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The Theory of the Feasibility Problems and Fixed Point Problems of Nonlinear Mappings

Sirawit Premjitpraphan and Atid Kangtunyakarn[1]

Department of Mathematics, Faculty of Science
King Mongkut's Institute of Technology Ladkrabang
Bangkok 10520, Thailand
e-mail : spreamjitpraphan@gmail.com (S. Premjitpraphan)
beawrock@hotmail.com (A. Kangtunyakarn)

Abstract : In this paper, the authors extend and improve some results of Hamdi [Hamdi, A., Liou, Y.C., Yao, Y. and Luo, C.: The common solutions of the split feasibility problems and fixed point problems. Journal of Inequalities and Applications (2015) 2015:385 DOI10.1186/s13660-015-0870-6] by using the concept of lemma 2.11 Suwannaut and Kangtunyakarn [Suwannaut, S. and Kangtunyakarn, A.: The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem. Fixed Point Theory and its Applications (2013) 2013:291]. Then they prove strong convergence theorem of the proposed iteration under some control condition. Moreover, we use S-mapping in application with our main result.

Keywords : the split feasibility problem; fixed point problem; $\mathcal{L}$-Lipschitzian; strongly positive.

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[1]Corresponding author.

1 Introduction

The split feasibility problem has become the inspiration in pure and applied mathematics. It attracted the author's attention due to its application in signal processing. The problem was introduced by Censor and Elfving(1994)([1]).

Let C and Q be nonempty closed convex subsets of real Hilbert space H_1 and H_2, respectively.

The split feasibility problem(SFP) was formulated so as to find a point u^* satisfy the properties :

$$u^* \in C \text{ and } Au^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

The split common fixed point problem(SCFP) was formulated such that

$$u^* \in F(T) \text{ and } Au^* \in F(S), \tag{1.2}$$

where $F(T)$ and $F(S)$ are fixed point sets of the operators $T : H_1 \to H_1$ and $S : H_2 \to H_2$.

Recently, the study of the split common fixed point problem(SCFP) has become popular among mathematicians. The problem, first analysed by Censor and Segal([2]), is a natural extension of the SFP and the convex feasibility problem.

In ([3]) Hamdi, Liou, Yao and Luo proved strong convergence theorem as following algorithm : $x_0 \in H_1$ and

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n \mathcal{B})\left(x_n - \delta A^*\left(Ax_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right) \end{cases}$$

for all $n \in \mathbb{N}$,
where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, $f : C \to H_1$ is ρ-contraction, $\mathcal{B}$ is strongly positive bounded linear operator on H_1, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$, $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. They showed that the sequence $\{x_n\}$ converges strongly to the unique fixed point of the contraction mapping $P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \mathcal{B}\right)$.

The purpose of this paper was to study the following split feasibility problem and fixed point problem :

$$\text{Find } u^* \in C \cap F(T) \text{ and } Au^* \in Q \cap F(S). \tag{1.3}$$

The set of solution of (1.3) is denoted by $\mathbf{\Gamma}$, that is,

$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}.$

It is immediately evident that (1.3) can be derived from SFP(1.1) and SCFP(1.2).

In this paper, we're motivated and inspired by Hamdi, Liou, Yao and Luo ([3]), we modified the split feasibility problem and fixed point problem by Hamdi, Liou, Yao and Luo ([3]) and used the concept from Lemma 2.11. we will introduce a new iteration to approach the solution of (1.3).
The proof of the strong convergence result is given later in the paper.

2 Preliminaries

Throughout this paper, we always assume that H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Using the notations of weak and strong convergence by $'' \rightharpoonup''$ and $'' \rightarrow''$, respectively.

Recall that a mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. The set of all elements of fixed point of a mapping T is denoted by $F(T) = \{x \in C : Tx = x\}$. Goebel and Kirk ([4]) showed that $F(T)$ is closed and convex. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad \lambda \in [0,1]$$

and

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

for all $x, y \in H$.

Lemma 2.1. [5] *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Definition 2.2. An operator A is a *strongly positive bounded linear operator* on H if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Definition 2.3. An operator $A : C \to H$ is called *$\mathcal{L}$-Lipschitzian* if

$$\|Ax - Ay\| \leq \mathcal{L}\|x - y\|, \quad \forall x, y \in C$$

for some constant $\mathcal{L} > 0$. If $\mathcal{L} \in [0,1]$, then A is called $\mathcal{L}$-contraction.

Definition 2.4. An operator $A : C \to C$ is called *pseudo-contractive* if

$$\langle Ax - Ay, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

Definition 2.5. An operator $A : C \to C$ is called *quasi-pseudo-contractive* if

$$\|Ax - y\|^2 \leq \|x - y\|^2 + \|Ax - x\|^2$$

for all $x \in C$ and $y \in F(A)$.

Definition 2.6. An operator $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \qquad \forall x, y \in C.$$

It is obvious that any α-inverse strongly monotone mapping A is $\frac{1}{\alpha}-Lipschitzian$.

Definition 2.7. An operator $A : C \to C$ is called *firmly nonexpansive* if

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 - \|(\mathcal{I} - A)x - (\mathcal{I} - A)y\|^2, \qquad \forall x, y \in C.$$

Definition 2.8. An operator A is said to be *demiclosed* if $\forall x_n \rightharpoonup \bar{u}$ and $A(x_n) \to u$ imply that $A(\bar{u}) = u$

Lemma 2.9. [6] *Let $\{\mathcal{Q}_n\} \subset [0, +\infty], \{v_n\} \subset [0, 1]$ and $\{\eta_n\}$ be three real number sequences. Suppose that $\{\mathcal{Q}_n\}, \{v_n\}$ and $\{\eta_n\}$ satisfy the following three conditions:*

(i) $\mathcal{Q}_{n+1} \leq (1 - v_n)\mathcal{Q}_n + \eta_n v_n$,

(ii) $\sum_{n=1}^{\infty} v_n = \infty$,

(iii) $\limsup_{n\to\infty} \eta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\eta_n v_n| < \infty$.

Then, $\lim_{n\to\infty} \mathcal{Q}_n = 0$.

Lemma 2.10. [7] *Let $\{\rho_n\}$ be a sequences of real numbers. Assume that there exists a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ such that $\rho_{n_k} \leq \rho_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = max\{i \leq n : \rho_{n_i} < \rho_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and

$$max\{\rho_{\tau(n)}, \rho_n\} \leq \rho_{\tau(n)+1},$$

for all $n \geq N_0$.

Lemma 2.11. [8] *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, ..., N$, let A_i be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\gamma_i > 0$ and $\bar{\gamma} = min_{i=1,2,...,N}\gamma_i$. Let $\{a_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N a_i = 1$. Then the following properties hold:*

(i) $\left\|\mathcal{I}-\rho\sum_{i=1}^{N}a_iA_i\right\|\leq 1-\rho\bar{\gamma}$ *and* $\mathcal{I}-\rho\sum_{i=1}^{N}a_iA_i$ *is a nonexpansive mapping for every* $0<\rho<\|A_i\|^{-1}$ *for* $i=1,2,...,N$.

(ii) $VI(C,\sum_{i=1}^{N}a_iA_i)=\bigcap_{i=1}^{N}VI(C,A_i)$.

Proposition 2.12. [9] *Let H be a real Hilbert space. Let $\mathcal{U}:H\to H$ be an $\mathcal{L}$-Lipschitzian operator with $\mathcal{L}>1$. Then*

$$F\left(((1-\zeta)\mathcal{I}+\zeta\mathcal{U})\mathcal{U}\right)=F\left(\mathcal{U}((1-\zeta)\mathcal{I}+\zeta\mathcal{U})\right)=F(\mathcal{U})$$

for all $\zeta\in(0,\frac{1}{\mathcal{L}})$.

Proposition 2.13. [9] *Let H be a real Hilbert space. Let $\mathcal{U}:H\to H$ be an $\mathcal{L}$-Lipschitzian quasi-pseudo-contractive operator. Then we have*

$$\|\mathcal{U}\left((1-\eta)x+\eta\mathcal{U}x\right)-u^*\|^2\leq\|x-u^*\|^2+(1-\eta)\left\|x-\mathcal{U}\left((1-\eta)x+\eta\mathcal{U}x\right)\right\|^2,$$

and the operator $(1-\xi)\mathcal{I}+\xi\mathcal{U}\left((1-\eta)\mathcal{I}+\eta\mathcal{U}\right)$ is quasi-nonexpansive when $0<\xi<\eta<\frac{1}{\sqrt{1+\mathcal{L}^2}+1}$, that is,

$$\|(1-\xi)x+\xi\mathcal{U}\left((1-\eta)x+\eta\mathcal{U}x\right)-u^*\|\leq\|x-u^*\|$$

for all $x\in H$ and $u^*\in F(\mathcal{U})$.

Proposition 2.14. [9] *Let H be a real Hilbert space. Let $\mathcal{U}:H\to H$ be an $\mathcal{L}$-Lipschitzian operator with $\mathcal{L}>1$. If $\mathcal{I}-\mathcal{U}$ is demiclosed at 0, then $\mathcal{I}-\mathcal{U}((1-\zeta)\mathcal{I}+\zeta\mathcal{U})$ is also demiclosed at 0 when $\zeta\in(0,\frac{1}{\mathcal{L}})$.*

3 Main Results

Theorem 3.1. *Let H_1 and H_2 are two real Hilbert space, let $C\subseteq H_1$ and $Q\subseteq H_2$ are two nonempty closed convex sets. Let $A:H_1\to H_2$ is a bounded linear operator with its adjoint A^*, D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i>0$ and $\bar{\gamma}=\min\limits_{i=1,2,...,N}\gamma_i$, $f:C\to H_1$ is a ρ-contraction, $S:Q\to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1>1$, $T:C\to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2>1$. Assume that $\mathbf{\Gamma}\neq\emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0\in H_1$*

$$\begin{cases} z_n=P_QAx_n,\\ v_n=(1-\xi_n)\,z_n+\xi_nS\left((1-\eta_n)\,z_n+\eta_nSz_n\right),\\ y_n=\alpha_n\gamma f(x_n)+\left(\mathcal{I}-\alpha_n\sum_{i=1}^{N}a_iD_i\right)\left(x_n-\delta A^*\left(Ax_n-v_n\right)\right),\\ u_n=P_Cy_n,\\ x_{n+1}=(1-\beta_n)u_n+\beta_nT\left((1-\gamma_n)\,u_n+\gamma_nTu_n\right), \quad for\ n\geq 1, \end{cases} \tag{3.1}$$

The parameters $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.

We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,
$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.
Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1+\mathcal{L}_1^2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1+\mathcal{L}_2^2}+1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z$.

Proof. Let $z^* = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right) z^*$, we have $z^* \in C \cap F(T)$ and $Az^* \in Q \cap F(S)$. From P_Q is firmly nonexpansive, thus

$$\begin{aligned}\|z_n - Az^*\|^2 &= \|P_Q Ax_n - P_Q Az^*\|^2\\ &\leq \|Ax_n - Az^*\|^2 - \|(\mathcal{I} - P_Q)Ax_n - (\mathcal{I} - P_Q)Az^*\|^2\\ &= \|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 . \end{aligned} \tag{3.2}$$

Applying Proposition 2.12, condition (ii) and (iii), we have

$$F\left(S((1-\eta_n)\mathcal{I} + \eta_n S)\right) = F(S)$$

and

$$F\left(T((1-\gamma_n)\mathcal{I} + \gamma_n T)\right) = F(T)$$

for all $n \in \mathbb{N}$.
By Proposition 2.13 and condition (ii), we have

$$\begin{aligned}\|v_n - Az^*\| &= \|[(1-\xi_n)\mathcal{I} + \xi_n S\left((1-\eta_n)\mathcal{I} + \eta_n S\right)] z_n - Az^*\|\\ &\leq \|z_n - Az^*\| . \end{aligned} \tag{3.3}$$

This together with (3.2), it implies that

$$\begin{aligned}\|v_n - Az^*\|^2 &\leq \|z_n - Az^*\|^2\\ &\leq \|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 \end{aligned} \tag{3.4}$$

By Proposition 2.13 and condition (iii), we have

$$\begin{aligned}\|x_{n+1} - z^*\| &= \|[(1-\beta_n)\mathcal{I} + \beta_n T((1-\gamma_n)\mathcal{I} + \gamma_n T)]u_n - z^*\| \\ &\le \|u_n - z^*\|. \end{aligned} \tag{3.5}$$

Since P_C is nonexpansive, we have

$$\begin{aligned}\|u_n - z^*\| &= \|P_C y_n - P_C z^*\| \\ &\le \|y_n - z^*\|. \end{aligned} \tag{3.6}$$

From definition of $\{y_n\}$, we obtain

$$\begin{aligned}\|y_n - z^*\| &= \left\|\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n)) - z^*\right\| \\ &= \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(z^*) + \alpha_n \gamma f(z^*) - \alpha_n \sum_{i=1}^N a_i D_i z^* + x_n - \delta A^*(Ax_n - v_n) \\ &\quad - \alpha_n \sum_{i=1}^N a_i D_i (x_n - \delta A^*(Ax_n - v_n)) + \alpha_n \sum_{i=1}^N a_i D_i z^* - z^*\| \\ &= \|\alpha_n \gamma (f(x_n) - f(z^*)) + \alpha_n \left(\gamma f(z^*) - \sum_{i=1}^N a_i D_i z^*\right) \\ &\quad + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right)(x_n - z^* - \delta A^*(Ax_n - v_n))\| \\ &\le \alpha_n \gamma \|f(x_n) - f(z^*)\| + \alpha_n \left\|\gamma f(z^*) - \sum_{i=1}^N a_i D_i z^*\right\| \\ &\quad + \left\|\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right\| \|x_n - z^* + \delta A^*(v_n - Ax_n)\| \\ &\le \alpha_n \gamma \rho \|x_n - z^*\| + \alpha_n \left\|\gamma f(z^*) - \sum_{i=1}^N a_i D_i z^*\right\| \\ &\quad + (1 - \alpha_n \bar{\gamma}) \|x_n - z^* + \delta A^*(v_n - Ax_n)\|. \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned} &\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\ =&\langle Ax_n - Az^*, v_n - Ax_n\rangle \\ =&\langle Ax_n - Az^* + v_n - Ax_n - (v_n - Ax_n), v_n - Ax_n\rangle \\ =&\langle Ax_n - Az^* + v_n - Ax_n, v_n - Ax_n\rangle - \langle v_n - Ax_n, v_n - Ax_n\rangle \\ =&\langle v_n - Az^*, v_n - Ax_n\rangle - \|v_n - Ax_n\|^2. \end{aligned} \tag{3.8}$$

and

$$\langle v_n - Az^*, v_n - Ax_n\rangle = \frac{1}{2}\left(\|v_n - Az^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right). \quad (3.9)$$

From (3.4), (3.8) and (3.9), we obtain

$$\begin{aligned}
&\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
=&\frac{1}{2}\left(\|v_n - Az^*\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right) - \|v_n - Ax_n\|^2 \\
\leq&\frac{1}{2}\left(\|Ax_n - Az^*\|^2 - \|Ax_n - z_n\|^2 + \|v_n - Ax_n\|^2 - \|Ax_n - Az^*\|^2\right) - \|v_n - Ax_n\|^2 \\
=&-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2. \quad (3.10)
\end{aligned}$$

From (3.10), we have

$$\begin{aligned}
&\|x_n - z^* + \delta A^*(v_n - Ax_n)\|^2 \\
=&\|x_n - z^*\|^2 + \delta^2\|A^*(v_n - Ax_n)\|^2 + 2\delta\langle x_n - z^*, A^*(v_n - Ax_n)\rangle \\
\leq&\|x_n - z^*\|^2 + \delta^2\|A^*\|^2\|v_n - Ax_n\|^2 + 2\delta\left(-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|v_n - Ax_n\|^2\right) \\
=&\|x_n - z^*\|^2 + \delta^2\|A\|^2\|v_n - Ax_n\|^2 - \delta\|z_n - Ax_n\|^2 - \delta\|v_n - Ax_n\|^2 \\
=&\|x_n - z^*\|^2 + \delta\left(\delta\|A\|^2 - 1\right)\|v_n - Ax_n\|^2 - \delta\|z_n - Ax_n\|^2. \quad (3.11)
\end{aligned}$$

From (3.11) and condition (iv), we have

$$\|x_n - z^* + \delta A^*(v_n - Ax_n)\|^2 \leq \|x_n - z^*\|^2.$$

So,

$$\|x_n - z^* + \delta A^*(v_n - Ax_n)\| \leq \|x_n - z^*\|. \quad (3.12)$$

From (3.7) and (3.12), we get

$$\begin{aligned}
&\|y_n - z^*\| \\
\leq&\alpha_n\gamma\rho\|x_n - z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| + (1 - \alpha_n\bar{\gamma})\|x_n - z^* + \delta A^*(v_n - Ax_n)\| \\
\leq&\alpha_n\gamma\rho\|x_n - z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| + (1 - \alpha_n\bar{\gamma})\|x_n - z^*\| \\
=&[1 - \alpha_n(\bar{\gamma} - \gamma\rho)]\|x_n - z^*\| + \alpha_n\left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|. \quad (3.13)
\end{aligned}$$

By definition of $\{x_n\}$, (3.5), (3.6) and (3.13), we get

$$\begin{aligned}\|x_{n+1}-z^*\| &\leq [1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\| + \alpha_n\left\|\gamma f(z^*)-\sum_{i=1}^{N} a_i D_i z^*\right\| \\ &= [1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\| + \alpha_n(\bar{\gamma}-\gamma\rho)\frac{\left\|\gamma f(z^*)-\sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}-\gamma\rho}.\end{aligned}$$

By induction, we get

$$\|x_{n+1}-z^*\| \leq \max\left\{\|x_0-z^*\|, \frac{\left\|\gamma f(z^*)-\sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}-\gamma\rho}\right\}.$$

Hence, the sequence $\{x_n\}$ is bounded.
Since P_C is firmly nonexpansive, we have

$$\begin{aligned}\|u_n-z^*\|^2 &= \|P_C y_n - z^*\|^2 \\ &= \|P_C y_n - P_C z^*\|^2 \\ &\leq \|y_n-z^*\|^2 - \|(\mathcal{I}-P_C)\,y_n - (\mathcal{I}-P_C)\,z^*\|^2 \\ &= \|y_n-z^*\|^2 - \|y_n - P_C y_n\|^2 \\ &= \|y_n-z^*\|^2 - \|u_n-y_n\|^2. \qquad (3.14)\end{aligned}$$

From (3.5), (3.13) and (3.14), we have

$$\begin{aligned}&\|x_{n+1}-z^*\|^2 \\ &\leq \|u_n-z^*\|^2 \\ &\leq \|y_n-z^*\|^2 - \|u_n-y_n\|^2 \\ &= \left([1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\| + \alpha_n\left\|\gamma f(z^*)-\sum_{i=1}^{N} a_i D_i z^*\right\|\right)^2 - \|u_n-y_n\|^2 \\ &= (1-\alpha_n(\bar{\gamma}-\gamma\rho))^2\,\|x_n-z^*\|^2 \\ &\quad + 2\alpha_n\,[1-\alpha_n(\bar{\gamma}-\gamma\rho)]\,\|x_n-z^*\|\left\|\gamma f(z^*)-\sum_{i=1}^{N} a_i D_i z^*\right\| \\ &\quad + \alpha_n^2\left\|\gamma f(z^*)-\sum_{i=1}^{N} a_i D_i z^*\right\|^2 - \|u_n-y_n\|^2.\end{aligned}$$

That is,

$$\|u_n - y_n\|^2 \leq \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n^2 \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|^2$$
$$+ 2\alpha_n \left[1 - \alpha_n(\bar{\gamma} - \gamma\rho)\right] \|x_n - z^*\| \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|. \quad (3.15)$$

Next, we focus our analysis on the fact that the sequence $\{\|x_n - z^*\|\}$ is either monotone decreasing at infinity(Case 1) or not(Case 2).
<u>*Case*1.</u> There exists $n_0 \in \mathbb{N}$ such that the sequence $\{\|x_n - z^*\|\}_{n \geq n_0}$ is decreasing.
<u>*Case*2.</u> For any $\bar{n_0} \in \mathbb{N}$, there exists an integer $\bar{m} \geq \bar{n_0}$ such that

$$\|x_{\bar{m}} - z^*\| \leq \|x_{\bar{m}+1} - z^*\|.$$

In $Case1$, we assume that there exists some integer $m > 0$ such that $\{\|x_n - z^*\|\}$ is decreasing for all $n \geq m$.
In this case, we get $\lim_{n\to\infty} \|x_n - z^*\|$ exists. From (3.15) and condition (i), we deduce

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \quad (3.16)$$

From (3.7) and condition (iv), we have

$$\begin{aligned}\|y_n - z^*\| \leq & \alpha_n \gamma\rho \|x_n - z^*\| + \alpha_n \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\| \\ & + (1 - \alpha_n\bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\| \\ = & \alpha_n \bar{\gamma} \left(\frac{\gamma\rho \|x_n - z^*\| + \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}} \right) \\ & + (1 - \alpha_n\bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|. \end{aligned} \quad (3.17)$$

Since $\{x_n\}$ is bounded, then there exists a constant $M > 0$ such that

$$\sup_n \left\{ \frac{\gamma\rho \|x_n - z^*\| + \left\|\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right\|}{\bar{\gamma}} \right\} < M.$$

By using property of convex function of $\|\cdot\|^2$ and (3.17), we have

$$\|y_n - z^*\|^2 \leq \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n\bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|^2. \quad (3.18)$$

From (3.5), (3.6), (3.11) and (3.18), thus

$$\begin{aligned}
&\|x_{n+1} - z^*\|^2 \\
\leq& \|u_n - z^*\|^2 \\
\leq& \|y_n - z^*\|^2 \\
\leq& \alpha_n \bar{\gamma} M^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - z^* + \delta A^* (v_n - Ax_n)\|^2 \\
\leq& \alpha_n \bar{\gamma} M^2 + (1-\alpha_n \bar{\gamma}) \Big(\|x_n - z^*\|^2 + \delta \Big(\delta \|A\|^2 - 1 \Big) \|v_n - Ax_n\|^2 - \delta \|z_n - Ax_n\|^2 \Big) \\
=& (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\|^2 + (1 - \alpha_n \bar{\gamma}) \delta \Big(\delta \|A\|^2 - 1 \Big) \|v_n - Ax_n\|^2 \\
&- \delta (1 - \alpha_n \bar{\gamma}) \|z_n - Ax_n\|^2 + \alpha_n \bar{\gamma} M^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1 - \alpha_n \bar{\gamma}) \delta \Big(1 - \delta \|A\|^2 \Big) \|v_n - Ax_n\|^2 + \delta (1 - \alpha_n \bar{\gamma}) \|z_n - Ax_n\|^2 \\
\leq& (1 - \alpha_n \bar{\gamma}) \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n \bar{\gamma} M^2 \\
\leq& \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n \bar{\gamma} M^2.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|z_n - Ax_n\| = 0. \tag{3.19}$$

Consider that

$$\begin{aligned}
\|v_n - z_n\| &= \|v_n - Ax_n + Ax_n - z_n\| \\
&\leq \|v_n - Ax_n\| + \|z_n - Ax_n\|.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.20}$$

Note that

$$\begin{aligned}
v_n - z_n &= (1 - \xi_n) z_n + \xi_n S ((1 - \eta_n) z_n + \eta_n S z_n) - z_n \\
&= \xi_n [S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - z_n].
\end{aligned}$$

From (3.20), then

$$\lim_{n\to\infty} \|z_n - S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n\| = 0. \tag{3.21}$$

Consider that

$$\begin{aligned}
&\|S ((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - S ((1 - \eta_n) \mathcal{I} + \eta_n S) Ax_n\| \\
\leq& \mathcal{L}_1 \|((1 - \eta_n) \mathcal{I} + \eta_n S) z_n - ((1 - \eta_n) \mathcal{I} + \eta_n S) Ax_n\| \\
=& \mathcal{L}_1 \|(1 - \eta_n) (z_n - Ax_n) + \eta_n (S z_n - S Ax_n)\| \\
\leq& \mathcal{L}_1 ((1 - \eta_n) \|z_n - Ax_n\| + \eta_n \|S z_n - S Ax_n\|) \\
\leq& \mathcal{L}_1 ((1 - \eta_n) \|z_n - Ax_n\| + \eta_n \mathcal{L}_1 \|z_n - Ax_n\|) \\
=& \mathcal{L}_1 (1 - \eta_n (1 - \mathcal{L}_1)) \|z_n - Ax_n\|.
\end{aligned} \tag{3.22}$$

From (3.22), thus

$$\begin{aligned}
&\|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\| \\
\le &\|Ax_n - z_n\| + \|z_n - S((1-\eta_n)\mathcal{I} + \eta_n S)z_n\| \\
&+ \|S((1-\eta_n)\mathcal{I} + \eta_n S)z_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\| \\
\le &\|Ax_n - z_n\| + \|z_n - S((1-\eta_n)\mathcal{I} + \eta_n S)z_n\| + \mathcal{L}_1(1-\eta_n(1-\mathcal{L}_1))\|z_n - Ax_n\|.
\end{aligned} \tag{3.23}$$

From (3.19), (3.21) and (3.23), then we have

$$\lim_{n\to\infty} \|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\| = 0. \tag{3.24}$$

Since

$$\begin{aligned}
&\|Ax_n - SAx_n\| \\
= &\|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n + S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n - SAx_n\| \\
\le &\|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\| + \|S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n - SAx_n\| \\
\le &\|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\| + \mathcal{L}_1\|((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n - Ax_n\| \\
= &\|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\| + \mathcal{L}_1\eta_n\|Ax_n - SAx_n\|.
\end{aligned}$$

It implies that

$$\|Ax_n - SAx_n\| \le \frac{1}{1-\mathcal{L}_1\eta_n}\|Ax_n - S((1-\eta_n)\mathcal{I} + \eta_n S)Ax_n\|.$$

By (3.24), we obtain

$$\lim_{n\to\infty} \|Ax_n - SAx_n\| = 0. \tag{3.25}$$

Consider that

$$
\begin{aligned}
\|y_n - x_n\| &= \left\|\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n)) - x_n\right\| \\
&= \left\|\alpha_n \gamma f(x_n) - \delta A^*(Ax_n - v_n) - \alpha_n \sum_{i=1}^{N} a_i D_i x_n + \delta \alpha_n \sum_{i=1}^{N} a_i D_i A^*(Ax_n - v_n)\right\| \\
&= \left\|\alpha_n \left(\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i x_n + \delta \sum_{i=1}^{N} a_i D_i A^*(Ax_n - v_n)\right) + \delta A^*(v_n - Ax_n)\right\| \\
&= \left\|\alpha_n \left(\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right) + \delta A^*(v_n - Ax_n)\right\| \\
&\le \alpha_n \left\|\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right\| + \delta \|A^*(v_n - Ax_n)\| \\
&\le \alpha_n \left\|\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right\| + \delta \|A^*\| \|v_n - Ax_n\| \\
&= \alpha_n \left\|\gamma f(x_n) - \sum_{i=1}^{N} a_i D_i (x_n - \delta A^*(Ax_n - v_n))\right\| + \delta \|A\| \|v_n - Ax_n\|.
\end{aligned}
$$

It follows from (3.19) and condition (i) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.26}$$

From definition of $\{x_n\}$, we have

$$
\begin{aligned}
\|x_{n+1} - z^*\|^2 &= \|(1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*\|^2 \\
&= \|(1-\beta_n)(u_n - z^*) + \beta_n [T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*]\|^2 \\
&= (1-\beta_n)\|u_n - z^*\|^2 + \beta_n \|T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|T((1-\gamma_n)u_n + \gamma_n T u_n) - u_n\|^2.
\end{aligned} \tag{3.27}
$$

Applying proposition 2.13, we have

$$
\begin{aligned}
&\|T((1-\gamma_n)u_n + \gamma_n T u_n) - z^*\|^2 \\
&\le \|u_n - z^*\|^2 + (1-\gamma_n)\|u_n - T((1-\gamma_n)u_n + \gamma_n T u_n)\|^2.
\end{aligned} \tag{3.28}
$$

From (3.6),(3.12), (3.18), (3.27) and (3.28), thus

$$\begin{aligned}
\|x_{n+1}-z^*\|^2 =&(1-\beta_n)\|u_n-z^*\|^2+\beta_n\|T((1-\gamma_n)u_n+\gamma_nTu_n)-z^*\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
\leq&(1-\beta_n)\|u_n-z^*\|^2+\beta_n(\|u_n-z^*\|^2\\
&+(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2)\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
=&\|u_n-z^*\|^2+\beta_n(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
\leq&\|y_n-z^*\|^2+\beta_n(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
\leq&\alpha_n\bar{\gamma}M^2+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|^2\\
&+\beta_n(1-\gamma_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
&-\beta_n(1-\beta_n)\|T((1-\gamma_n)u_n+\gamma_nTu_n)-u_n\|^2\\
=&\alpha_n\bar{\gamma}M^2+(1-\alpha_n\bar{\gamma})\|x_n-z^*+\delta A^*(v_n-Ax_n)\|^2\\
&-\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\\
\leq&\alpha_n\bar{\gamma}M^2+\|x_n-z^*\|^2\\
&-\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2.
\end{aligned}$$

It implies that

$$\beta_n(\gamma_n-\beta_n)\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|^2\leq\alpha_n\bar{\gamma}M^2+\|x_n-z^*\|^2-\|x_{n+1}-z^*\|^2.$$

By condition (i) and (iii), we get

$$\lim_{n\to\infty}\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|=0. \tag{3.29}$$

Observe that

$$\begin{aligned}
\|u_n-Tu_n\|\leq&\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\|T((1-\gamma_n)u_n+\gamma_nTu_n)-Tu_n\|\\
\leq&\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\mathcal{L}_2\|(1-\gamma_n)u_n+\gamma_nTu_n-u_n\|\\
=&\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|+\mathcal{L}_2\gamma_n\|u_n-Tu_n\|.
\end{aligned}$$

Thus,

$$\|u_n-Tu_n\|\leq\frac{1}{1-\mathcal{L}_2\gamma_n}\|u_n-T((1-\gamma_n)u_n+\gamma_nTu_n)\|.$$

This together with (3.29) implies that,

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0. \tag{3.30}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \leq 0,$$

where $z^* = P_{\mathbf{\Gamma}}(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i)z^*$.
Choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle = \lim_{i\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{n_i} - z^*\rangle. \quad (3.31)$$

Since the sequence $\{y_n\}$ is bounded, without loss of generality, we have a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z$. Subsequently, we derive from above conclusion that

$$\begin{cases} x_{n_i} \rightharpoonup z, \\ y_{n_i} \rightharpoonup z, \\ u_{n_i} \rightharpoonup z \end{cases} \quad (3.32)$$

and

$$\begin{cases} Ax_{n_i} \rightharpoonup Az, \\ Ay_{n_i} \rightharpoonup Az, \\ Au_{n_i} \rightharpoonup Az. \end{cases} \quad (3.33)$$

Note that $u_{n_i} = P_C y_{n_i} \in C$ and (3.32), thus $z \in C$.
From demiclosedness of $(\mathcal{I} - T)$ and $(\mathcal{I} - T)u_{n_i} \to 0$, then $z \in F(T)$.
Therefore, $z \in C \cap F(T)$.
Note that $z_{n_i} = P_Q Ax_{n_i} \in Q$ and from (3.19) and (3.33), we have $z_{n_i} \rightharpoonup Az$.
Thus, $Az \in Q$.
From demiclosedness of $(\mathcal{I} - S)$ and $(\mathcal{I} - S)Ax_{n_i} \to 0$, then $Az \in F(S)$.
Therefore, $Az \in Q \cap F(S)$. That is $z \in \mathbf{\Gamma}$.
Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle &= \lim_{i\to\infty}\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{n_i} - z^*\rangle \\ &= \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, z - z^*\rangle \\ &\leq 0. \end{aligned} \quad (3.34)$$

Consider that

$$\begin{aligned}
\|y_n - z^*\|^2 \leq& \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^* - \delta A^*(Ax_n - v_n)\|^2 \\
&+ 2\langle \alpha_n \gamma (f(x_n) - f(z^*)) + \alpha_n \left(\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*\right), y_n - z^*\rangle \\
=& \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^* - \delta A^*(Ax_n - v_n)\|^2 \\
&+ 2\alpha_n\gamma\langle f(x_n) - f(z^*), y_n - z^*\rangle + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq& \left\|\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right\|^2 \|x_n - z^*\|^2 + 2\alpha_n\gamma \|f(x_n) - f(z^*)\| \|y_n - z^*\| \\
&+ 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq& (1 - \alpha_n\bar{\gamma})^2 \|x_n - z^*\|^2 + 2\alpha_n\gamma\rho \|x_n - z^*\| \|y_n - z^*\| \\
&+ 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
\leq& (1 - \alpha_n\bar{\gamma})^2 \|x_n - z^*\|^2 + \alpha_n\gamma\rho \left(\|x_n - z^*\|^2 + \|y_n - z^*\|^2\right) \\
&+ 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
=& (1 - \alpha_n\bar{\gamma})^2 \|x_n - z^*\|^2 + \alpha_n\gamma\rho \|x_n - z^*\|^2 + \alpha_n\gamma\rho \|y_n - z^*\|^2 \\
&+ 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle.
\end{aligned}$$

It follow that

$$\begin{aligned}
&(1 - \alpha_n\gamma\rho) \|y_n - z^*\|^2 \\
\leq& \left(1 - 2\alpha_n\bar{\gamma} + {\alpha_n}^2\bar{\gamma}^2 + \alpha_n\gamma\rho\right) \|x_n - z^*\|^2 + 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\
=& (1 + \alpha_n\gamma\rho - 2\alpha_n\bar{\gamma}) \|x_n - z^*\|^2 + {\alpha_n}^2\bar{\gamma}^2 \|x_n - z^*\|^2 \\
&+ 2\alpha_n\langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle
\end{aligned}$$

$$= (1 - \alpha_n\gamma\rho + 2\alpha_n\gamma\rho - 2\alpha_n\bar{\gamma}) \|x_n - z^*\|^2 + {\alpha_n}^2\bar{\gamma}^2 \|x_n - z^*\|^2 + 2\alpha_n\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle$$

then,

$$\|y_n - z^*\|^2 \le \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right] \|x_n - z^*\|^2 + \frac{\bar{\gamma}^2{\alpha_n}^2}{1 - \gamma\rho\alpha_n} \|x_n - z^*\|^2 + \frac{2\alpha_n}{1 - \gamma\rho\alpha_n}\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle.$$

Therefore,

$$\begin{aligned} &\|x_{n+1} - z^*\|^2 \\ &\le \|y_n - z^*\|^2 \\ &\le \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right] \|x_n - z^*\|^2 + \frac{\bar{\gamma}^2{\alpha_n}^2}{1 - \gamma\rho\alpha_n} \|x_n - z^*\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \gamma\rho\alpha_n}\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle \\ &= \left[1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\right] \|x_n - z^*\|^2 \\ &\quad + \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_n}\left[\frac{\bar{\gamma}^2\alpha_n}{2(\bar{\gamma} - \gamma\rho)} \|x_n - z^*\|^2 + \frac{1}{\bar{\gamma} - \gamma\rho}\langle\gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_n - z^*\rangle\right]. \end{aligned} \tag{3.35}$$

Applying (3.34), (3.35) and Lemma 2.9, we obtain $x_n \to z^*$ as $n \to \infty$.
In $Case 2$, we assume that there exists some integer $\bar{n_0}$ such that

$$\|x_{\bar{n_0}} - z^*\| \le \|x_{\bar{n_0}+1} - z^*\|.$$

Setting $w_n = \|x_n - z^*\|$, then

$$w_{\bar{n_0}} \le w_{\bar{n_0}+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = max\{l \in \mathbb{N} \mid n_0 \le l \le n, w_l \le w_{l+1}\}.$$

It is clear that τ_n is a nondecreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty.$$

and

$$w_{\tau(n)} \leq w_{\tau(n)+1}$$

for all $n \geq n_0$.
By a similar argument of Case 1, that is

$$\lim_{n\to\infty} \left\|u_{\tau(n)} - y_{\tau(n)}\right\| = 0,$$
$$\lim_{n\to\infty} \left\|x_{\tau(n)} - y_{\tau(n)}\right\| = 0,$$
$$\lim_{n\to\infty} \left\|SAx_{\tau(n)} - Ax_{\tau(n)}\right\| = 0$$

and

$$\lim_{n\to\infty} \left\|u_{\tau(n)} - Tu_{\tau(n)}\right\| = 0.$$

This implies that $w_w(y_{\tau(n)}) \subset \mathbf{\Gamma}$.
We obtain

$$\limsup_{n\to\infty} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau_n} - z^* \rangle \leq 0. \tag{3.36}$$

From $w_{\tau(n)} \leq w_{\tau(n)+1}$ and (3.35), we have

$$\begin{aligned} w^2_{\tau(n)} \leq & w^2_{\tau(n)+1} \\ & \leq \left[1 - \frac{2\alpha_{\tau(n)}(\bar{\gamma} - \gamma\rho)}{1 - \gamma\rho\alpha_{\tau(n)}}\right] w^2_{\tau(n)} + \frac{\bar{\gamma}^2\alpha^2_{\tau(n)}}{1 - \gamma\rho\alpha_{\tau(n)}} w^2_{\tau(n)} \\ & \quad + \frac{2\alpha_{\tau(n)}}{1 - \gamma\rho\alpha_{\tau(n)}} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau(n)} - z^* \rangle. \end{aligned} \tag{3.37}$$

It implies that

$$w^2_{\tau(n)} \leq \frac{2}{2(\bar{\gamma} - \gamma\rho) - \bar{\gamma}^2\alpha_{\tau(n)}} \langle \gamma f(z^*) - \sum_{i=1}^{N} a_i D_i z^*, y_{\tau(n)} - z^* \rangle. \tag{3.38}$$

Combining (3.36) and (3.38), we have

$$\limsup_{n\to\infty} w_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} w_{\tau(n)} = 0, \tag{3.39}$$

From (3.39), implies that

$$\lim_{n\to\infty} w_{\tau(n)+1} = 0.$$

Applying Lemma 2.10 ,we have

$$max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}.$$

It implies that

$$w_n \leq w_{\tau(n)+1}. \tag{3.40}$$

Since w_n is nondecreasing sequence and $n \leq \tau(n)$,

$$w_n \leq w_{\tau(n)}. \tag{3.41}$$

From (3.40) and (3.41), we obtain

$$0 \leq w_n \leq max\{w_{\tau(n)}, w_{\tau(n)+1}\}.$$

Therefore, $w_n \to 0$. That is, $x_n \to z^*$. This complete the proof. □

By using our main result, we obtain the following results in Hilbert spaces.

Corollary 3.2. *Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}_1$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_1 > 1$, $T : C \to C$ is an $\mathcal{L}_2$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L}_2 > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$*

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1 - \xi_n) z_n + \xi_n S\left((1 - \eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + (\mathcal{I} - \alpha_n D)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1 - \beta_n) u_n + \beta_n T\left((1 - \gamma_n) u_n + \gamma_n T u_n\right), \quad for\ n \geq 1, \end{cases} \tag{3.42}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,
$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}$.
Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \mathcal{L}_1^2} + 1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1 + \mathcal{L}_2^2} + 1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D\|^{-1}$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}(\gamma f + \mathcal{I} - D)\, z$.

Proof. Putting $D = D_1 = D_2 = D_3 = ... = D_N$ in Theorem 3.1, we get the desired conclusions. □

Corollary 3.3. *Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\dots,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is an $\mathcal{L}$-Lipschitzian quasi-pseudo-contractive operator with $\mathcal{L} > 1$. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be sequences generated by $x_0 \in H_1$*

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n)\, z_n + \xi_n S\left((1-\eta_n)\, z_n + \eta_n S z_n\right), \\ x_{n+1} = P_C\left[\alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(Ax_n - v_n))\right], for\; n \geq 1 \end{cases} \tag{3.43}$$

The parameters $\{\alpha_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,
$\mathbf{\Gamma} = \{x \mid x \in C, Ax \in Q \cap F(S)\}$.
Suppose that $S - \mathcal{I}$ is demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \mathcal{L}_1^2} + 1}$,

(iii) $0 < \delta < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(iv) $0 < \gamma < \dfrac{1}{\|A\|^2}$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right)$.

Proof. Putting $T \equiv \mathcal{I}$ in Theorem 3.1, we get the desired conclusions. □

4 Application

Lemma 4.1. [10] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x-y\|, \quad \forall x, y \in C.$$

By Lemma 4.1, applying T, S are κ,$\bar{\kappa}$-strict pseudo-contractive mappings, we obtain this theorem.

Theorem 4.2. *Let H_1 and H_2 are two real Hilbert space, let $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex sets. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction, $S : Q \to Q$ is a $\bar{\kappa}$-strict pseudo-contractive mapping, $T : C \to C$ is a κ-strict pseudo-contractive mapping. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$*

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n S\left((1-\eta_n) z_n + \eta_n S z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^N a_i D_i\right)\left(x_n - \delta A^*\left(A x_n - v_n\right)\right), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n T\left((1-\gamma_n) u_n + \gamma_n T u_n\right), \quad for\ n \geq 1, \end{cases} \tag{4.1}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0,1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap F(T), Ax \in Q \cap F(S)\}.$$

Suppose that $T - \mathcal{I}$ and $S - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \dfrac{1}{\sqrt{1 + \left(\frac{1+\bar{\kappa}}{1-\bar{\kappa}}\right)^2} + 1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \dfrac{1}{\sqrt{1 + \left(\frac{1+\kappa}{1-\kappa}\right)^2} + 1}$,

(iv) $0 < \delta, \gamma < \dfrac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ *converge strongly to the unique fixed point of the contraction mapping* $z = P_{\Gamma}\Big(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\Big)$.

Proof. By using Theorem 3.1 and Lemma 4.1, we obtain the conclusion. □

In 2009, Kangtunyakarn and Suantai([11]) introduced the S-mapping generated by a finite family of κ-strictly pseudo contractive mappings and real numbers as follows:

Definition 4.3. Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo contractions of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I,\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,\\
&\;\,.\\
&\;\,.\\
&\;\,.\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called S-mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 4.4. [11] *Let* C *be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ*-strict pseudo contractions of* C *into* C *with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and* $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ *for all* $j = 1, 2, ..., N - 1$ *and* $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ *for all* $j = 1, 2, ..., N$. *Let* S *be the mapping generated by* $T_1, T_2, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$. *Then* $F(S) = \bigcap_{i=1}^N F(T_i)$ *and* S *is a nonexpansive mapping.*

Theorem 4.5. *Let* C *and* Q *are nonempty closed convex subset of real Hilbert spaces. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i*-strict pseudo contractions of* C *into* C *with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and* $\kappa = max\{\kappa_i : i = 1, 2, ..., N\}$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (\kappa, 1)$ *for all* $j = 1, 2, ..., N - 1$ *and* $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1)$ $\alpha_2^j \in [\kappa, 1)$ *for all* $j = 1, 2, ..., N$. *Let* S *be the* S*-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$. *Let* $\{\bar{T}_i\}_{i=1}^N$ *be a finite family of* $\bar{\kappa}_i$*-strict pseudo contractions of* Q *into* Q *with*

$\bigcap_{i=1}^{N} F(\bar{T}_i) \neq \emptyset$ and $\bar{\kappa} = max\{\bar{\kappa_i} : i = 1, 2, ..., N\}$ and let $\beta_j = (\beta_1^j, \beta_2^j, \beta_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\beta_1^j + \beta_2^j + \beta_3^j = 1$, β_1^j, $\beta_3^j \in (\bar{\kappa}, 1)$ for all $j = 1, 2, ..., N-1$ and $\beta_1^N \in (\bar{\kappa}, 1], \beta_3^N \in [\bar{\kappa}, 1)$ $\beta_2^j \in [\bar{\kappa}, 1)$ for all $j = 1, 2, ..., N$. Let $\bar{S}$ be the S-mapping generated by $\bar{T}_1, \bar{T}_2, ..., \bar{T}_N$ and $\beta_1, \beta_2, ..., \beta_N$. Let $A : H_1 \to H_2$ is a bounded linear operator with its adjoint A^*, D_i is strongly positive bounded linear operator on H_1 with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,...,N} \gamma_i$, $f : C \to H_1$ is a ρ-contraction. Assume that $\mathbf{\Gamma} \neq \emptyset$ and let $\{x_n\}$ be a sequences generated by $x_0 \in H_1$

$$\begin{cases} z_n = P_Q A x_n, \\ v_n = (1-\xi_n) z_n + \xi_n \bar{S}\left((1-\eta_n) z_n + \eta_n \bar{S} z_n\right), \\ y_n = \alpha_n \gamma f(x_n) + \left(\mathcal{I} - \alpha_n \sum_{i=1}^{N} a_i D_i\right)(x_n - \delta A^*(A x_n - v_n)), \\ u_n = P_C y_n, \\ x_{n+1} = (1-\beta_n) u_n + \beta_n S\left((1-\gamma_n) u_n + \gamma_n S u_n\right), \quad for\ n \geq 1, \end{cases} \tag{4.2}$$

The parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{\eta_n\}$ are real sequences in $[0, 1]$, δ and γ are two positive constants.
We use $\mathbf{\Gamma}$ to denote the set of solution of problem (1.3), that is,

$$\mathbf{\Gamma} = \{x \mid x \in C \cap \bigcap_{i=1}^{N} F(T_i), Ax \in Q \cap \bigcap_{i=1}^{N} F(\bar{T}_i)\}.$$

Suppose that $S - \mathcal{I}$ and $\bar{S} - \mathcal{I}$ are demiclosed at 0. Assume that the following conditions are satisfied :

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a_1 < \xi_n < b_1 < \eta_n < c_1 < \frac{1}{\sqrt{2}+1}$,

(iii) $0 < a_2 < \beta_n < b_2 < \gamma_n < c_2 < \frac{1}{\sqrt{2}+1}$,

(iv) $0 < \delta, \gamma < \frac{1}{\|A\|^2}$ *and* $\bar{\gamma} > \gamma\rho$,

(v) $0 < \alpha_n < \|D_i\|^{-1}$ *for* $i = 1, 2, ..., N$.

Then the sequence $\{x_n\}$ converge strongly to the unique fixed point of the contraction mapping $z = P_{\mathbf{\Gamma}}\left(\gamma f + \mathcal{I} - \sum_{i=1}^{N} a_i D_i\right)$.

Proof. By using Theorem 3.1 and Lemma 4.4, we obtain the conclusion. □

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