi(n)} - T x_{\psi(n)}\|^2} + \gamma \|y_{\psi(n)} - T y_{\psi(n)}\|^2 \\ &\quad \leq 2\alpha_{\psi(n)} \langle f(z) - z, y_{\psi(n)} - z \rangle - s_{\psi(n)+1} + s_{\psi(n)} \qquad (3.26) \\ &\quad \leq 2\alpha_{\psi(n)} \langle f(z) - z, y_{\psi(n)} - z \rangle + \|x_{\psi(n)} - x_{\psi(n)+1}\| (\sqrt{s_{\psi(n)}} + \sqrt{s_{\psi(n)+1}}).\end{aligned}$$

Hence $\dfrac{\sigma p^2_{\psi(n)}(x_{\psi(n)})}{\|\nabla p_{\psi(n)}(x_{\psi(n)})\|^2 + \|x_{\psi(n)} - T x_{\psi(n)}\|^2} \to 0$ and $\|y_{\psi(n)} - T y_{\psi(n)}\|^2 \to 0$. Then we have $p_{\psi(n)}(x_{\psi(n)}) \to 0$ as $n \to \infty$ since $\{\|\nabla p_{\psi(n)}(x_{\psi(n)})\|\}$ is bounded. By the same argument to the proof in Case1, we have $\omega_w(x_{\psi(n)}) \subset S \cap F(T)$. We see that

$$\begin{aligned}\|x_{\psi(n)+1} - x_{\psi(n)}\| &= \|\beta_{\psi(n)} y_{\psi(n)} + (1 - \beta_{\psi(n)}) T y_{\psi(n)} - x_{\psi(n)}\| \\ &\leq \beta_{\psi(n)} \|y_{\psi(n)} - x_{\psi(n)}\| + (1 - \beta_{\psi(n)}) \|T y_{\psi(n)} - x_{\psi(n)}\| \\ &\leq \beta_{\psi(n)} \|y_{\psi(n)} - x_{\psi(n)}\| + (1 - \beta_{\psi(n)}) \|T y_{\psi(n)} - y_{\psi(n)}\| \\ &\quad + (1 - \beta_{\psi(n)}) \|y_{\psi(n)} - x_{\psi(n)}\|. \end{aligned} \tag{3.27}$$

It follows that

$$\lim_{n\to\infty} \|x_{\psi(n)+1} - x_{\psi(n)}\| = 0. \tag{3.28}$$

Moreover, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, y_{\psi(n)} - z\rangle &= \limsup_{n\to\infty}\langle f(z) - z, x_{\psi(n)} - z\rangle \\ &= \lim_{k\to\infty}\langle f(z) - z, x_{\psi(n_k)} - z\rangle \\ &= \langle f(z) - P_{S\cap F(T)}f(z), x^* - P_{S\cap F(T)}f(z)\rangle \\ &\leq 0. \end{aligned}\tag{3.29}$$

Since $s_{\psi(n)} \leq s_{\psi(n)+1}$, and from (3.15) we have

$$\alpha_{\psi(n)}(1-a)s_{\psi(n)} = 2\alpha_{\psi(n)}\langle f(z) - z, y_{\psi(n)} - z\rangle. \tag{3.30}$$

It follows that

$$s_{\psi(n)} \leq \frac{2}{(1-a)}\langle f(z) - z, y_{\psi(n)} - z\rangle, n > n_0. \tag{3.31}$$

From (3.29) and (3.31), we have

$$\limsup_{n\to\infty} s_{\psi(n)} \leq 0, \tag{3.32}$$

consequently $s_{\psi(n)} \to 0$, and (3.28) implies that

$$\begin{aligned}\sqrt{s_{\psi(n)+1}} &\leq \|(x_{\psi(n)} - z) + (x_{\psi(n)+1} - x_{\psi(n)})\| \\ &\leq \sqrt{s_{\psi(n)}} + \|x_{\psi(n)+1} - x_{\psi(n)}\| \\ &\to 0, \text{ as } n \to \infty. \end{aligned}\tag{3.33}$$

From (3.28) and (3.33), we obtain $s_{\psi(n)+1} \to 0$. By (3.14), we conclude that $s_n \to 0$. Therefore $x_n \to z$. □

3.2 Weak Convergence Theorem

In this section, we prove the weak convergence theorem.

Algorithm 3.3. Let $T : H \to H$ be a nonexpansive mapping and start an initial guess $x_1 \in H$ arbitrarily. Assume that the *nth* iterate x_n has been constructed. If $\nabla p_n(x_n) = 0$ and $x_n = Tx_n$ then stop. Otherwise continue and calculate the $(n+1)th$ iterate x_{n+1} by the following manner:

$$\begin{aligned} y_n &= x_n - \tau_n \nabla p_n(x_n), \\ x_{n+1} &= \beta_n y_n + (1-\beta_n)Ty_n,\ n \geq 1, \end{aligned}\tag{3.34}$$

where $(\beta_n) \subset (0,1)$,∇p_n is given as (1.4), $\tau_n = \dfrac{\rho_n p_n(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}$, $0 < \rho_n < 4$.

Theorem 3.4. *Assume that* (β_n) *and* (ρ_n) *satisfy the assumptions:*

(a1) $\inf_n \rho_n(4-\rho_n) > 0$;

(a2) $\inf_n \beta_n(1-\beta_n) > 0$.

Then the sequence (x_n) *generated by Algorithm* 3.3 *converges weakly to a point of* $S \cap F(T)$.

Proof. Consider

$$\begin{aligned} \|x_{n+1}-z\|^2 &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\|^2 \\ &= \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 \\ &\quad -\beta_n(1-\beta_n)\|(y_n-z)-(Ty_n-z)\|^2 \\ &\le \beta_n\|y_n-z\|^2 + (1-\beta_n)\|y_n-z\|^2 - \beta_n(1-\beta_n)\|y_n-Ty_n\|^2 \\ &= \|y_n-z\|^2 - \beta_n(1-\beta_n)\|y_n-Ty_n\|^2. \end{aligned} \tag{3.35}$$

From (3.9) we have

$$\begin{aligned} \|y_n-z\|^2 &= \|x_n - \tau_n\nabla p_n(x_n) - z\|^2 \\ &\le \|x_n-z\|^2 - \rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2}. \end{aligned} \tag{3.36}$$

It follows that, by (3.35) and (3.36)

$$\begin{aligned} \|x_{n+1}-z\|^2 \le& \|x_n-z\|^2 - \rho_n(4-\rho_n)\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \\ &- \beta_n(1-\beta_n)\|y_n-Ty_n\|^2. \end{aligned} \tag{3.37}$$

Thus (x_n) is decreasing and hence $\lim_{n\to\infty}\|x_n - z\|$ exists. So (x_n) is a bounded sequence. By our assumptions, there exist $\sigma,\gamma > 0$ such that $\rho_n(4-\rho_n) \ge \sigma$ and $\beta_n(1-\beta_n) \ge \gamma$ for all n. Setting $s_n = \|x_n - z\|^2$ by (3.37) we have

$$\frac{\sigma p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} + \gamma\|y_n - Ty_n\|^2 \le s_n - s_{n+1}. \tag{3.38}$$

Since (s_n) is convergent, so $\frac{p_n^2(x_n)}{\|\nabla p_n(x_n)\|^2 + \|x_n - Tx_n\|^2} \to 0$ and $\|y_n - Ty_n\|^2 \to 0$. This implies that $p_n(x_n) \to 0$ since (x_n) is bounded.

Next we show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Consider

$$\begin{aligned} \|y_n - x_n\| &= \|x_n - \tau_n\nabla p_n(x_n) - x_n\| \\ &= \tau_n\|\nabla p_n(x_n)\| \to 0, \text{ as } n\to\infty. \end{aligned} \tag{3.39}$$

Hence $\|y_n - x_n\| \to 0$ and by (3.19) we obtain $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. As the same proof in Theorem 3.2 and by the demiclosedness principle, we can show that $\omega_w(x_n) \subset S \cap F(T)$. Hence, by Lemma 2.5 (i), the sequence (x_n) converges weakly to a point in $S \cap F(T)$. □

4 Numerical Examples

In this section, we provide some numerical examples and illustrate its performance by using Algorithm 3.1 in Theorem 3.2. We present numerical results for solving the MSFP and the fixed point problem for nonexpansive mappings in Hilbert spaces.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^3$, $r = t = 2$ and $l_1 = l_2 = \lambda_1 = \lambda_2 = \frac{1}{2}$. Define

$$\begin{aligned} C_1 &= \{x = (a,b,c)^T \in \mathbb{R}^3 : a^2 + b^2 - c \leq 0\}, \\ C_2 &= \{x = (a,b,c)^T \in \mathbb{R}^3 : a^2 + b^2 + c - 9 \leq 0\}, \\ Q_1 &= \{x = (a,b,c)^T \in \mathbb{R}^3 : a^2 + b^2 + c^2 - 9 \leq 0\}, \\ Q_2 &= \{x = (a,b,c)^T \in \mathbb{R}^3 : \frac{a^2}{9} + \frac{b^2}{4} + \frac{c^2}{4} - 3 \leq 0\} \end{aligned}$$

and

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -7 & 2 & 0 \\ 8 & 9 & 1 \end{pmatrix}. \text{ Find } x^* \in C_1 \cap C_2 \text{ such that } Ax^* \in Q_1 \cap Q_2.$$

Let $T : H \to H$ be defined by $Tx = (x_1, -x_2, 4 - x_3)$ where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Choose $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{3n+1}$ for all $n \in \mathbb{N}$ and $f(x) = \frac{1}{2}x$ where $x \in \mathbb{R}^3$. We choose the sequence $\{p_n\}$ as follows:
Case 1: $\rho_n = \frac{0.2n}{n+1}$; Case 2: $\rho_n = \frac{1.5n}{n+1}$; Case 3: $\rho_n = \frac{2n}{n+1}$; Case 4: $\rho_n = \frac{3.5n}{n+1}$.

The stopping criterion is defined by

$$\begin{aligned} E_n = \quad & \frac{1}{2}\left(\|x_n - P_{C_1^n} x_n\|^2 + \|x_n - P_{C_2^n} x_n\|^2\right) \\ & + \frac{1}{2}\left(\|Ax_n - P_{Q_1^n} Ax_n\|^2 + \|Ax_n - P_{Q_2^n} Ax_n\|^2\right) \\ & + \|x_n - Tx_n\|^2 < 10^{-2}. \end{aligned}$$

We choose different choices of x_1 as
Choice 1 : $x_1 = (-11, 6, -10)^T$;
Choice 2 : $x_1 = (2, -5, 1)^T$;
Choice 3 : $x_1 = (8, 3, 12)^T$;
Choice 4 : $x_1 = (4, -1, 6)^T$.

The numerical experiments, using our Algorithm 3.1 for each choice are reported in the following Table 1.

Table 1: Algorithm 3.1 with different cases of ρ_n and different choices of x_1

		$\rho_n = \frac{0.2n}{n+1}$	$\rho_n = \frac{1.5n}{n+1}$	$\rho_n = \frac{2n}{n+1}$	$\rho_n = \frac{3.5n}{n+1}$
Choice 1	No. of Iter.	39	10	8	6
	cpu (Time)	0.037197	0.007181	0.006680	0.004980
Choice 2	No. of Iter.	27	8	7	6
	cpu (Time)	0.019552	0.006403	0.005182	0.003835
Choice 3	No. of Iter.	59	13	10	7
	cpu (Time)	0.063695	0.014288	0.011978	0.005490
Choice 4	No. of Iter.	43	11	9	6
	cpu (Time)	0.036036	0.009403	0.007173	0.006262

The convergence behavior of the error E_n for each choice of ρ_n and x_1 is shown in Figure 1-4, respectively.

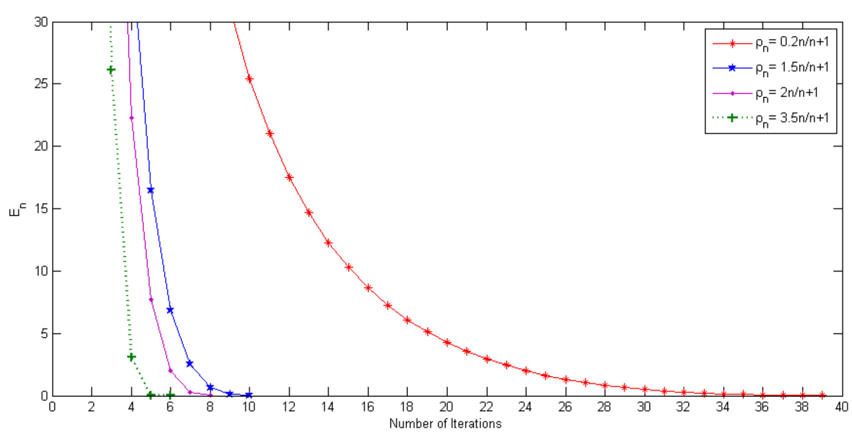

Figure 1: Error plotting E_n for Choice 1 in Example 4.1

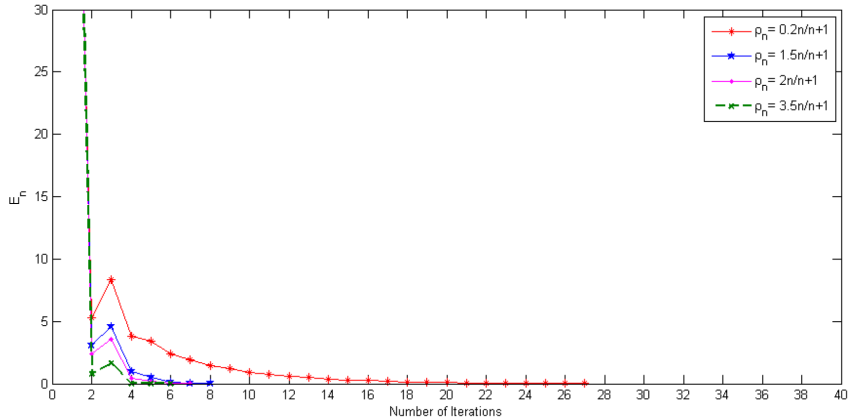

Figure 2: Error plotting E_n for Choice 2 in Example 4.1

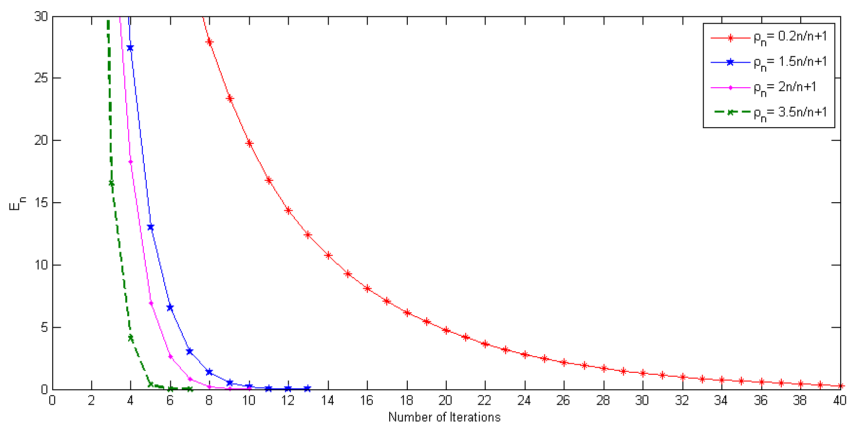

Figure 3: Error plotting E_n for Choice 3 in Example 4.1

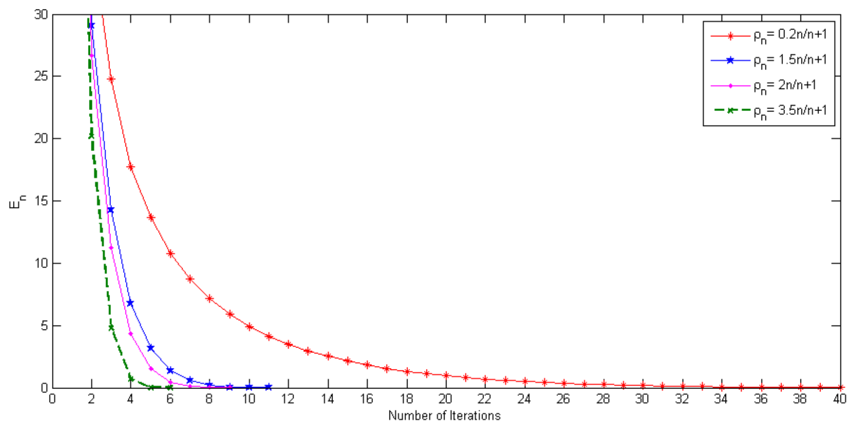

Figure 4: Error plotting E_n for Choice 4 in Example 4.1

Acknowledgements : The authors would like to thank Unit of Excellence (UOE 62001). P. Cholamjiak was supported by Thailand Research Fund and University of Phayao under grant no. RSA6180084.

References

[1] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problem, Inverse Prob. 21 (2005) 2071-2084.

[2] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projection in product space, Numer. Algorithms 8 (1994) 221-239.

[3] C. Byrne, Iterative oblique projection onto convex sets and the split feasibity problem, Inverse Prob. 18 (2002) 441-453.

[4] W. Cholamjiak, P. Cholamjiak, S. Suantai, An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces, J. Fixed Point Theory Appl. 20 (2018) https://doi.org/10.1007/s11784-018-0526-5.

[5] W. Cholamjiak, P. Cholamjiak, S. Suantai, Convergence of iterative schemes for solving fixed point problems for multi-valued nonself mappings and equilibrium problems, J. Nonlinear Sci. Appl 8 (6) (2015) 1245-1256.

[6] P. Cholamjiak, W. Cholamjiak, Y.J. Cho, S. Suantai, Weak and strong convergence to common fixed points of a countable family of multi-valued mappings in Banach spaces, Thai Journal of Mathematics 9 (3) (2012) 505-520.

[7] Y. Dang, Y. Gao, The strong convergence of a KM-CQ-like algorithm for a split feasibility problem, Inverse Prob. 27 (2011) 015007.

[8] G. López, V. Martin, H.K. Xu, Iterative algorithms for the multiple-sets split feasibility problem, Biomedical mathematics: promising directions in imaging therapy planning and inverse problems, Y. Censor, M. Jiang, G. Wang, Madison (WI) Medical Physics Publishing (2009) 243-279.

[9] K. Sitthithakerngkiet, J. Deepho, P. Kumam, Modified hybrid steepest method for the split feasibility problem in image recovery of inverse problems, Numer. Funct. Anal. Optim. 38 (4) (2017) 507-522.

[10] S. Suantai, Y. Shehu, P. Cholamjiak, O.S. Iyiola, Strong convergence of a self-adaptive method for the split feasibility problem in Banach spaces, J. Fixed Point Theory Appl. 20 (2) (2018) 68.

[11] S. Suantai, Y. Shehu, P. Cholamjiak, Nonlinear iterative methods for solving the split common null point problem in Banach spaces, Optimization Methods and Software (2018) 1-22.

[12] S. Suantai, N. Pholasa, P. Cholamjiak, The modified inertial relaxed CQ algorithm for solving the split feasibility problems, Journal of Industrial and Management Optimization (2018) 3-11.

[13] H.K. Xu, A variable Krasonosel'skii-Mann algorithm and the multiple-set split feasibility problem, Inverse Prob. 22 (2006) 2021-2034.

[14] H.K. Xu, Iterative algorithms for nonlinnear operators, J. Lond. Math. Soc. 66 (2002) 240-256.

[15] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Prob. 26 (2010) 105018.

[16] Q. Yang, The relaxed CQ algorithm for solving the split feasibility problem, Inverse Prob. 20 (2004) 1261-1266.

[17] J. Zhao, Q. Yang, Self-adaptive projection methods for the multiple-sets split feasibility problem, Inverse Prob. 27 (2011) 035009.

[18] J. Zhao, Y. Zhang, Q. Yang, Modified projection methods for the split feasibility problem and multiple-sets feasibility problem, Appl. Math. Comput. 219 (2012) 1644-1653.

[19] W. Zhang, D. Han, Z. Li, A self-adaptive projection method for solving the multiple-sets split feasibility problem, Inverse prob. 25 (2009) 115001.

[20] G. López, V. Martín-Márquez, F.H. Wang, H.K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, Inverse Prob. (2012) doi:10.1088/0266-5611/28/8/085004.

[21] S. He, Z. Zhao, B. Luo, A relaxed self-adaptive CQ algorithm for the multiple-sets split feasibility problem, Optimization 64 (9) (2015) 1907-1918.

[22] K. Goebel, W.A. Kirk, Topics on Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[23] H.K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66 (2002) 240-256.

[24] P.E. Maingé, Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 325 (2007) 469-479.

[25] P.E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008) 899-912.

[26] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problem, SIAM Rev. 38 (1996) 367-426.

(Received 15 February 2018)
(Accepted 5 February 2019)