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Green's Relations and Natural Partial Order on the Regular Subsemigroup of Transformations Preserving an Equivalence Relation and Fixed a Cross-Section

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Abstract : Let X be an arbitrary nonempty set and $T(X)$ the full transformation semigroup on X. For an equivalence relation E on X and a cross-section R of the partition X/E induced by E, let

$$T_E(X,R) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x,y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Then the set $Reg(T_E(X,R))$ of all regular elements of $T_E(X,R)$ is a regular subsemigroup of $T(X)$. In this paper, we describe Green's relations for elements of the semigroup $Reg(T_E(X,R))$. Also, we discuss the natural partial order on this semigroup and characterize when two elements in $Reg(T_E(X,R))$ are related under this order.

Keywords : transformation semigroup; equivalence relation; Green's relations; regular element; natural partial order.

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1 Introduction and Preliminaries

An element x of a semigroup S is called *regular* if there exists $y \in S$ such that $x = xyx$. If all its elements of S are regular, we call S a *regular semigroup*. The set of all regular elements of S is denoted by $Reg(S)$.

In 1951, Green [1] defined the equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ on a semigroup S by the rules that, for $a, b \in S$,

$(a, b) \in \mathcal{R}$ if and only if $aS^1 = bS^1$,

$(a, b) \in \mathcal{L}$ if and only if $S^1a = S^1b$, and

$(a, b) \in \mathcal{J}$ if and only if $S^1aS^1 = S^1bS^1$

where S^1 is the semigroup with identity obtained from S by adjoining an identity if necessary. Then he also defined the equivalence relations $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. These five equivalence relations are known as *Green's relations*. Hence $\mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

In 1952, Vagner [2] defined the *natural partial order* for any inverse semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = be \text{ for some idempotent } e \in S. \tag{1.1}$$

Later, Nambooripad [3] extended this partial order $\leq$ on a regular semigroup S by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some idempotents } e, f \in S. \tag{1.2}$$

For an inverse semigroup S this relation is just the natural partial order (1.1).

In 1986, Mitsch [4] extended the above partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1. \tag{1.3}$$

This natural partial order coincides with the relation (1.2) if the semigroup S is regular.

Let $T(X)$ be the full transformation semigroup on a set X under the usual composition of mappings. In 1955, Miller and Doss [5] proved that $T(X)$ is a regular semigroup and described its Green's relations. Over the past decades, notions of regularity and Green's relations of subsemigroups of $T(X)$ have been widely considered, see [6], [7] and [8]. In [6], the author introduced a family of subsemigroups of $T(X)$ defined by

$$T_E(X) = \{\alpha \in T(X) : \forall a, b \in X, (a, b) \in E \Rightarrow (a\alpha, b\alpha) \in E\}$$

where E is an arbitrary equivalence relation on X. The author investigated the regularity and Green's relations for $T_E(X)$. Also, the natural partial order on $T_E(X)$ was described in [9].

For an equivalence relation E on a set X, let R be a cross-section of the partition X/E induced by E (i.e., $|R \cap A| = 1$ for all $A \in X/E$). In [10], Araújo and Konieczny defined a subsemigroup of $T(X)$ as follows:

$$T(X, E, R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Clearly, $T(X, E, R) \subseteq T_E(X)$. The semigroup $T(X, E, R)$ is the centralizer of the idempotent transformation with kernel E and image R. They determined the structure of $T(X, E, R)$ in terms of Green's relations and described the regular elements of $T(X, E, R)$ in [7]. Moreover, the natural partial order on $T(X, E, R)$ was discussed in [11]. Now, we consider the following subset of $T_E(X)$:

$$T_E(X, R) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Then $T_E(X, R)$ is a subsemigroup of $T(X, E, R)$. In [8], the authors investigated regular and E-inversive elements of the semigroup $T_E(X, R)$. The regularity of the semigroup $T_E(X, R)$ was characterized as follows:

Theorem 1.1. ([8]) *Let $\alpha \in T_E(X, R)$. Then α is regular if and only if $\alpha|_R$ is an injection.*

Theorem 1.2. ([8]) *$T_E(X, R)$ is a regular semigroup if and only if R is finite.*

Moreover, in [12], the authors showed that $Reg(T_E(X, R))$ is a regular subsemigroup of $T_E(X, R)$.

Theorem 1.3. ([12]) *$Reg(T_E(X, R))$ is the largest regular subsemigroup of $T_E(X, R)$.*

The purpose of this paper is to investigate Green's relations on the semigroup $Reg(T_E(X, R))$. Moreover, we study the natural partial order on $Reg(T_E(X, R))$ and characterize when two elements of $Reg(T_E(X, R))$ are related under this order. Also, their maximal, minimal and covering elements are described.

In what follows, let E be an equivalence relation on a set X and R a cross-section of the partition of X. Denote by X/E the quotient set and E_r the E-class containing r for all $r \in R$.

2 Green's Relations

In this section, we focus on Green's relations for regular elements of the semigroup $T_E(X, R)$. First, we need the following lemmas.

Lemma 2.1. ([6]) *Let $\alpha \in T(X)$. Then $\alpha \in T_E(X)$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $A\alpha \subseteq B$.*

Lemma 2.2. ([12]) *Let $\alpha, \beta \in T_E(X, R)$. Then $\alpha\beta \in Reg(T_E(X, R))$ if and only if α and β are elements in $Reg(T_E(X, R))$.*

Lemma 2.3. *Let $\alpha \in T_E(X,R)$ and $r,s \in R$. If $x \in E_r$ with $x\alpha \in E_s$, then $E_r\alpha \subseteq E_s$ and $r\alpha = s$.*

Proof. Suppose that $x \in E_r$ with $x\alpha \in E_s$. Let $y \in E_r$. Then $(x,y) \in E$. Since $\alpha \in T_E(X)$, $(x\alpha, y\alpha) \in E$ and so $y\alpha \in E_s$. Hence $E_r\alpha \subseteq E_s$. Since $r\alpha \in E_s \cap R$, it follows that $r\alpha = s$. □

For $\alpha \in T(X)$, the symbol $\pi(\alpha)$ will denote the decomposition of X induced by the map α, namely

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}.$$

Hence $\pi(\alpha) = X/\ker\alpha$, where $\ker\alpha = \{(x,y) \in X \times X : x\alpha = y\alpha\}$.

Lemma 2.4. *Let $\alpha, \beta \in Reg(T_E(X,R))$. Then $\alpha = \beta\mu$ for some $\mu \in Reg(T_E(X,R))$ if and only if $\ker\beta \subseteq \ker\alpha$.*

Proof. Suppose that $\alpha = \beta\mu$ for some $\mu \in Reg(T_E(X,R))$. Let $(x,y) \in \ker\beta$. Then $x\beta = y\beta$ and so $x\alpha = x\beta\mu = y\beta\mu = y\alpha$. This shows that $(x,y) \in \ker\alpha$. Hence $\ker\beta \subseteq \ker\alpha$.

Conversely, suppose that $\ker\beta \subseteq \ker\alpha$. For every $y \in X\beta \setminus R$, we choose and fix an element $a_y \in X \setminus R$ such that $a_y\beta = y$. And every $r \in R$, we choose and fix an element $a_r \in R$ such that $a_r\beta = r$ (since $R\beta = R$). For each $r \in R$, we define a map $\mu_r : E_r \to X$ by

$$x\mu_r = \begin{cases} a_x\alpha & \text{if } x \in X\beta, \\ a_r\alpha & \text{otherwise.} \end{cases}$$

We define the map $\mu : X \to X$ by $\mu|_{E_r} = \mu_r$ for all $r \in R$. Since R is a cross-section of the partition X/E induced by E, we have that μ is well-defined and so $\mu \in T(X)$. We show $\mu \in T_E(X,R)$ and $\alpha = \beta\mu$ in the following. Let $r \in R$. Then $r = a_r\beta$ for some $a_r \in R$. Claim that $E_r\mu \subseteq E_{a_r\alpha}$. Let $y \in E_r$. If $y \notin X\beta$, then $y\mu = y\mu_r = a_r\alpha \in E_{a_r\alpha}$. If $y \in X\beta$, then $y = a_y\beta$ for some $a_y \in X$. Thus $a_y \in E_s$ for some $s \in R$. Since $a_y \in E_s$ and $a_y\beta = y \in E_r$ by Lemma 2.3, we get that $s\beta = r = a_r\beta$. By assumption, we have $s\alpha = a_r\alpha$. This implies that

$$y\mu = y\mu_r = a_y\alpha \in E_s\alpha \subseteq E_{s\alpha} = E_{a_r\alpha}.$$

Hence $E_r\mu \subseteq E_{a_r\alpha}$, so we have the claim. It follows from Lemma 2.1 that $\mu \in T_E(X)$. Obviously, $R\mu \subseteq R$. For the reverse inclusion, let $r \in R$. Then $s\alpha = r$ for some $s \in R$. Thus $s\beta = t$ for some $t \in R$ and so there exists $a_t \in R$ such that $s\beta = t = a_t\beta$. By assumption, we deduce that $r = s\alpha = a_t\alpha = t\mu_t = t\mu$. It implies that $R \subseteq R\mu$ and hence $\mu \in T_E(X,R)$. Finally, we will show that $\alpha = \beta\mu$. Let $x \in X$. Then $x\beta \in X\beta$ and $x\beta \in E_r$ for some $r \in R$ and so $a_{x\beta}\beta = x\beta$ for some $a_{x\beta} \in X$. Thus $(a_{x\beta}, x) \in \ker\beta$ so that $x\alpha = a_{x\beta}\alpha = (x\beta)\mu_r = x\beta\mu$ by assumption. Therefore $\alpha = \beta\mu$. By Lemma 2.2, we have $\mu \in Reg(T_E(X,R))$. □

Using Lemma 2.4, we can establish the next result.

Theorem 2.5. *Let $\alpha, \beta \in Reg(T_E(X,R))$. Then $(\alpha,\beta) \in \mathcal{R}$ if and only if* $\ker\alpha = \ker\beta$.

Next, we consider the relation $\mathcal{L}$, the following lemmas are needed.

Lemma 2.6. ([12]) *Let $\alpha \in Reg(T_E(X,R))$ and $x, y \in X$. Then $(x,y) \in E$ if and only if $(x\alpha, y\alpha) \in E$.*

Lemma 2.7. *Let $\alpha, \beta \in Reg(T_E(X,R))$. Then the following statements are equivalent.*

(*i*) *$\alpha = \lambda\beta$ for some $\lambda \in Reg(T_E(X,R))$.*

(*ii*) *For every $A \in X/E$, there exists some $B \in X/E$ such that $A\alpha \subseteq B\beta$.*

(*iii*) *$X\alpha \subseteq X\beta$.*

Proof. $(i) \Rightarrow (ii)$ Assume that $\alpha = \lambda\beta$ for some $\lambda \in Reg(T_E(X,R))$. Let $r \in R$. By Lemma 2.1, there exists $s \in R$ such that $E_r\lambda \subseteq E_s$. By assumption, we have $E_r\alpha = E_r\lambda\beta \subseteq E_s\beta$.

$(ii) \Rightarrow (iii)$ Assume that (ii) holds. Let $y \in X\alpha$. Then $y = x\alpha$ for some $x \in X$. Let $A \in X/E$ such that $x \in A$. By assumption, there exists some $B \in X/E$ such that $A\alpha \subseteq B\beta$. It follows that $y = x\alpha \in A\alpha \subseteq B\beta \subseteq X\beta$. Hence $X\alpha \subseteq X\beta$.

$(iii) \Rightarrow (i)$ Suppose that $X\alpha \subseteq X\beta$. For each $x \in X \setminus R$, we choose and fix $x' \in X$ such that $x\alpha = x'\beta$. If $x \in R$, then $x\alpha \in R\alpha = R = R\beta$. Thus we choose and fix $x' \in R$ such that $x\alpha = x'\beta$. Define $\lambda : X \to X$ by

$$x\lambda = x' \text{ for all } x \in X.$$

Let $(x,y) \in E$. Then $(x'\beta, y'\beta) = (x\alpha, y\alpha) \in E$ where $x', y' \in X$. Hence by Lemma 2.6, we have $(x\lambda, y\lambda) = (x', y') \in E$. Consequently, $\lambda \in T_E(X)$. Clearly, $R\lambda \subseteq R$. On the other hand, let $r \in R$. Then $r\beta \in R$ and $r\beta = s\alpha = s'\beta$ for some $s, s' \in R$. By Theorem 1.1, $r = s'$, hence $s\lambda = s' = r$. Thus $\lambda \in T_E(X,R)$. If $x \in X$, then $x\lambda\beta = x'\beta = x\alpha$. Hence $\alpha = \lambda\beta$. Since $\alpha, \beta \in Reg(T_E(X,R))$ by Lemma 2.2, it follows that $\lambda \in Reg(T_E(X,R))$. □

The following theorem is a direct consequence of Lemma 2.7.

Theorem 2.8. *Let $\alpha, \beta \in Reg(T_E(X,R))$. Then the following statements are equivalent.*

(*i*) *$(\alpha,\beta) \in \mathcal{L}$.*

(*ii*) *For every $A \in X/E$, there exist $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.*

(*iii*) *$X\alpha = X\beta$.*

The following result is evident from Theorems 2.5 and 2.8.

Theorem 2.9. *Let* $\alpha, \beta \in Reg(T_E(X,R))$. *Then* $(\alpha, \beta) \in \mathcal{H}$ *if and only if* $\ker \alpha = \ker \beta$ *and* $X\alpha = X\beta$.

Theorem 2.10. *Let* $\alpha, \beta \in Reg(T_E(X,R))$. *Then* $(\alpha, \beta) \in \mathcal{D}$ *if and only if there is a bijection* $\varphi : X\alpha \to X\beta$ *satisfying*

(*i*) $R\varphi = R$ *and*

(*ii*) *for every* $A \in X/E$, *there exists* $B \in X/E$ *such that* $(A\alpha)\varphi \subseteq B\beta$.

Proof. Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then there exists $\gamma \in Reg(T_E(X,R))$ such that $(\alpha, \gamma) \in \mathcal{R}$ and $(\gamma, \beta) \in \mathcal{L}$.

Next, we shall construct a bijection $\varphi : X\alpha \to X\beta$ such that $R\varphi = R$ and for every $A \in X/E$, there exists $B \in X/E$ such that $(A\alpha)\varphi \subseteq B\beta$. By Theorem 2.8, we observe that $X\beta = X\gamma$. For each $x\alpha \in X\alpha$, define $\varphi : X\alpha \to X\gamma$ by $(x\alpha)\varphi = x\gamma$. If $x\alpha = y\alpha$, then $(x, y) \in \ker \alpha$ and so $(x\alpha)\varphi = x\gamma = y\gamma = (y\alpha)\varphi$ since $\ker \alpha \subseteq \ker \gamma$. Hence φ is well-defined. Similarly, since $\ker \gamma \subseteq \ker \alpha$, we can show that φ is an injection. Since $x\gamma = (x\alpha)\varphi$ for all $x \in X$, φ is a surjection. Since $R\alpha = R = R\gamma$, $R\varphi = (R\alpha)\varphi = R\gamma = R$. Hence (*i*) holds. For each $A \in X/E$, by Theorem 2.8, there exists $B \in X/E$ such that $(A\alpha)\varphi = A\gamma \subseteq B\beta$. Therefore, (*ii*) holds.

Conversely, assume that $\varphi : X\alpha \to X\beta$ is a bijection satisfying (*i*) and (*ii*). Define $\gamma : X \to X$ by $x\gamma = (x\alpha)\varphi$ for all $x \in X$. By (*i*), we deduce that $R\gamma = (R\alpha)\varphi = R\varphi = R$. Let $A \in X/E$. From (*ii*), we have $(A\alpha)\varphi \subseteq B\beta$ for some $B \in X/E$. Since $\beta \in T_E(X)$, there exists $C \in X/E$ such that $B\beta \subseteq C$ by Lemma 2.1. It follows that $A\gamma = (A\alpha)\varphi \subseteq B\beta \subseteq C$. This implies that $\gamma \in T_E(X)$. Hence $\gamma \in T_E(X,R)$. If $r\gamma = s\gamma$ for some $r, s \in R$, then $(r\alpha)\varphi = (s\alpha)\varphi$. Thus $r\alpha = s\alpha$ since φ is an injection. By the regularity of α, $r = s$. Hence $\gamma \in Reg(T_E(X,R))$. Since φ is injective, for every $x, y \in X$, we have

$$x\gamma = y\gamma \Leftrightarrow (x\alpha)\varphi = (y\alpha)\varphi \Leftrightarrow x\alpha = y\alpha.$$

This shows that $\ker \alpha = \ker \gamma$. Since φ is surjective, $X\gamma = (X\alpha)\varphi = X\beta$. It follows that $(\alpha, \gamma) \in \mathcal{R}$ and $(\gamma, \beta) \in \mathcal{L}$, by Theorems 2.5 and 2.8, respectively. Hence $(\alpha, \beta) \in \mathcal{D}$. □

Finally, we characterize Green's relation $\mathcal{J}$ for regular elements of $T_E(X,R)$.

Lemma 2.11. *Let* $\alpha, \beta \in Reg(T_E(X,R))$. *Then* $\alpha = \lambda\beta\mu$ *for some* $\lambda, \mu \in Reg(T_E(X,R))$ *if and only if there is a mapping* $\varphi : X\beta \to X\alpha$ *satisfying*

(*i*) $\varphi|_R : R \to R$ *is a bijection,*

(*ii*) *for every* $x, y \in X\beta$, $(x, y) \in E$ *implies that* $(x\varphi, y\varphi) \in E$ *and*

(*iii*) *for every* $A \in X/E$, *there exists* $B \in X/E$ *such that* $A\alpha \subseteq (B\beta)\varphi$.

Proof. Assume that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in Reg(T_E(X,R))$. For each $x \in X$, we let $r_x \in R$ with $(x, r_x) \in E$. Define $\varphi : X\beta \to X\alpha$ by

$$x\varphi = \begin{cases} x\mu & \text{if } x \in X\lambda\beta, \\ r_x\mu & \text{otherwise.} \end{cases}$$

Let $x \in X\beta$. If $x \in X\lambda\beta$, then $x = x'\lambda\beta$ for some $x' \in X$. By assumption, we have $x\varphi = x\mu = x'\lambda\beta\mu = x'\alpha \in X\alpha$. If $x \notin X\lambda\beta$, then $r_x \in X\lambda\beta$ and so $r_x = s\lambda\beta$ for some $s \in R$. By assumption, we obtain that $x\varphi = r_x\mu = s\lambda\beta\mu = s\alpha \in X\alpha$. This shows that φ is well-defined. If $r, s \in R$ such that $r\varphi = s\varphi$, then $r\mu = s\mu$, since $R = R\lambda\beta$. Thus $r'\alpha = s'\alpha$ such that $r = r'\lambda\beta$ and $s = s'\lambda\beta$ where $r', s' \in R$. By the regularity of α, $r' = s'$, and so $r = s$. Since $R\mu = R$, $R\varphi = R\mu = R$. Therefore, $\varphi|_R : R \to R$ is a bijection. Let $x, y \in X\beta$ be such that $(x, y) \in E$. Then $r_x = r_y$ and $x, y \in E_{r_x}$. By Lemma 2.1, there is $A \in X/E$ such that $x\mu, y\mu, r_x\mu \in E_{r_x}\mu \subseteq A$. This implies that $x\varphi, y\varphi \in A$ and hence $(x\varphi, y\varphi) \in E$. Thus (ii) holds. Finally, let $A \in X/E$. By Lemma 2.1, there exists $B \in X/E$ such that $A\lambda \subseteq B$. By assumption and the definition of φ, we then have $A\alpha = A\lambda\beta\mu \subseteq (B\beta \cap X\lambda\beta)\mu = (B\beta \cap X\lambda\beta)\varphi \subseteq (B\beta)\varphi$. Hence (iii) holds.

Conversely, assume that $\varphi : X\beta \to X\alpha$ is a mapping satisfying the conditions (i), (ii) and (iii). Let $r \in R$. To show that $(E_r \cap X\beta)\varphi \subseteq E_{r\varphi}$, let $x \in E_r \cap X\beta$. Then $(x, r) \in E$ and $x, r \in X\beta$. By (ii), $(x\varphi, r\varphi) \in E$. Define $\mu_r : E_r \to E_{r\varphi}$ by

$$x\mu_r = \begin{cases} x\varphi & \text{if } x \in X\beta, \\ r\varphi & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by $\mu|_{E_r} = \mu_r$ for all $r \in R$. Since R is a cross-section of the partition X/E induced by E, it follows that μ is well-defined. For each $r \in R$, $E_r\mu_r \subseteq E_{r\varphi}$ for some $E_{r\varphi} \in X/E$ and by Lemma 2.1, we have $\mu \in T_E(X)$. It follows from (i) that $R\mu = R\varphi = R$. Hence $\mu \in T_E(X,R)$.

For each $r \in R$, by (iii) we choose and fix $r' \in R$ such that $E_r\alpha \subseteq (E_{r'}\beta)\varphi$. If $(r'\beta)\varphi = a\alpha$ for some $a \in X$, then since $r'\beta \in R$ and $R\varphi = R$, $a\alpha \in R$ and so $E_r\alpha \subseteq (E_{r'}\beta)\varphi \subseteq E_{a\alpha}$. Thus $r\alpha = a\alpha = (r'\beta)\varphi$ by Lemma 2.3. Let $x \in E_r$. Then we choose and fix $b_x \in E_{r'}$ (if $x = r$, we choose $b_x = r'$) such that $x\alpha = (b_x\beta)\varphi$. Define $\lambda : X \to X$ by $x\lambda = b_x$ for all $x \in X$. For each $r \in R$, we get that $E_r\lambda \subseteq E_{r'}$. By Lemma 2.1, we obtain that $\lambda \in T_E(X)$. Obviously, $R\lambda \subseteq R$. On the other hand, let $r \in R$. Then $r\beta \in R$ and so $(r\beta)\varphi = s\alpha$ for some $s \in R$. Thus $(r\beta)\varphi = s\alpha = (b_s\beta)\varphi$ where $b_s \in E_{s'}$ and $s' \in R$. Since $\varphi|_R$ is injective, $r\beta = b_s\beta$ and by the regularity of β, it follows that $r = b_s$. Hence $s\lambda = b_s = r$, which implies the equality. This proves that $\lambda \in T_E(X,R)$. Furthermore, for $x \in X$,

$$x\lambda\beta\mu = b_x\beta\mu = (b_x\beta)\varphi = x\alpha,$$

which implies that $\alpha = \lambda\beta\mu$. It follows from Lemma 2.2 that $\lambda, \mu \in Reg(T_E(X,R))$. □

By the above lemma, we have the following result immediately.

Theorem 2.12. *Let $\alpha, \beta \in Reg(T_E(X, R))$. Then $(\alpha, \beta) \in \mathcal{J}$ if and only if there exist mappings $\varphi : X\beta \to X\alpha$ and $\psi : X\alpha \to X\beta$ satisfying*

(*i*) *$\varphi|_R, \psi|_R : R \to R$ are bijections,*

(*ii*) *for every $x, y \in X\beta$, $(x, y) \in E$ implies that $(x\varphi, y\varphi) \in E$,*

(*iii*) *for every $x, y \in X\alpha$, $(x, y) \in E$ implies that $(x\psi, y\psi) \in E$ and*

(*iv*) *for every $A \in X/E$, there exist $B, C \in X/E$ such that $A\alpha \subseteq (B\beta)\varphi$ and $A\beta \subseteq (C\alpha)\psi$.*

3 Natural Partial Order

In this section, we investigate the condition under which $\alpha \leq \beta$ for two elements $\alpha, \beta \in Reg(T_E(X, R))$.

By Theorem 1.3, the natural partial order $\leq$ defined on $Reg(T_E(X, R))$ as follows: for $\alpha, \beta \in Reg(T_E(X, R))$,

$$\alpha \leq \beta \text{ if and only if } \alpha = \lambda\beta = \beta\mu \text{ for some idempotents } \lambda, \mu \in Reg(T_E(X, R)).$$

Theorem 3.1. *Let $\alpha, \beta \in Reg(T_E(X, R))$. Then $\alpha \leq \beta$ if and only if*

(*i*) $\ker \beta \subseteq \ker \alpha$,

(*ii*) *for every $x \in X$, if $x\beta \in X\alpha$, then $x\alpha = x\beta$ and*

(*iii*) *$X\alpha \subseteq X\beta$.*

Proof. Suppose that $\alpha \leq \beta$. Then there exist idempotents $\lambda, \mu \in Reg(T_E(X, R))$ such that $\alpha = \lambda\beta = \beta\mu$. It follows from Lemma 2.4 that $\ker \beta \subseteq \ker \alpha$. Thus (*i*) holds. Let $x \in X$ be such that $x\beta \in X\alpha$. Then $x\beta = y\alpha$ for some $y \in X$ and thus $x\alpha = x\beta\mu = y\alpha\mu = y\beta\mu\mu = y\beta\mu = y\alpha = x\beta$. Hence (*ii*) holds. From Lemma 2.7, we then have (*iii*) holds.

Conversely, we assume the conditions (*i*), (*ii*) and (*iii*) hold. We will construct idempotents $\lambda, \mu \in Reg(T_E(X, R))$ such that $\alpha = \lambda\beta = \beta\mu$. Define $\mu \in Reg(T_E(X, R))$ as in the proof of Lemma 2.4. Then $\alpha = \beta\mu$. It remains to show that μ is idempotent. Let $x \in X$. By the definition of μ and (*iii*), $x\mu \in X\mu \subseteq X\alpha \subseteq X\beta$, and hence $x\mu\mu = a_{x\mu}\alpha$ where $a_{x\mu} \in X$ with $a_{x\mu}\beta = x\mu$. Since $a_{x\mu}\beta = x\mu \in X\mu \subseteq X\alpha$ and (*ii*), we deduce that $x\mu = a_{x\mu}\beta = a_{x\mu}\alpha = x\mu\mu$. This shows that μ is idempotent.

Next, we find an idempotent $\lambda \in Reg(T_E(X, R))$ with $\alpha = \lambda\beta$. For each $x \in X$, if $x\beta \in X\alpha$, then by (*ii*), $x\alpha = x\beta$. Thus we let $x' = x$. Otherwise, we choose and fix $x' \in X$ such that $x\alpha = x'\beta$ by (*iii*). Define $\lambda : X \to X$ by

$$x\lambda = x' \text{ for all } x \in X.$$

Let $(x, y) \in E$. Then $(x'\beta, y'\beta) = (x\alpha, y\alpha) \in E$ for some $x', y' \in X$. It follows from Lemma 2.6 that $(x\lambda, y\lambda) = (x', y') \in E$. Consequently, $\lambda \in T_E(X)$. Since

$R\beta = R = R\alpha \subseteq X\alpha$, $R\lambda = R$. Therefore $\lambda \in T_E(X,R)$. Let $x \in X$. We then have $x\lambda\beta = x'\beta = x\alpha$. Hence $\alpha = \lambda\beta$. It remains to show that λ is idempotent. Let $x \in X$. Then $(x\lambda)\beta = x\lambda\beta = x\alpha \in X\alpha$. It follows that $(x\lambda)' = x\lambda$ and hence $x\lambda\lambda = (x\lambda)' = x\lambda$. Thus $\lambda^2 = \lambda$. Further $\lambda \in Reg(T_E(X,R))$.

From the discussion above, $\alpha \leq \beta$ as required. □

As an immediate consequence of Theorem 3.1, we have the following results.

Corollary 3.2. *Let $\alpha, \beta \in Reg(T_E(X,R))$ and $\alpha \leq \beta$. Then the following statements hold:*

(i) If $X\alpha = X\beta$, then $\alpha = \beta$.

(ii) For every $P \in \pi(\alpha)$, there exists $P' \in \pi(\beta)$ such that $P' \subseteq P$ and $P\alpha = P'\beta$.

(iii) If $\pi(\alpha) = \pi(\beta)$, then $\alpha = \beta$.

(iv) For every $U \in X/E$, $U\alpha \subseteq U\beta$.

Proof. (i) It is obtained directly from Theorem 3.1 (ii).

(ii) Let $P \in \pi(\alpha)$ and $x \in P$. Then by Theorem 3.1(iii), $x\alpha = x'\beta$ for some $x' \in X$. Let $P' = (x'\beta)\beta^{-1}$. Then $P' \in \pi(\beta)$ and $P\alpha = P'\beta$. If $y \in P'$, then $x\alpha = x'\beta = y\beta$. By Theorem 3.1 (ii), we have $y\alpha = y\beta = x\alpha$ and so $y \in (x\alpha)\alpha^{-1} = P$. Hence $P' \subseteq P$.

(iii) It is an immediate consequence of (ii).

(iv) Let $U \in X/E$. By Lemma 2.6, there exists $A \in X/E$ such that $U = A\alpha^{-1}$. By Lemma 2.7, there exists $V \in X/E$ such that $U\alpha \subseteq V\beta$. Let $x \in U$. Then $x\alpha = y\beta$ for some $y \in V$. It follows from Theorem 3.1 (ii) that $y\alpha = y\beta = x\alpha \in U\alpha \subseteq A$. Therefore $y \in A\alpha^{-1} = U$, which implies that $U \cap V \neq \emptyset$, so $U = V$. Hence $U\alpha \subseteq U\beta$. □

Let ρ be a partial order on a semigroup S. An element $c \in S$ is said to be *right compatible* with ρ if $(ac, bc) \in \rho$ for all $(a,b) \in \rho$. *Left compatibility* with ρ is defined dually.

Corollary 3.3. *Let $\gamma \in Reg(T_E(X,R))$. If γ is an injection, then γ is right compatible with $\leq$ on $Reg(T_E(X,R))$.*

Proof. Assume that γ is injective. Let $\alpha, \beta \in Reg(T_E(X,R))$ be such that $\alpha \leq \beta$. Let $x, y \in X$ be such that $x\beta\gamma = y\beta\gamma$. Then $x\beta = y\beta$ because γ is an injection. Since $\ker\beta \subseteq \ker\alpha$, it follows that $x\alpha = y\alpha$. Thus $x\alpha\gamma = y\alpha\gamma$. This shows that $\ker\beta\gamma \subseteq \ker\alpha\gamma$. Let $x \in X$ be such that $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\alpha\gamma$ for some $y \in X$. By assumption, $x\beta = y\alpha \in X\alpha$. From Theorem 3.1 (ii), we get $x\beta = x\alpha$ and hence $x\beta\gamma = x\alpha\gamma$. Since $X\alpha \subseteq X\beta$, $X\alpha\gamma \subseteq X\beta\gamma$. The desired result then follows from Theorem 3.1. Therefore, γ is right compatible. □

Corollary 3.4. *Let $\gamma \in Reg(T_E(X,R))$. If γ is a surjection, then γ is left compatible with $\leq$ on $Reg(T_E(X,R))$.*

Proof. Suppose that γ is a surjection. Let $\alpha, \beta \in Reg(T_E(X,R))$ be such that $\alpha \leq \beta$. We need to show that $\gamma\alpha \leq \gamma\beta$. Since $\ker\beta \subseteq \ker\alpha$, it follows that $\ker\gamma\beta \subseteq \ker\gamma\alpha$. Let $x\gamma\beta \in X\gamma\alpha$. Then $x\gamma\beta \in X\alpha$. Since $\alpha \leq \beta$, $x\gamma\beta = x\gamma\alpha$ by Theorem 3.1 (ii). Since $X\gamma = X$ and $X\alpha \subseteq X\beta$, $X\gamma\alpha \subseteq X\gamma\beta$. The desired result follows from Theorem 3.1. Therefore, γ is left compatible. □

For $\alpha \in T_E(X)$, we let

$$E(\alpha) = \{A\alpha^{-1} : A \in X/E \text{ and } A\alpha^{-1} \neq \emptyset\}.$$

Then $E(\alpha)$ is also a partition of X. From [9] and [11], for $\alpha \in T_E(X)$, $A \in E(\alpha)$ is *saturated* if $A\alpha \in X/E$, that is, $A\alpha = B$ for some $B \in X/E$.

Lemma 3.5. *For every $\alpha \in Reg(T_E(X,R))$, $X/E = E(\alpha)$.*

Proof. Let $A \in X/E$. Then $A = E_r$ for some $r \in R$. Then by Lemma 2.1, $E_r\alpha \subseteq E_{r'}$ for some $r' \in R$. Thus $E_r \subseteq E_{r'}\alpha^{-1} \in E(\alpha)$. For each $x \in E_{r'}\alpha^{-1}$, we have $(r\alpha, x\alpha) \in E$. It follows from Lemma 2.6 that $(r,x) \in E$. Hence $E_r = E_{r'}\alpha^{-1}$. Consequently, $A \in E(\alpha)$. For the reverse inclusion, let $A \in E(\alpha)$. Then $A = E_r\alpha^{-1}$ for some $r \in R$. Thus $A\alpha \subseteq E_r$ and hence $r = r'\alpha$ for some $r' \in R$. This implies that $E_{r'} \subseteq A$. And for each $a \in A$, we have $(a\alpha, r'\alpha) \in E$. By Lemma 2.6, $(a, r') \in E$. Therefore, $A = E_{r'} \in X/E$ and hence $A \in X/E$. Thus $X/E = E(\alpha)$ as required. □

For each $\alpha \in Reg(T_E(X,R))$, by Lemma 3.5, $A \in X/E$ is said to be *saturated* of α if $A\alpha \in X/E$.

The following results prove useful in characterizing a maximal element in $Reg(T_E(X,R))$.

Lemma 3.6. *Let $\alpha \in Reg(T_E(X,R))$. If $U \in X/E$ is non-saturated of α such that $\alpha|_U$ is not an injection, then α is not maximal.*

Proof. Let $U \in X/E$ be non-saturated of α such that $\alpha|_U$ is not an injection. Then $U\alpha \subset A$ for some $A \in X/E$. Let $a \in A \setminus U\alpha$. Then $a \notin R$. Since $\alpha|_U$ is not injective, there are distinct elements $u_1, u_2 \in U$ such that $u_1\alpha = u_2\alpha$. Since $|A \cap R| = 1$, we assume $u_1 \notin R$. Let $\beta \in T(X)$ be defined by

$$x\beta = \begin{cases} a & \text{if } x = u_1, \\ x\alpha & \text{otherwise.} \end{cases}$$

Since $(a, u_2\alpha) \in E$ and $\alpha \in T_E(X)$, we deduce that $\beta \in T_E(X)$. Since $\alpha|_R : R \to R$ is a bijection, $\beta|_R$ is also. Since $R\beta = R$ and by Theorem 1.1, $\beta \in Reg(T_E(X,R))$. Claim that $\alpha \leq \beta$. Let $x\beta = y\beta$. Then $x\beta = a$ or $x\beta = x\alpha$. If $y\beta = x\beta = a$, then $x = u_1 = y$ and hence $x\alpha = y\alpha$. If $y\beta = x\beta = x\alpha \neq a$, then $y \neq u_1$, so $x\alpha = y\beta = y\alpha$. Consequently, $\ker\beta \subseteq \ker\alpha$. If $x\beta \in X\alpha$, then $x\beta \neq a$, so that $x\beta = x\alpha$. Moreover, $X\alpha \subseteq X\alpha \cup \{a\} = X\beta$. By virtue of Theorem 3.1, $\alpha \leq \beta$. Since $\alpha \neq \beta$, it follows that α is not maximal in $Reg(T_E(X,R))$. □

Lemma 3.7. *Let $\alpha, \beta \in Reg(T_E(X,R))$ be such that $\alpha \leq \beta$ and $U \in X/E$. Then the following statements hold:*

(*i*) *If U is saturated of α, then $U\alpha = U\beta$.*

(*ii*) *If $\alpha|_U$ is an injection, then $U\alpha = U\beta$.*

Proof. (*i*) Assume that U is saturated of α. Then there exists $A \in X/E$ such that $U\alpha = A$. Since $\alpha \leq \beta$ by Corollary 3.2 (*iv*), it follows that $A = U\alpha \subseteq U\beta$. Since X/E is a partition of X and by Lemma 2.1, we deduce that $U\beta \subseteq A$. Hence $U\alpha = U\beta$.

(*ii*) Suppose that $\alpha|_U$ is an injection. By Lemma 2.6, there exists $A \in X/E$ such that $U = A\alpha^{-1}$. By Corollary 3.2 (*iv*), we get that $U\alpha \subseteq U\beta$. Also, $U\beta \subseteq A$. Now we show $U\alpha = U\beta$. Indeed, if there is some $y \in U\beta \setminus U\alpha$, then $y \in A \setminus U\alpha$. Let $x \in U$ with $y = x\beta$. Then $x \neq r \in U \cap R$ and $x\alpha \neq x\beta$. Since $\alpha|_U$ is injective, we have $x\alpha \neq r\alpha = r\beta$ and $x\alpha \neq z\beta$ for any $z \in U \setminus \{x, r\}$. So $x\alpha \in U\alpha \setminus U\beta$, which implies that $U\alpha \nsubseteq U\beta$, a contradiction. Hence $U\alpha = U\beta$. □

From Lemmas 3.6 and 3.7, we characterize a maximal element in $Reg(T_E(X,R))$ as follows.

Theorem 3.8. *Let $\alpha \in Reg(T_E(X,R))$. If α is a surjection, then α is maximal.*

Proof. Suppose that α is a surjection. Let $\beta \in Reg(T_E(X,R))$ be such that $\alpha \leq \beta$. By Theorem 3.1, $X\alpha \subseteq X\beta$. It follows from assumption that $X\alpha = X\beta$. Hence by Corollary 3.2 (*i*), we conclude that $\alpha = \beta$. Consequently, α is a maximal element of $Reg(T_E(X,R))$. □

The converse of Theorem 3.8 is not necessarily true. We now show that there exists a maximal element of $Reg(T_E(X,R))$ which is not surjective.

Example 3.9. Let $X = \{1,2,\ldots,8\}$, $X/E = \{\{1,2\},\{3,4,5\},\{6,7,8\}\}$ and $R = \{1,3,6\}$. Define $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 1 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Then $\alpha \in T_E(X,R)$ and α is not surjective. By Theorem 1.2, α is regular. Let $\beta \in Reg(T_E(X,R))$ be such that $\alpha \leq \beta$. Then by Theorem 3.1, we have $X\alpha \subseteq X\beta$. Thus $X\beta = X\alpha$ or $X\beta = X$. Suppose that $X\beta = X$. Then $x_1\beta = 6, x_2\beta = 7$ and $x_3\beta = 8$ for some $x_1, x_2, x_3 \in X$. By Lemma 2.6, $(x_1, x_2), (x_1, x_3) \in E$. Since $x_1\beta, x_3\beta \in X\alpha$, by Theorem 3.1 (*ii*), we deduce that $x_1\alpha = x_1\beta = 6$ and $x_3\alpha = x_3\beta = 8$, which implies that $x_1 = 1$ and $x_3 = 2$. It follows that $x_2 \in \{1,2\} = \{x_1, x_3\}$. This is a contradiction with β is a mapping. Hence $X\beta = X\alpha$. We can conclude that $\alpha = \beta$ by Corollary 3.2 (*i*). Consequently, α is a maximal element.

Theorem 3.10. *Let $\alpha \in Reg(T_E(X,R))$. Suppose that α is not surjective. Then α is maximal if and only if for every non-saturated A of α, $\alpha|_A$ is an injection.*

Proof. Suppose that α is maximal and not surjective. Let $A \in X/E$ be non-saturated of α. By Lemma 3.6, $\alpha|_A$ must be an injection.

Conversely, suppose that each non-saturated A of α, $\alpha|_A$ is an injection. Let $\beta \in Reg(T_E(X,R))$ be such that $\alpha \leq \beta$. Then by Theorem 3.1, we obtain that $X\alpha \subseteq X\beta$. On the other hand, let $A \in X/E$. If A is saturated of α, then by Lemma 3.7 (i), we deduce that $A\alpha = A\beta$. If A is non-saturated of α, then $\alpha|_A$ is an injection. By Lemma 3.7 (ii), we obtain that $A\alpha = A\beta$. Hence $X\beta \subseteq X\alpha$ by Lemma 2.7. Therefore $X\alpha = X\beta$. From Corollary 3.2 (i), it follows that $\alpha = \beta$ and thus α is a maximal element of $Reg(T_E(X,R))$. □

Next, we characterize minimal elements of $Reg(T_E(X,R))$.

Theorem 3.11. *Let $\alpha \in Reg(T_E(X,R))$. Then α is minimal if and only if $X\alpha = R$.*

Proof. Assume that α is minimal. For each $x \in X$, let $r_x \in R$ be such that $(x, r_x) \in E$. Define $\beta : X \to X$ by $x\beta = r_x\alpha$ for all $x \in X$. Then $\beta \in Reg(T_E(X,R))$ and $X\beta = R$. If $a\alpha = b\alpha$, then by Lemma 2.6, we have $(a,b) \in E$. Thus $r_a = r_b$ and so $a\beta = r_a\alpha = r_b\alpha = b\beta$. Hence $\ker\alpha \subseteq \ker\beta$. Let $x \in X$ be such that $x\alpha \in X\beta$. Then $x\alpha = x'\beta = r_{x'}\alpha$ for some $x' \in X$ and by Lemma 2.6, $(x, r_{x'}) \in E$, whence $r_x = r_{x'}$. Therefore $x\alpha = x'\beta = r_{x'}\alpha = r_x\alpha = x\beta$. Hence (ii) in Theorem 3.1 holds. Obviously, $X\beta = R = R\alpha \subseteq X\alpha$. It follows from Theorem 3.1 that $\beta \leq \alpha$. By assumption, $\alpha = \beta$. Hence $X\alpha = X\beta = R$.

Conversely, suppose that $X\alpha = R$. Let $\beta \in Reg(T_E(X,R))$ be such that $\beta \leq \alpha$. By Theorem 3.1, it follows that $X\alpha = R = R\beta \subseteq X\beta \subseteq X\alpha$. Thus $X\alpha = X\beta$. It follows from Corollary 3.2 (i) that $\alpha = \beta$. Hence α is minimal. □

Let $\leq$ be a partial order on a semigroup S. An element $b \in S$ is called an *upper cover* for $a \in S$ if $a < b$ and there exists no $c \in S$ such that $a < c < b$. A *lower cover* is defined dually.

Finally, the following results are concerned with the existence of an upper cover and a lower cover for elements of $Reg(T_E(X,R))$.

Theorem 3.12. *Let $\alpha \in Reg(T_E(X,R))$. If α is not maximal, then α has an upper cover.*

Proof. Suppose that α is not maximal. So α is not surjective. Let $a \in X \setminus X\alpha$. Then there exists $A \in X/E$ such that $a \in A$. Let $U = A\alpha^{-1}$. Then $U\alpha \subseteq A$. By Lemma 3.5, $U \in X/E$. Since $a \notin X\alpha$, $U\alpha \neq A$. Therefore U is non-saturated. By Theorem 3.10, we have that $\alpha|_U$ is not an injection. Let β be defined as in the proof of Lemma 3.6, and we will show that β is an upper cover of α. Suppose that $\alpha < \gamma \leq \beta$ for some $\gamma \in Reg(T_E(X,R))$. Then by Theorem 3.1, we obtain that $X\alpha \subset X\gamma \subseteq X\beta = X\alpha \cup \{a\}$ and thus $X\gamma = X\beta$. It follows from Corollary 3.2 (i) that $\gamma = \beta$. Hence β is an upper cover of α. □

Theorem 3.13. *Let $\alpha \in Reg(T_E(X,R))$. If α is not minimal, then α has a lower cover.*

Proof. If α is not minimal, then $X\alpha \neq R$. Let $y \in X\alpha \setminus R$. Then $y \in E_r$ for some $r \in R$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} r & \text{if } x \in y\alpha^{-1}, \\ x\alpha & \text{otherwise.} \end{cases}$$

Then β is well-defined and $\alpha \neq \beta$. Let $r' \in R$ be such that $r'\alpha = r$. For each $x \in y\alpha^{-1}$, $(x\alpha, r'\alpha) = (y, r) \in E$. By Lemma 2.6, $(x, r') \in E$. Thus $y\alpha^{-1} \subseteq E_{r'}$. If $x \in E_{r'} \setminus y\alpha^{-1}$, then $(x\beta, r'\beta) = (x\alpha, r'\alpha) = (x\alpha, r) \in E$. Hence $E_{r'}\beta \subseteq E_r$. For each $s \in R \setminus \{r'\}$, by Lemma 2.1, there exists $s' \in R$ such that $E_s\beta = E_s\alpha \subseteq E_{s'}$. From Lemma 2.1, we obtain that $\beta \in T_E(X)$. Since $R \subseteq X \setminus y\alpha^{-1}$, we get $R\beta = R\alpha = R$ and hence $\beta \in T_E(X, R)$. Since α is regular, β is also.

Now, we will show that $\beta \leq \alpha$ by using Theorem 3.1. Let $a, b \in X$ be such that $a\alpha = b\alpha$. If $a\alpha = y$, then $a\beta = r = b\beta$. Otherwise, $a\beta = a\alpha = b\alpha = b\beta$. Hence $\ker\alpha \subseteq \ker\beta$. Suppose that $x\alpha \in X\beta$ where $x \in X$. Since $X\beta = X\alpha \setminus \{y\}, x\alpha \neq y$ and hence $x \notin y\alpha^{-1}$. Therefore, $x\beta = x\alpha$. Obviously, $X\beta = X\alpha \setminus \{y\} \subseteq X\alpha$. Hence $\beta \leq \alpha$.

Finally, to show that β is a lower cover for α, let $\gamma \in Reg(T_E(X, R))$ be such that $\beta \leq \gamma < \alpha$. Then by Theorem 3.1, $X\alpha \setminus \{y\} = X\beta \subseteq X\gamma \subset X\alpha$, which implies $X\beta = X\gamma$. By Corollary 3.2 (i), we conclude that $\beta = \gamma$. Consequently, α has a lower cover. □

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