



Fuzzy Version of Some Fixed Point Theorems On Expansion Type Maps in Fuzzy FM Metric Spaces

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Abstract : Ever since the introduction of fuzzy sets by Zadeh [1], the fuzziness invaded almost all the branches of crisp mathematics. Deng [2], Kaleva and Seikalla [3] and Kramosil and Michalek [4], have introduced the concept of fuzzy metric space in different ways. In order to define the Hausdorff topology of fuzzy metric space, George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [4]. In this paper effort has been made to obtain some results on fixed points of expansion type mapping in fuzzy metric space. Our results are the fuzzy version of some fixed point theorems for expansion type mappings on metric spaces.

Keywords : Fuzzy metric spaces, Expansion type maps, Common fixed point.

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1 Introduction

Ever since the introduction of fuzzy sets by Zadeh [1], the fuzziness invaded almost all the branches of crisp mathematics. Deng [2], Kaleva and Seikalla [3] and Kramosil and Michalek [4], have introduced the concept of fuzzy metric space in different ways. In order to define the Hausdorff topology of fuzzy metric space, George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [4].

Definition 1.1 [6]. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if $([0, 1], *)$ is an abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t -norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 [5]. The 3-tuple $(X, M, *)$ is called a fuzzy metric space (FM-space) if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set in $X^2 \times [0, \infty]$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$.

(Fm-1) $M(x, y, 0) > 0$,

(Fm-2) —

(Fm-3) $M(x, y, t) = M(y, x, t)$,

(Fm-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(Fm-5) $M(x, y, \cdot) : [0, \infty] \rightarrow [0, 1]$ is continuous.

Below in the following, we give some definitions which will be useful in the sequel for proving certain fixed point theorems in fuzzy metric spaces.

Definition 1.3 [7]. Let $(X, M, *)$ be a fuzzy metric space.

1. A sequence $\{x_n\}$ in X is said to be converge to a point $x \in X$ (denoted by $\lim x_n = x$) if $\lim M(x_n, x, t) = 1$ for all $t > 0$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.
3. An FM-space in which every Cauchy sequence is convergent is said to be complete.

Remark 1.1 Since $*$ is continuous, it follows from (FM – 4) that the limit of a sequence in FM-space is uniquely determined.

Rhoades [8] summarized contractive maps of different type discussed on their fixed point. He considered 250 types of mappings and analyzed the relationship amongst them. He also obtained some general theorems on fixed points. These 250 types of mappings are based on 25 types where $d(Tx, Ty)$ is governed by $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(y, Tx)$, $d(x, Ty)$.

In fuzzy metric spaces, our analysis shows that the authors have tried to establish the fuzzy version of some theorems on contraction in metric spaces e.g., in [7], Grabiec presented the fuzzy version of Banach contraction theorem as follows: “Let $(X, M, *)$ be a complete FM-space and f be a self map of X such that

$$M(fx, fy, kt) \geq M(x, y, t)$$

for all $x, y \in X$, $t > 0$ and where $k \in (0, 1)$, then f has a unique fixed point in X .”

In 1976, Rosenholtz [9] discussed local expansion mappings. Let (X, d) be a metric, T is a local expansion if every point in X has a neighbourhood B on which T is expansion. In fact, Rosenholtz proved “If (X, d) be a complete metric space and $T : X \rightarrow X$ be a self map of X onto itself satisfying $d(Tx, Ty) > \lambda d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $\lambda > 1$, then T has fixed point in X .”

2 Preliminaries

In 1984, Wang, Li, Gao and Iseki [10] proved that

Theorem (A). *Let T be a self map of complete metric space X onto itself and if there exists a constant $\lambda > 1$ s.t.*

$$d(Tx, Ty) > \lambda d(x, y)$$

for all $x, y \in X$ then T has a unique fixed point in X .

Theorem (B). *If there exists non-negative real numbers α, β, γ with $\alpha + \beta + \gamma > 1$ and $\alpha < 1$ s.t.*

$$d(Tx, Ty) \geq \alpha d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y)$$

for each $x, y \in X$ with $x \neq y$ and T is onto, then T has a fixed point.

Theorem (C). *If there exists a constant $\alpha < 1$ s.t.*

$$d(Tx, Ty) \geq \alpha \min\{d(x, Tx), d(y, Ty), d(x, y)\}$$

for all $s, y \in X$ and T is onto and continuous, then T has a fixed point.

Throughout this paper, $(X, M, *)$ will denote the fuzzy metric space in the sense of definition (1.2) with the following conditions:

(Fm-6) $\lim M(x, y, t) = 1$ for all $x, y \in X$.

We need the following lemmas:

Lemma 2.1 [7]. *For all $x, y \in X$, $M(x, y, \cdot)$ is non-decreasing.*

Lemma 2.2 [7]. *Let $\{y_n\}$ be a sequence in an FM-space $(X, M, *)$ with the condition (Fm-6). If there exists a number $k \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n=1, 2, \dots$, then $\{y_n\}$ is a Cauchy sequence in X .

Lemma 2.3 [7]. *For all $x, y \in X$, $t > 0$ and $0 < k < 1$,*

$$M(x, y, kt) \geq M(x, y, t)$$

, then $x = y$.

Remark 2.1 *In Lemma (2.2) and Lemma (2.3), the condition may be replaced respectively by:*

1. If there exist some $k > 1$ s.t.

$$M(y_{n+2}, y_{n+1}, t) \geq M(y_{n+1}, y_n, kt),$$

$\{y_n\}$ is a Cauchy sequence.

2. If there exists some $k > 1$ s.t.

$$M(x, y, t) \geq M(x, y, kt),$$

then $x = y$.

3 Main Results

Our main object is to obtain some fuzzy version of the theorems on expansion type maps in metric spaces. As, the fuzzy version of the theorem (A) is given by the following theorem (3.1)

Theorem 3.1 Let $(X, M, *)$ be a complete fuzzy metric space and f be a self map of X onto itself. There exist a constant $k > 1$ s.t.

$$M(fx, fy, kt) \leq M(x, y, t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$, then f has a unique fixed point in X .

Proof: Let $x_0 \in X$, as f is onto there is an element $x_1 \in f^{-1}x_0$. In the same way $x_n \in f^{-1}x_{n-1}$ for all $n = 2, 3, 4, \dots$, thus we get a sequence $\{x_n\}$. If $x_m = x_{m-1}$ for some m , then x_m is a fixed point of f . Now suppose $x_n \neq x_{n-1}$ for all $n = 1, 2, 3, \dots$, then it follows from (1) that

$$M(x_n, x_{n+1}, kt) = M(fx_{n+1}, fx_{n+2}, kt) \leq M(x_{n+1}, x_{n+2}, t)$$

for all $t > 0$ and for all $n = 0, 1, 2, \dots$. Therefore, in view of Remark (2.1)(i) on Lemma (2.2), $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, x_n has a limit $u \in X$. As f is onto, there is an element $v \in X$ s.t. $v \in f^{-1}u$. Now

$$M(x_n, u, kt) = M(fx_{n+1}, fv, kt) \leq M(x_{n+1}, v, t)$$

which as $n \rightarrow \infty$, gives $M(u, v, t) = 1$ for all $t > 0$. Therefore by (Fm-2), it follows that $u = v$ yielding thereby $fu = u$, and so u is the fixed point of f . Let u and v be the two fixed points of f i.e. $fu = u$ and $fv = v$, then (1) yields

$$M(u, v, kt) = M(fu, fv, t) \leq M(u, v, t)$$

for all $t > 0$. Hence, in view of Remark (2.1)(ii) on Lemma (2.3), we obtain $u = v$, which shows the uniqueness of u as a fixed point of f . This completes the proof.

Theorem 3.2 Let $(X, M, *)$ be a complete FM-space with $t * t \leq t$ for all $t \in [0, 1]$ and f be a mapping from X onto itself. There exists a number $k > 1$ s.t.

$$M(fx, fy, kt) \leq M(x, y, t) * M(x, fx, t) * M(y, fy, t) \tag{3.2}$$

for all $x, y \in X$ and $t > 0$, then f has a unique fixed point in X .

Proof : A sequence $\{x_n\}$ is developed similarly as in Theorem (3.1). If $x_{m-1} = x_m$ for some m , f has a fixed point x_m . Suppose $x_{n-1} \neq x_n$ for every positive integer n , then from (2)

$$\begin{aligned} M(x_n, x_{n+1}, kt) &= M(fx_{n+1}, fx_{n+2}, kt) \\ &\leq M(x_{n+1}, x_{n+2}, t) * M(x_{n+1}, fx_{n+1}, t) * M(x_{n+2}, fx_{n+2}, t) \\ &= M(x_{n+1}, x_{n+2}, t) * M(x_{n+1}, x_n, t) * M(x_{n+2}, x_{n+1}, t) \end{aligned}$$

yielding thereby

$$M(x_n, x_{n+1}, kt) \leq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t). \tag{3.3}$$

Now, suppose

$$M(x_{n+1}, x_{n+2}, t) < M(x_n, x_{n+1}, t)$$

then it follows from (3) that

$$M(x_n, x_{n+1}, kt) \leq M(x_{n+1}, x_{n+2}, t)$$

for all $t > 0$ which in view of Remark (2.1)(i) on Lemma (2.3), implies $x_n = x_{n+1}$, a contradiction.

Now, let

$$M(x_n, x_{n+1}, t) \leq M(x_{n+1}, x_{n+2}, t),$$

in this case, it is noting from (3) that

$$M(x_n, x_{n+1}, kt) \leq M(x_{n+1}, x_{n+2}, t)$$

for all $t > 0$. Thus, in view of Remark (2.1)(i) on Lemma (2.2), $\{x_n\}$ is a Cauchy sequence in X which is complete, therefore there exists some $u \in X$ s.t. $x_n \rightarrow u$. Since f is onto, there is an element $v \in f^{-1}u$. Now,

$$M(x_n, u, kt) = M(fx_{n+1}, fv, kt) \leq M(x_{n+1}, v, t) * M(x_{n+1}, x_n, t) * M(v, u, t)$$

which, as letting $n \rightarrow \infty$, gives $M(u, v, t) = 1$ for all $t > 0$. Therefore by (Fm-2), it is noting that $u = v$ and so $fu = u$ i.e. u is a fixed point of f . The uniqueness of u as a fixed point of f can be shown easily from (2). Hence the theorem proved.

Theorem 3.3 Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \leq t$ for all $t \in [0, 1]$ and f, g be two self maps of X onto itself. If there exists a number $k > 1$ s.t.

$$M(fx, gy, kt) \leq M(x, y, t) * M(x, fx, t) * M(y, gy, t). \tag{3.4}$$

Proof : Choose an element $x_0 \in X$; as f is onto there is an element $x_1 \in f^{-1}x_0$. since g is onto, there exists an element x_2 satisfying $x_2 \in g^{-1}x_1$. Thus in general, a sequence $\{x_n\}$ is defined as $x_{2n+1} \in f^{-1}x_{2n}, x_{2n+2} \in g^{-1}x_{2n+1}$ for all $n = 0, 1, 2, \dots$. Now, we have two cases as follows:

Case(1) When $x_m \neq x_{m+1}$ for all $m = 0, 1, 2, \dots$. In this case, it follows from (4) that

$$\begin{aligned} M(x_{2n}, x_{2n+1}, kt) &= M(fx_{2n+1}, gx_{2n+2}, kt) \\ &\leq M(x_{2n+1}, x_{2n+2}, t) * M(x_{2n+1}, x_{2n}, t) * M(x_{2n+2}, x_{2n+1}, t) \\ &\leq M(x_{2n+1}, x_{2n+2}, t) * M(x_{2n}, x_{2n+1}, t) \end{aligned} \quad (3.5)$$

Suppose

$$M(x_{2n+1}, x_{2n+2}, t) < M(x_{2n}, x_{2n+1}, t),$$

then from (5) we obtain

$$M(x_{2n}, x_{2n+1}, kt) \leq M(x_{2n}, x_{2n+1}, t)$$

which, in view of Remark (2.1)(ii) on Lemma (2.3), implies $x_{2n} = x_{2n+1}$ which is a contradiction. Therefore, let

$$M(x_{2n+1}, x_{2n+2}, t) \geq M(x_{2n}, x_{2n+1}, t),$$

then (5) yields

$$M(x_{2n}, x_{2n+1}, kt) \leq M(x_{2n+1}, x_{2n+2}, t)$$

for all $t > 0$. Similarly, it can be shown that

$$M(x_{2n+1}, x_{2n+2}, kt) \leq M(x_{2n+2}, x_{2n+1}, t)$$

for all $t > 0$. Thus, in general we obtain

$$M(x_n, x_{n+1}, kt) \leq M(x_{n+1}, x_{n+2}, t)$$

for all $t > 0$ and $n = 0, 1, 2, \dots$. Hence, in view of remark (2.1)(i) on lemma (2.2), $\{x_n\}$ is a Cauchy sequence in X which is complete, therefore $\{x_n\}$ has a limit point in X . Since $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$, $x_{2n} \rightarrow u$ and $x_{2n+1} \rightarrow u$ as $n \rightarrow \infty$. As f and g are onto, there exist $v, w \in X$ satisfying $v \in f^{-1}u$ and $w \in g^{-1}u$. Now,

$$\begin{aligned} M(x_{2n}, u, kt) &= M(fx_{2n+1}, gw, kt) \\ &\leq M(x_{2n+1}, w, t) * M(x_{2n+1}, x_{2n}, t) * M(w, gw, t) \end{aligned}$$

which, as $n \rightarrow \infty$, gives $M(u, w, t) = 1$ for all $t > 0$. Thus by (Fm-2), it follows that $u = w$. In the similar pattern, taking $x = v$ and $y = x_{2n+2}$ in (4), and therefore proceeding as above, we obtain $u = v$. Therefore, $u = v = w$ which immediately implies $fu = gu = u$ and so u is a common fixed point of f and g . Now, let u and v be two common fixed point of f and g i.e. $fu = gu = u$ and $fv = gv = v$, then

$$\begin{aligned} M(u, v, kt) &= M(fu, gv, kt) \\ &= \text{leq} M(u, v, t) * M(u, fu, t) * M(v, gv, t) \\ &= M(u, v, t) * 1 * 1 = M(u, v, t) \end{aligned}$$

for all $t > 0$. Further by an application of Remark (2.1)(ii) on Lemma (2.3), we obtain $u = v$.

Case(II) When $x_{m-1} = x_m$ for some m . Here m may be even or odd positive integer. Without loss of generality, suppose m is an even integer, say $m = 2p$, then $x_{2p-1} = x_{2p}$ i.e. $gx_{2p} = fx_{2p-1}$ which implies $x_{2p} = x_{2p+1}$ (as we have $fx \neq gy$ if $x \neq y$). Therefore we have $x_{2p-1} = x_{2p} = x_{2p+1} = \dots$. Which shows that $\{x_n\}$ is a convergent sequence and so Cauchy sequence in X . The rest of the proof is similar to as in case (1) and this completes the proof.

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