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On Convergence Theorems for Two Generalized Nonexpansive Multivalued Mappings in Hyperbolic Spaces

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Abstract : The purpose of this paper is to establish Δ-convergence and strong convergence theorems for the mixed Agarwal-O'Regan-Sahu type iterative scheme [1] to approximate a common fixed point for two generalized nonexpansive multivalued mappings in hyperbolic spaces. The results presented in this paper extend and improve some recent results in the literature.

Keywords : hyperbolic spaces; Δ-convergence theorems; strong convergence theorems; common fixed points; condition (E).
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1 Introduction and Preliminaries

Fixed point theory has an important role for the various problems in Mathematics. A metric space embedded with a convex structure is a nonlinear framework for fixed point theory which one of convex structure is practicable in hyperbolic spaces.

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The class of hyperbolic spaces contains normed linear spaces and convex subsets, Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov [2] and the Hilbert ball equipped with the hyperbolic metric [3]. Strong convergence of modified viscosity implicit approximation methods for asymptotically nonexpansive mappings in complete CAT(0) spaces has been studied by Pakkaranang et al. [4]. Throughout in this paper we consider a hyperbolic space which is defined by Kohlenbach [5] in 2005.

A *hyperbolic space* [5] is a metric space (X, d) together with a mapping $W : X^2 \times [0, 1] \to X$ satisfying the following statements:

(W1) $d(u, W(x, y, \alpha)) \le \alpha d(u, x) + (1 - \alpha)d(u, y)$;

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;

(W3) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$;

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \le \alpha d(x, y) + (1 - \alpha)d(z, w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

A hyperbolic space (X, d, W) is said to be *uniformly convex* [6] if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d(W(x, y, \frac{1}{2}), u) \le (1 - \delta)r,$$

provided $d(x, u) \le r$, $d(y, u) \le r$ and $d(x, y) \ge \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is well known as a modulus of uniformly convexity of X. We call η *monotone* if it decreases with r (for a fixed ε), i.e., for any given $\varepsilon > 0$ and for any $r_2 \ge r_1 > 0$, we have $\eta(r_2, \varepsilon) \le \eta(r_1, \varepsilon)$.

A nonempty subset K of a hyperbolic space X is *convex* if $W(x, y, \alpha) \in K$ for any $x, y \in K$ and $\alpha \in [0, 1]$.

Let (X, d) be a metric space, K be a nonempty subset of X and T be a multivalued mapping of K into set of all subsets of K. The set of fixed points of T denoted by $F(T) = \{x \in K : x \in Tx\}$.

A nonempty subset K of X is said to be *proximal*, if for each $x \in X$, there exists an element $y \in K$ such that

$$d(x, y) = \operatorname{dist}(x, K) := \inf_{z \in K} d(x, z).$$

Remark 1.1. [7] It is well known that each weakly compact convex subset of Banach space is proximal.

Let $CB(X)$ be the collection of all nonempty closed bounded subsets of X and $P(X)$ be the collection of all nonempty proximal bounded subsets of X. The *Hausdorff metric* H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A)\}, \text{ for all } A, B \in CB(X).$$

We need the following definitions, lemmas, propositions that will be used in the sequel.

Definition 1.2. [8] A multivalued mapping $T : K \to CB(K)$ is said to be

(i) *nonexpansive* if $H(Tx, Ty) \leq d(x, y)$, for all $x, y \in K$;

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$, for all $x \in K$ and $p \in F(T)$.

Definition 1.3. [9] A multivalued mapping $T : K \to CB(K)$ is said to satisfy *condition* (E_μ) provided that

$$\text{dist}(x, Ty) \leq \mu \text{dist}(x, Tx) + d(x, y), \text{ for all } x, y \in K.$$

We say that T satisfies *condition* (E) whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

Proposition 1.4. [9] *If $T : K \to CB(K)$ is multivalued mapping satisfying condition (E) with $F(T) \neq \emptyset$, then T is a multivalued quasi-nonexpansive mapping.*

Lemma 1.5. [9] *Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping. Then T satisfies condition (E_1).*

We need the following definition of convergence in hyperbolic spaces [10] which is called *Δ-convergence*. It plays an important role in the main results and we recall some definitions and lemmas.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. We can define a function $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \text{ for all } x \in X.$$

The asymptotic radius of a bounded sequence $\{x_n\}$ with respect to a nonempty subset K of X is defined and denoted by

$$r_K(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a nonempty subset K of X is defined and denoted by

$$AC_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}), \text{ for all } y \in K\}.$$

Recall that a sequence $\{x_n\}$ in X is said to *Δ-converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the *Δ*-lim of $\{x_n\}$.

Lemma 1.6. [11] *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 1.7. [12] *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq c$, $\limsup_{n\to\infty} d(y_n, x) \leq c$, $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$ for some $c \geq 0$, then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 1.8. [7] *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η, then X possesses the Opial property, i.e., for any sequence $\{x_n\} \subset X$ with Δ-$\lim_{n\to\infty} x_n = x$ and for any $y \in X$ with $x \neq y$, then*

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Lemma 1.9. [7] *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η, K be a subset of X and $\{x_n\}$ be a bounded sequence in X with $AC(\{x_n\}) = \{p\}$. Suppose that $\{u_n\}$ is a subsequence of $\{x_n\}$ with $AC(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ is convergent, then $p = u$.*

In 2016, Kim et al. [9] established the existence of a fixed point for generalized nonexpansive multivalued mappings in hyperbolic spaces as the following lemma.

Lemma 1.10. [9] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty closed convex subset of X, and $T : K \to P(K)$ be a multivalued mapping satisfying the condition (E) with convex values. If $\{x_n\}$ is a sequence in K such that Δ- $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$, then z is a fixed point of T.*

Moreover, they [9] established Δ-convergence and strong convergence theorems for the iterative sequence induced by Chang et al. [1].

Theorem 1.11. [9] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty closed convex subset of X, and $T : K \to P(K)$ be a multivalued mapping satisfing the condition (E) with convex values. Suppose that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. For arbitrarily chosen $x_0 \in K$, sequence $\{x_n\}$ is defined by*

$$x_{n+1} = W(u_n, v_n, \alpha_n), \tag{1.1}$$

$$y_n = W(x_n, u_n, \beta_n),$$

where $v_n \in Ty_n, u_n \in Tx_n$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following condition:

(C1) there exist constants $a, b \in (0, 1)$ with $0 < b(1-a) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$.

Then the sequence $\{x_n\}$ *defined by* (1.1) *is* Δ*-convergent to a point in* $F(T)$*.*

Lemma 1.12. [9] *Let* (X, d, W) *be a complete uniformly convex hyperbolic space,* K *be a nonempty closed convex subset of* X*. Let* $T : K \to P(K)$ *be a multivalued mapping with* $F(T) \neq \emptyset$ *and let* $P_T : K \to 2^k$ *be a multivalued mapping defined by*

$$P_T(x) := \{y \in Tx : d(x, y) = \mathrm{dist}(x, Tx)\}, x \in K. \tag{1.2}$$

Then the following conclusions hold:

(1) $F(T) = F(P_T)$*;*

(2) $P_T(p) = \{p\}$*, for each* $p \in F(T)$*;*

(3) *for each* $x \in K$*,* $P_T(x)$ *is a closed subset of* $T(x)$ *and so it is compact;*

(4) $d(x, Tx) = d(x, P_T(x))$ *for each* $x \in K$*.*

On the other hand, in the case of a single-valued mapping $T : K \to K$, we known that Mann and Ishikawa iteration processes are defined as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{N}$$

and

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{aligned}$$

respectively.

In 2007, Agarwal-O'Regan-Sahu [13] introduced the iteration process:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)T x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}. \end{aligned}$$

They presented that their process is dependent of Mann and Ishikawa and converges faster than of these, for more details see [13].

In 2014, Chang et al. [1] introduced the mixed Agarwal-O'Regan-Sahu type iterative scheme for the multivalued nonexpansive in the setting of hyperbolic spaces as follows:

$$\begin{aligned} x_{n+1} &= W(u_n, v_n, \alpha_n), \\ y_n &= W(x_n, u_n, \beta_n), \end{aligned} \tag{1.3}$$

where $v_n \in T_1 y_n, u_n \in T_2 x_n$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences.

The purpose of this paper is to establish Δ-convergence and strong convergence theorems for the mixed Agarwal-O'Regan-Sahu type iterative scheme [1] to approximate a common fixed point for two generalized nonexpansive multivalued mappings in hyperbolic spaces. The results presented in this paper extend and improve some recent results in the literature.

2 Main Results

In this section, we establish Δ-convergence and strong convergence theorems for the mixed Agarwal-O'Regan-Sahu type iterative scheme [1] to approximate a common fixed point for two generalized nonexpansive multivalued mappings in hyperbolic spaces.

Theorem 2.1. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty closed convex subset of X, and $T_i : K \to P(K)$ $(i = 1, 2)$ be a multivalued mapping satisfying the condition (E) with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in F$ $(i = 1, 2)$. For arbitrarily chosen $x_1 \in K$, sequence $\{x_n\}$ is the Agarwal-O'Regan-Sahu type sequence defined by (1.3) where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following condition:*

(C1) there exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$.

Then the sequence $\{x_n\}$ defined by (1.3) is Δ-convergent to a point in F.

Proof. We divide the proof into 3 steps as follows:
Step 1: First, we prove that $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F$.
By using Proposition 1.4 we obtain that each multivalued mapping T satisfying condition (E) with $F(T) \neq \emptyset$ is a multivalued quasi-nonexpansive mapping. Hence for each $p \in F$, by (1.3), we have

$$\begin{aligned} d(y_n, p) &= d(W(x_n, u_n, \beta_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(u_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n \mathrm{dist}(u_n, T_2 p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n H(T_2 x_n, T_2 p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{2.1}$$

This implies that

$$\begin{aligned} d(x_{n+1}, p) &= d(W(u_n, v_n, \alpha_n), p) \\ &\leq (1-\alpha_n)d(u_n, p) + \alpha_n d(v_n, p) \\ &\leq (1-\alpha_n)\mathrm{dist}(u_n, T_2 p) + \alpha_n \mathrm{dist}(v_n, T_1 p) \\ &\leq (1-\alpha_n)H(T_2 x_n, T_2 p) + \alpha_n H(T_1 y_n, T_1 p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{2.2}$$

Therefore that sequence $\{d(x_n, p)\}$ is non-increasing and bounded below. It follows that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.
Step 2: We now prove that

$$\lim_{n\to\infty} \mathrm{dist}(x_n, T_i x_n) = 0, \text{ for each } i = 1, 2. \tag{2.3}$$

By Step 1, we know that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Suppose that $\lim_{n\to\infty} d(x_n, p) = c \geq 0$. If $c = 0$, then we obtain that

$$\begin{aligned} \mathrm{dist}(x_n, T_i x_n) &\leq d(x_n, p) + \mathrm{dist}(T_i x_n, p) \\ &\leq d(x_n, p) + H(T_i x_n, T_i p) \\ &\leq d(x_n, p) + d(x_n, p) \\ &\leq 2d(x_n, p). \end{aligned}$$

This implies that $\lim_{n\to\infty} \mathrm{dist}(x_n, T_i x_n) = 0$, for each $i = 1, 2$. Therefore (2.3) is true. If $c > 0$, then by using (2.1), we obtain that

$$\limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c. \tag{2.4}$$

Since

$$d(v_n, p) = \mathrm{dist}(v_n, T_1 p) \leq H(T_1 y_n, T_1 p) \leq d(y_n, p),$$

we have

$$\limsup_{n\to\infty} d(v_n, p) \leq c. \tag{2.5}$$

Similarly, since

$$d(u_n, p) = \mathrm{dist}(u_n, T_2 p) \leq H(T_2 x_n, T_2 p) \leq d(x_n, p),$$

we obtain that

$$\limsup_{n\to\infty} d(u_n, p) \leq c. \tag{2.6}$$

Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, by using (2.5), (2.6) and Lemma 1.7, we have

$$\lim_{n\to\infty} d(u_n, v_n) = 0. \tag{2.7}$$

On the other hand, by (1.3), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(u_n, v_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(u_n, p) + \alpha_n d(v_n, p) \\ &\leq (1-\alpha_n)\mathrm{dist}(u_n, T_2 p) + \alpha_n \mathrm{dist}(v_n, T_1 p) \\ &\leq (1-\alpha_n) H(T_2 x_n, T_2 p) + \alpha_n H(T_1 y_n, T_1 p) \\ &\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(y_n, p) \\ &= d(x_n, p) - \alpha_n d(x_n, p) + \alpha_n d(y_n, p). \end{aligned}$$

This implies that

$$d(x_n, p) \leq \frac{d(x_n, p) - d(x_{n+1}, p)}{\alpha_n} + d(y_n, p) \leq \frac{d(x_n, p) - d(x_{n+1}, p)}{a} + d(y_n, p). \tag{2.8}$$

Taking inferior limit to the both sides of the above inequality, we obtain that

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{2.9}$$

By using (2.4) and (2.8), we have

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{2.10}$$

By Lemma 1.7, (2.6), (2.10) and $\lim_{n\to\infty} d(x_n, p) = c$, then we have

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \tag{2.11}$$

Therefore

$$\text{dist}(x_n, T_2x_n) \leq d(u_n, x_n) \to 0 \text{ as } n \to \infty. \tag{2.12}$$

On the other hand, it follows from (1.3) and (2.11) that

$$d(y_n, x_n) = d(W(x_n, u_n, \beta_n), x_n) \leq \beta_n d(u_n, x_n), \tag{2.13}$$

and

$$\begin{aligned} \text{dist}(y_n, T_1y_n) &\leq d(y_n, v_n) \\ &= d(W(x_n, u_n, \beta_n), v_n) \\ &\leq (1-\beta_n)d(x_n, v_n) + \beta_n d(u_n, v_n) \\ &\leq (1-\beta_n)[d(x_n, u_n) + d(u_n, v_n)] + \beta_n d(u_n, v_n). \end{aligned}$$

Taking limit $n \to \infty$ and using (2.7) and (2.11) in above inequality, we have

$$\lim_{n\to\infty} \text{dist}(y_n, T_1y_n) = 0. \tag{2.14}$$

Since T_1 satisfies the condition (E), we have

$$\begin{aligned} \text{dist}(x_n, T_1x_n) &\leq d(x_n, y_n) + \text{dist}(y_n, T_1x_n) \\ &\leq d(x_n, y_n) + \mu\text{dist}(y_n, T_1y_n) + d(y_n, x_n) \\ &= 2d(x_n, y_n) + \mu\text{dist}(y_n, T_1y_n). \end{aligned}$$

Taking limit $n \to \infty$ and using (2.13) and (2.14), we have

$$\lim_{n\to\infty} \text{dist}(x_n, T_1x_n) = 0.$$

This implies that, $\lim_{n\to\infty} \text{dist}(x_n, T_ix_n) = 0,\ \ i = 1, 2.$
Step 3: We finally prove that the sequence $\{x_n\}$ is Δ-convergent to a point in $F = \bigcap_{i=1}^{2} F(T_i)$. Denote $W_\Delta(\{x_n\}) = \bigcup_{\{u_n\}\subset\{x_n\}} AC(\{u_n\})$. We now prove that $W_\Delta(\{x_n\}) \subset F$. Let $u \in W_\Delta(\{x_n\})$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $AC(\{u_n\}) = \{u\}$. By applying Lemma 1.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = p \in K$. Since $\lim_{n\to\infty} \text{dist}(v_n, T_iv_n) = 0\ \ (i = 1, 2)$, it follows from Lemma 1.10, we have $p \in F$. Since $\{d(u_n, p)\}$ converges from Lemma 1.9, we have $u = p \in F$. It follows that $W_\Delta(\{x_n\}) \subset F$. We next prove that $W_\Delta(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $AC(\{u_n\}) = \{u\}$ and $AC(\{x_n\}) = \{x\}$. Since $u \in W_\Delta(\{x_n\}) \subset F$, we have $\lim_{n\to\infty} d(x_n, u)$ exists. By Lemma 1.9, we obtain that $x = u \in W_\Delta(\{x_n\})$. Therefore $W_\Delta(\{x_n\})$ consists of exactly one point. Hence the sequence $\{x_n\}$ is Δ-convergent to element of F. □

We now illustrate the strong convergence theorems in the setting of complete uniformly convex hyperbolic spaces.

Theorem 2.2. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty compact convex subset of X and $T_i : K \to CB(X)$ $(i = 1, 2)$ be a multivalued mapping satisfying the condition (E) with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in F$ $(i = 1, 2)$. Then the sequence $\{x_n\}$ defined by* (1.3) *converges strongly to a point in F.*

Proof. For all $x \in K$ and for all $i = 1, 2$, by the assumption, we obtain that $T_i x$ is a bounded closed and convex subset K. Since K is compact and $T_i x$ is a nonempty compact and convex subset of K, we have $T_i x$ is a bounded proximal subset in K. This implies that $T_i : K \to P(K)$ for all $i = 1, 2$. Therefore all conditions of Theorem 2.1 are satisfied. From (2.2) and (2.3) we obtain that

$$\lim_{n\to\infty} d(x_n, p) \text{ exists and } \lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n) = 0,$$

for all $p \in F$ and for all $i = 1, 2$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging strong to $q \in K$. Since T_1 satisfies the condition (E), there exists $\mu \geq 1$ and then

$$\begin{aligned} \operatorname{dist}(q, T_1 q) &\leq d(q, x_{n_k}) + \operatorname{dist}(x_{n_k}, T_1 q) \\ &\leq d(q, x_{n_k}) + \mu \operatorname{dist}(x_{n_k}, T_1 x_{n_k}) + d(x_{n_k}, q) \\ &\leq 2d(q, x_{n_k}) + \mu \operatorname{dist}(x_{n_k}, T_1 x_{n_k}). \end{aligned}$$

By taking the limit as $k \to \infty$, we have $\operatorname{dist}(q, T_1 q) = 0$ and it follows that $q \in T_1 q$. Similarly, we can prove that $q \in T_2 q$. This implies that $q \in F$. Since $\lim_{n\to\infty} d(x_n, q)$ exists, we have

$$\lim_{n\to\infty} d(x_n, q) = \lim_{k\to\infty} d(x_{n_k}, q) = 0.$$

This show that $\{x_n\}$ converges strongly to a point in F. □

Theorem 2.3. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty compact convex subset of X and $T_i : K \to CB(X)$ $(i = 1, 2)$ be a multivalued mapping satisfying the condition (E) with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in F$ $(i = 1, 2)$. Then the sequence $\{x_n\}$ defined by* (1.3) *converges strongly to a point in F if and only if $\liminf_{n\to\infty} dist(x_n, F) = 0$.*

Proof. Assume that $\lim_{n\to\infty} d(x_n, p) = 0$ for some $p \in F$. Since

$$\operatorname{dist}(x_n, F) \leq d(x_n, p), \text{ for all } n \in \mathbb{N}.$$

By taking limit inferior $n \to \infty$ in the above inequality, we have

$$\liminf_{n\to\infty} \operatorname{dist}(x_n, F) = 0.$$

Conversely, suppose that $\liminf_{n\to\infty} \operatorname{dist}(x_n, F) = 0$. From (2.2), we have

$$d(x_{n+1}, p) \leq d(x_n, p), \text{ for all } p \in F.$$

It follows that $\operatorname{dist}(x_{n+1}, F) \leq \operatorname{dist}(x_n, F)$ for all $n \in \mathbb{N}$.
Therefore $\lim_{n\to\infty} \operatorname{dist}(x_{n+1}, F)$ exists and $\lim_{n\to\infty} \operatorname{dist}(x_{n+1}, F) = 0$. This implies that, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and sequence $\{p_k\}$ in F such that

$$d(x_{n_k}, p_k) < \frac{1}{2^k},$$

for all $k \in \mathbb{N}$. From (2.2), we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Therefore

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}. \end{aligned}$$

It follows that $\{p_k\}$ is a Cauchy sequence in K and it converges to some $w \in K$. Since

$$\operatorname{dist}(p_k, T_i w) \leq H(T_i p_k, T_i w) \leq d(p_k, w)$$

and $p_k \to w$ as $k \to \infty$, we have $\operatorname{dist}(w, T_i w) = 0$. Therefore $w \in T_i w$ for each $i = 1, 2$. This yields $w \in F$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n\to\infty} d(x_n, w)$ exists, we have $\{x_n\}$ converges strongly to w. Therefore the proof is complete. □

Theorem 2.4. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty compact convex subset of X and $T_i : K \to CB(X)$ $(i = 1, 2)$ be a multivalued mapping satisfying the condition (E) with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in F$ $(i = 1, 2)$. Let $\{x_n\}$ be defined by (1.3). Suppose that there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, where $f(r) > 0$ for all $r > 0$ such that*

$$\operatorname{dist}(x_n, T_i x_n) \geq f(\operatorname{dist}(x_n, F)), \ i = 1, 2.$$

Then the sequence $\{x_n\}$ converges strongly to a point in F.

Proof. In the proof of Theorem 2.1, we obtain that $\mathrm{dist}(x_n, T_i x_n) = 0,\ i = 1, 2$. By the assumption, we have

$$\lim_{n\to\infty} f(\mathrm{dist}(x_n, F)) \leq \lim_{n\to\infty} \mathrm{dist}(x_n, T_i x_n) = 0,\ i = 1, 2.$$

It follows that

$$\lim_{n\to\infty} f(\mathrm{dist}(x_n, F)) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is an increasing function with $f(0) = 0$, we can conclude that $\lim_{n\to\infty} \mathrm{dist}(x_n, F) = 0$. By Theorem 2.3, we obtain that $\{x_n\}$ converges strongly to a point in F. Therefore the proof is complete. □

Theorem 2.5. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty closed convex subset of X and $T_i : K \to CB(K)$ $(i = 1, 2)$ be a multivalued mapping with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$. Let P_{T_i} be a multivalued mapping satisfying the condition (E) $(i = 1, 2)$. For arbitrarily chosen $x_1 \in K$, sequence $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = W(u_n, v_n, \alpha_n), \tag{2.15}$$
$$y_n = W(x_n, u_n, \beta_n),$$

where $u_n \in P_{T_2} x_n, v_n \in P_{T_1} y_n = P_{T1}(W(x_n, u_n, \beta_n))$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the condition (C1). Then the sequence $\{x_n\}$ defined by (2.15) is Δ-convergent to a point in F.

Proof. By Lemma 1.12, we know that the mapping P_{T_i}, $i = 1, 2$ defined by (1.2) has the following property, for each $i = 1, 2$, $P_{T_i} : K \to P(K)$ is multivalued mappings with $\bigcap_{i=1}^{2} F(P_{T_i}) = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$ and

$$P_{T_i}(p) = \{p\},\ \text{for each } p \in F.$$

By replacing the mappings T_i by P_{T_i} in Theorem 2.1, $i = 1, 2$, we obtain that all the conditions in Theorem 2.1 are satisfied. This implies that $\{x_n\}$ is Δ-convergent to a point in F. □

Using the result in Step 1 and Step 2 of Theorem 2.1 with the sequence $\{x_n\}$ defined by (2.15) and the same technique as in the proof of Theorem 18 and Theorem 19 of Kim et al. [9], we get the following strong convergence results for multivalued mappings P_{T_i} satisfying the condition (E), $i = 1, 2$ without proving.

Theorem 2.6. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty closed convex subset of X and $T_i : K \to CB(K)$ $(i = 1, 2)$ be a multivalued mapping with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$. Let P_{T_i} $(i = 1, 2)$ be a multivalued mapping satisfying the condition (E) $(i = 1, 2$. For arbitrarily $x_1 \in K$, let $\{x_n\}$ be a sequence in K defined by (2.15). Then the sequence $\{x_n\}$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} dist(x_n, F)) = 0$.*

Theorem 2.7. *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η, K be a nonempty closed convex subset of X and $T_i : K \to CB(K)$ $(i = 1, 2)$ be a multivalued mapping with convex values. Suppose that $F = \bigcap_{i=1}^{2} F(T_i) \neq \emptyset$. Let P_{T_i} $(i = 1, 2)$ be a multivalued mapping satisfying the condition (E). Let $\{x_n\}$ be a sequence in K defined by (2.15). Suppose that there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, where $f(r) > 0$ for all $r > 0$ such that*

$$\operatorname{dist}(x_n, T_i x_n) \geq f(\operatorname{dist}(x_n, F),\ i = 1, 2.$$

Then the sequence $\{x_n\}$ converges strongly to a point in F.

3 Numerical Example

Let $X = \mathbb{R}$ with metric $d(x, y) = |x - y|$ and $K = [0, 4]$. Define $W : X^2 \times [0, 1] \to X$ by $W(x, y, \alpha) := \alpha x + (1 - \alpha)y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. Then (X, d, W) is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and K is a nonempty closed convex subset of X. Let $T_1, T_2 : K \to CB(K)$ be defined by

$$T_1 x = \begin{cases} [0, \frac{x}{4}], & x < 4; \\ \{1\}, & x = 4, \end{cases}$$

and

$$T_2 x = [\frac{x}{6}, \frac{x}{2}].$$

Set $\alpha_n = \beta_n = \frac{1}{2}$, for all $n \geq 1$. First, we will show that it satisfies the condition (C1). Choose $a = \frac{1}{5}$, $b = \frac{1}{2}$. Then we obtain that

$$\{\alpha_n\} = \{\beta_n\} = \{\frac{1}{2}\} \subseteq [\frac{1}{5}, \frac{1}{2}] = [a, b].$$

By definitions of T_1 and T_2, it is clear that $F = F(T_1) \cap F(T_2) = \{0\}$.
Next, we will show that T_1 and T_2 satisfy condition (E).

If $x, y < 4$, then $\operatorname{dist}(x, T_1 x) = \operatorname{dist}(x, [0, \frac{x}{4}]) = |x - \frac{x}{4}| = |\frac{3}{4}x|$. Therefore

$$\begin{aligned} \operatorname{dist}(x, T_1 y) = \operatorname{dist}(x, [0, \frac{y}{4}]) = |x - \frac{y}{4}| &= |x - \frac{x}{4} + \frac{x}{4} - \frac{y}{4}| \\ &\leq |x - \frac{x}{4}| + |\frac{x}{4} - \frac{y}{4}| \\ &= |\frac{3}{4}x| + \frac{1}{4}|x - y| \\ &\leq |\frac{3}{4}x| + |x - y| = \operatorname{dist}(x, T_1 x) + d(x, y). \end{aligned}$$

If $x, y = 4$, then $\operatorname{dist}(x, T_1 x) = \operatorname{dist}(x, \{1\}) = |x - 1|$. Therefore

$$\begin{aligned} \operatorname{dist}(x, T_1 y) = \operatorname{dist}(x, \{1\}) &= |x - 1| \\ &\leq |x - 1| + |x - y| \\ &= \operatorname{dist}(x, T_1 x) + d(x, y). \end{aligned}$$

If $x < 4$ and $y = 4$, then $\text{dist}(x, T_1x) = \text{dist}(x, [0, \frac{x}{4}]) = |x - \frac{x}{4}| = |\frac{3}{4}x|$. Therefore

$$\begin{aligned}
\text{dist}(x, T_1y) = \text{dist}(x, \{1\}) = |x - 1| &= |x - \frac{x}{4} + \frac{x}{4} - 1| \\
&\leq |x - \frac{x}{4}| + |\frac{x}{4} - 1| \\
&= |x - \frac{x}{4}| + \frac{1}{4}|x - 4| \\
&= |\frac{3}{4}x| + \frac{1}{4}|x - y| \\
&\leq |\frac{3}{4}x| + |x - y| = \text{dist}(x, T_1x) + d(x, y).
\end{aligned}$$

If $x = 4$ and $y < 4$, then $\text{dist}(x, T_1x) = \text{dist}(x, \{1\}) = |x - 1|$. Therefore

$$\begin{aligned}
\text{dist}(x, T_1y) = \text{dist}(x, [0, \frac{y}{4}]) = |x - \frac{y}{4}| &= |x - 1 + 1 - \frac{y}{4}| \\
&\leq |x - 1| + |1 - \frac{y}{4}| \\
&= |x - 1| + \frac{1}{4}|4 - y| \\
&= |x - 1| + \frac{1}{4}|x - y| \\
&\leq |x - 1| + |x - y| \\
&= \text{dist}(x, T_1x) + d(x, y).
\end{aligned}$$

Then we can conclude that, for all $x, y \in K$,

$$\text{dist}(x, T_1y) \leq \mu\text{dist}(x, T_1x) + d(x, y), \text{ for some } \mu = 1.$$

We observe that

$$\text{dist}(x, T_2x) \ = \ \text{dist}(x, [\frac{x}{6}, \frac{x}{2}]) \ = \ |x - \frac{x}{2}| \ = \ |\frac{x}{2}|$$

and

$$\text{dist}(x, T_2y) = \text{dist}(x, [\frac{y}{6}, \frac{y}{2}]).$$

If $\text{dist}(x, T_2y) = |x - \frac{y}{6}|$, then we have

$$\begin{aligned}
\text{dist}(x, T_2y) = |x - \frac{y}{6}| &= |x - \frac{x}{6} + \frac{x}{6} - \frac{y}{6}| \\
&\leq |x - \frac{x}{6}| + |\frac{x}{6} - \frac{y}{6}| \\
&= |\frac{5}{6}x| + \frac{1}{6}|x - y| \\
&\leq \frac{5}{3}|\frac{x}{2}| + |x - y| = \frac{5}{3}\text{dist}(x, T_2x) + d(x, y).
\end{aligned}$$

If $\text{dist}(x, T_2y) = |x - \frac{y}{2}|$, then we have

$$\begin{aligned}
\text{dist}(x, T_2y) = |x - \frac{y}{2}| &= |x - \frac{x}{2} + \frac{x}{2} - \frac{y}{2}| \\
&\leq |x - \frac{x}{2}| + |\frac{x}{2} - \frac{y}{2}| \\
&= |\frac{x}{2}| + \frac{1}{2}|x - y| \\
&\leq |\frac{x}{2}| + |x - y| \\
&\leq \frac{5}{3}|\frac{x}{2}| + |x - y| = \frac{5}{3}\text{dist}(x, T_2x) + d(x, y).
\end{aligned}$$

Then for $x, y \in K$, we have

$$\text{dist}(x, T_2y) \leq \mu\text{dist}(x, T_2x) + d(x, y), \text{ for some } \mu = \frac{5}{3}.$$

Hence T_1 and T_2 are multivalued quasi-nonexpansive mappings.
From $\{\alpha_n\} = \{\beta_n\} = \{\frac{1}{2}\}$, for $x_0 = \frac{3}{4}$ and using (1.3), we have $T_2x_0 = [\frac{1}{8}, \frac{3}{8}]$, taking $u_0 = \frac{1}{4} \in T_2x_0$, we obtain that

$$\begin{aligned}
y_0 = W(x_0, u_0, \frac{1}{2}) &= \frac{1}{2}x_0 + (1 - \frac{1}{2})u_0 \\
&= \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} \\
&= \frac{3}{8} + \frac{1}{8} = \frac{1}{2^3}(1 + 3) = \frac{1}{2^3}(4).
\end{aligned}$$

Now we compute $T_1y_0 = [0, \frac{\frac{1}{2^3}(4)}{4}]$, taking $v_0 = \frac{1}{2^3} \in T_1y_0$, then

$$\begin{aligned}
x_1 = W(u_0, v_0, \frac{1}{2}) &= \frac{1}{2}u_0 + (1 - \frac{1}{2})v_0 \\
&= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2^3} = \frac{1}{2^3} + \frac{1}{2^4} \\
&= \frac{1}{2^3}[1 + \frac{1}{2}].
\end{aligned}$$

From definition of T_2, we have $T_2x_1 = [\frac{\frac{1}{2^3}(1+\frac{1}{2})}{6}, \frac{\frac{1}{2^3}(1+\frac{1}{2})}{2}]$, taking $u_1 = \frac{1}{2^4} \in T_2x_1$, then

$$\begin{aligned}
y_1 = W(x_1, u_1, \frac{1}{2}) &= \frac{1}{2} \cdot \frac{1}{2^3}[1 + \frac{1}{2}] + \frac{1}{2} \cdot \frac{1}{2^4} \\
&= \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^5} = \frac{1}{2^4}[1 + \frac{1}{2} + \frac{1}{2}].
\end{aligned}$$

Hence $T_1y_1 = [0, \frac{\frac{1}{2^4}(1+\frac{1}{2}+\frac{1}{2})}{4}]$, taking $v_1 = \frac{1}{2^5} \in T_1y_1$, then

$$\begin{aligned}
x_2 = W(u_1, v_1, \frac{1}{2}) &= \frac{1}{2}u_1 + (1 - \frac{1}{2})v_1 \\
&= \frac{1}{2} \cdot \frac{1}{2^4} + \frac{1}{2} \cdot \frac{1}{2^5} = \frac{1}{2^5} + \frac{1}{2^6} = \frac{1}{2^5}[1 + \frac{1}{2}].
\end{aligned}$$

From definition of T_2, we have $T_2x_2 = [\frac{\frac{1}{2^5}(1+\frac{1}{2})}{6}, \frac{\frac{1}{2^5}(1+\frac{1}{2})}{2}]$, taking $u_2 = \frac{1}{2^6} \in T_2x_2$, then

$$\begin{aligned} y_2 = W(x_2, u_2, \frac{1}{2}) &= \frac{1}{2} \cdot \frac{1}{2^5}[1 + \frac{1}{2}] + \frac{1}{2} \cdot \frac{1}{2^6} \\ &= \frac{1}{2^7} + \frac{1}{2^6} + \frac{1}{2^7} = \frac{1}{2^6}[1 + \frac{1}{2} + \frac{1}{2}]. \end{aligned}$$

Hence $T_1y_2 = [0, \frac{\frac{1}{2^6}(1+\frac{1}{2}+\frac{1}{2})}{4}]$, taking $v_2 = \frac{1}{2^7} \in T_1y_2$, then

$$\begin{aligned} x_3 = W(u_2, v_2, \frac{1}{2}) &= \frac{1}{2}u_2 + (1 - \frac{1}{2})v_2 \\ &= \frac{1}{2} \cdot \frac{1}{2^6} + \frac{1}{2} \cdot \frac{1}{2^7} \\ &= \frac{1}{2^7} + \frac{1}{2^8} = \frac{1}{2^7}[1 + \frac{1}{2}]. \end{aligned}$$

Inductively, we have $x_{n+1} = \frac{1}{2^{2n+3}}(1 + \frac{1}{2})$. Therefore all conditions of Theorem 2.2 are satisfied. Then the sequence $\{x_n\}$ converges strongly to a point in F which is 0.

Using the algorithm (1.3) with the initial point $x_0 = \frac{3}{4}$, $u_0 = \frac{1}{4}$ we have numerical result appeared in Table 1.

n	x_n
1	0.187500
2	0.046875
3	0.011718
4	0.002929
5	0.000732
6	0.000183
7	0.000045
8	0.000011
9	0.000002
10	0.000000

Table 1: The values of the sequence $\{x_n\}$

4 Conclusion Remarks

Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6 and Theorem 2.7, are improvements and extensions of the corresponding results in the following senses:

(1) Our results propose for two generalized nonexpansive mappings instead of one generalized nonexpansive mapping in Kim et al. [9] (Theorem 12, Theorem 13, Theorem 14, Theorem 15, Theorem 17, Theorem 18 and Theorem 19).

(2) We suppose that $T_i : K \to P(K)$ insteads of $T_i : K \to C(K)$, $(i = 1, 2)$ in Chang et al. [1] (Theorem 2.1), where K is a nonempty closed convex subset of X, where $P(K)$ is the collection of all nonempty proximal bounded and closed subsets of K and $C(K)$ is the collection of all nonempty compact subset of K.

(3) We suppose that a mapping T_i is a multivalued mapping satisfying the condition (E) $(i = 1, 2)$ insteads of multivalued nonexpansive mappings in Chang et al. [1] (Theorem 2.1).

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