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Stein's Method for Cauchy Approximation

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Abstract: In this paper, we use Stein's method to find the necessary and sufficient conditions for Cauchy distribution and give a bound for Cauchy approximation. The random variables which considered need not to be independent.

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1 Introduction

Stein[11] introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Stein's method has been applied with much success in the area of normal approximation(See, for example, Erickson[7], Bolthausen[5], Baldi, Rinott and Stein[1] and Barbour[2]). This method was extended from the normal distribution to the Poisson distribution by Chen[6]. Chen's work has resulted in advances in the theory of Poisson approximation and has helped to develop and improve upon a body of interesting applications and examples.(For theoretical developments, see Barbour and Eagleson[3,4], Holst and Janson[8]). Many authors developed Stein's method to other approximations, for examples, Peköz[10] applied this method to geometric approximation.

In 2002, Neammanee[9] applied Stein's method for Cauchy approximation to the distribution function of sums of independent random variables. In this paper, we further develop the Stein's technique to find the bound in more general situation, i.e. we need not to assume independence of random variables. We organize this paper as follows. Main results are stated in section 2 while proof of main results is given in section 3.

2 Main results

Let $X_1, X_2, \ldots, X_n$ be random variables. Neammanee[9] used Stein's method to find a bound between the distribution function F_n of $X_1 + X_2 + \cdots + X_n$ and the Cauchy distribution F,

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+t^2}dt. \tag{2.1}$$

In his work, he assume the independence of X_n's but it is not necessary in this paper. To applied Stein's method we need the followings construction.
Let $\mathcal{H} = \{h : \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{\infty} \frac{|h(x)|}{1+x^2}dx < \infty\}$ and for each $h \in \mathcal{H}$,

$$Cau(h) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{1+x^2}dx. \tag{2.2}$$

From Neammanee[9],we know that the Stein's equation for Cauchy distribution F, is

$$f'(w) - \frac{2wf(w)}{1+w^2} = h(w) - Cau(h) \tag{2.3}$$

and

$$f_{w_0}(w) = \begin{cases} \pi(1+w^2)F(w)(1-F(w_0)) \text{ if } \ w \le w_0 \\ \pi(1+w^2)F(w_0)(1-F(w)) \text{ if } \ w \ge w_0 \end{cases} \tag{2.4}$$

is a solution of (2.3) when we choose $h(w) = I_{(-\infty,w_0]}$ and

$$I_{(-\infty,w_0]}(w) = \begin{cases} 1 & \text{if } \ w \le w_0 \\ 0 & \text{if } \ w > w_0. \end{cases}$$

The main results are the followings.

Theorem 2.1. *Let $X_1, X_2, \ldots, X_n$ be random variables such that $EX_i = 0$, $EX_i^4 < \infty$ and S_i be a subset of $\{1,2,\ldots,n\}$ for all $i = 1,2,\ldots,n$. Let $W = X_1 + X_2 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,*

$$\begin{aligned}
&\Big|P(W \le w_0) - F(w_0)\Big| \\
&\le 3\sqrt{E[1 - \frac{2}{1+W^2}\sum_{i=1}^{n}\sum_{j\in S_i} X_iX_j]^2} \\
&\quad + 4\pi \min\Big\{E\sum_{i=1}^{n}|X_i|\Big|\sum_{j\in S_i}X_i\Big|, \sqrt{E\sum_{i=1}^{n}X_i^2 E\Big|\sum_{i=1}^{n}X_i\sum_{j\in S_i}X_j\Big|^2}\Big\}F(w_0)(1-F(w_0)) \\
&\quad + 2\pi E\sum_{i=1}^{n}\Big|E^{\{X_j|j\notin S_i\}}X_i\Big| + 12(\pi+1)E\sum_{i=1}^{n}|X_i|\Big(\sum_{j\in S_i}X_j\Big)^2
\end{aligned}$$

where $E^{\mathcal{B}}X$ is the conditional expectation of X with respect to $\mathcal{B}$.

In the case of X_n's are independent we have the following corollary and example.

Corollary 2.2. *Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $EX_i = 0$, $EX_i^4 < \infty$ for all $i = 1, 2, \ldots, n$. Let $W = X_1 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,*

$$\begin{aligned}
&\Big|P(W \le w_0) - F(w_0)\Big| \\
&\le 3\sqrt{E\Big[1 - \frac{2}{1+W^2}\sum_{i=1}^{n} X_i^2\Big]^2} \\
&\quad + 4\pi \min\Big\{\sum_{i=1}^{n} EX_i^2, \sqrt{n\Big(\sum_{i=1}^{n} EX_i^2\Big)\Big(\sum_{i=1}^{n} EX_i^4\Big)}\Big\} F(w_0)(1 - F(w_0)) \\
&\quad + 12(\pi + 1)\sum_{i=1}^{n} E|X_i|^3.
\end{aligned}$$

Example 2.3. Let $Y_1, Y_2, \ldots, Y_n$ be identically independent random variables with zero means, $EY_i^2 = \frac{1}{2}$ and $E|Y_i|^5 < \infty$. Let $X_i = \frac{Y_i}{\sqrt{n}}$ and $W = X_1 + X_2 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,

$$|P(W \le w_0) - F(w_0)| < \frac{C}{\sqrt[4]{n}} + C \min\{\frac{1}{2}, \frac{1}{\sqrt{2}}\sqrt{E|Y_i|^4}\} F(w_0)(1 - F(w_0)).$$

We note that Corollary 2.2 and Example 2.3 are main results of Neammanee[9]. Moreover, we also use Stein's method to find the necessary and sufficient conditions in order to a random variable W be Cauchy.

Theorem 2.4. *Let W be a random variable. Then W has a Cauchy distribution if and only if for all functions $f : \mathbb{R} \to \mathbb{R}$ such that f' exists a.e., continuous a.e. and $E|f'(W)| < \infty$, we have*

$$Ef'(W) = 2E\frac{Wf(W)}{1+W^2}.$$

Throughout this paper, C stands for an absolute constant with possibly different values in different places.

3 Proof of Main Results

To prove Theorem 2.1, we first introduce the basic assumption.

Basic assumption: Let $(\widetilde{\Omega}, \widetilde{\mathcal{B}}, \widetilde{P})$ be a probability space and let $\mathcal{B}$ and $\mathcal{C}$ be sub-σ-algebras of $\widetilde{\mathcal{B}}$. The random variable G is $\widetilde{\mathcal{B}}$-measurable and random variable $\widetilde{W}$ is $\mathcal{C}$-measurable. Assuming

$$E|G| < \infty,$$

we define

$$W = E^{\mathcal{B}}G.$$

Lemma 3.1. *In addition to the basic assumption, let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(w)| \leq C(1+w^2)$ for some $C > 0$. Then*

$$E\frac{Wf(W)}{1+W^2} = EG\Big[\frac{f(W)}{1+W^2} - \frac{f(\widetilde{W})}{1+\widetilde{W}^2}\Big] + E(E^{\mathcal{C}}G)\frac{f(\widetilde{W})}{1+\widetilde{W}^2}. \tag{3.1}$$

Moreover,

$$P(W \leq w_0) = F(w_0) + E\Big[f'_{w_0}(W) - 2G\Big(\frac{f_{w_0}(W)}{1+W^2} - \frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big)\Big] - 2E(E^{\mathcal{C}}G)\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} \tag{3.2}$$

for $w_0 \in \mathbb{R}$.

Proof. Since $E\frac{Gf(\widetilde{W})}{1+\widetilde{W}^2} = EE^{\mathcal{C}}\frac{Gf(\widetilde{W})}{1+\widetilde{W}^2} = E(E^{\mathcal{C}}G)\frac{f(\widetilde{W})}{1+\widetilde{W}^2}$, so

$$\begin{aligned} E\frac{Wf(W)}{1+W^2} &= E(E^{\mathcal{B}}G)\frac{f(W)}{1+W^2} \\ &= EE^{\mathcal{B}}\Big(\frac{Gf(W)}{1+W^2}\Big) \\ &= E\frac{Gf(W)}{1+W^2} \\ &= EG\Big[\frac{f(W)}{1+W^2} - \frac{f(\widetilde{W})}{1+\widetilde{W}^2}\Big] + E(E^{\mathcal{C}}G)\frac{f(\widetilde{W})}{1+\widetilde{W}^2}. \end{aligned}$$

Then (3.1) holds and (3.2) follows from (3.1) and (2.3) when $h = I_{(-\infty,w_0]}$. □

Lemma 3.2. *Under the basic assumption for any $w_0 \in \mathbb{R}$, we have*

$$\begin{aligned} P(W \leq w_0) &= F(w_0) + Ef'_{w_0}(W)\Big[1 - 2\frac{G(W-\widetilde{W})}{1+W^2}\Big] + 4E\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2} \\ &+ 2EG\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w \leq \widetilde{W}) - I(w \leq W)]\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)dw \\ &- 4EG\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w \leq \widetilde{W}) - I(w \leq W)]\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'dw \\ &- 2E\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} \end{aligned}$$

Proof. Let $w_0 \in \mathbb{R}$. For $W < \widetilde{W}$, we see that

$$
\begin{aligned}
&\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2} \\
&= \int_W^{\widetilde{W}} \Big[\Big(\frac{f_{w_0}(w)}{1+w^2}\Big)' - \frac{f'_{w_0}(W)}{1+W^2} + \frac{2Wf_{w_0}(W)}{(1+W^2)^2}\Big] dw \\
&= \int_W^{\widetilde{W}} \Big[\frac{f'_{w_0}(w)}{1+w^2} - \frac{2wf_{w_0}(w)}{(1+w^2)^2} - \frac{f'_{w_0}(W)}{1+W^2} + \frac{2Wf_{w_0}(W)}{(1+W^2)^2}\Big] dw \\
&= \int_W^{\widetilde{W}} \int_W^{w} \Big(\frac{f'_{w_0}(y)}{1+y^2}\Big)' dydw - 2\int_W^{\widetilde{W}} \int_W^{w} \Big(\frac{yf_{w_0}(y)}{(1+y^2)^2}\Big)' dydw \\
&= \int_W^{\widetilde{W}} \int_y^{\widetilde{W}} \Big(\frac{f'_{w_0}(y)}{1+y^2}\Big)' dwdy - 2\int_W^{\widetilde{W}} \int_y^{\widetilde{W}} \Big(\frac{yf_{w_0}(y)}{(1+y^2)^2}\Big)' dwdy \\
&= \int_W^{\widetilde{W}} (\widetilde{W}-y)\Big(\frac{f'_{w_0}(y)}{1+y^2}\Big)' dy - 2\int_W^{\widetilde{W}} (\widetilde{W}-y)\Big(\frac{yf_{w_0}(y)}{(1+y^2)^2}\Big)' dy.
\end{aligned}
$$

and by the same argument we can show that

$$
\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}
$$

$$
= \int_{\widetilde{W}}^{W} (w-\widetilde{W})\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)' dw - 2\int_{\widetilde{W}}^{W} (w-\widetilde{W})\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)' dw
$$

for $\widetilde{W} < W$.

So

$$
\begin{aligned}
&\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2} - \frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2} + \frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2} \\
&= \int_{-\infty}^{\infty} (\widetilde{W}-w)[I(w\le\widetilde{W}) - I(w\le W)]\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)' dw \\
&- 2\int_{-\infty}^{\infty} (\widetilde{W}-w)[I(w\le\widetilde{W}) - I(w\le W)]\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)' dw.
\end{aligned} \tag{3.3}
$$

By Lemma 3.1, we have

$$\begin{aligned}
P(W\le w_0) &= F(w_0)+E\Big[f'_{w_0}(W)-2\frac{G(W-\widetilde{W})f'_{w_0}(W)}{1+W^2}+2\frac{G(W-\widetilde{W})f'_{w_0}(W)}{1+W^2}\\
&\quad+4\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}-4\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}\\
&\quad-2G\Big(\frac{f_{w_0}(W)}{1+W^2}-\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\Big)\Big]-2E\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\\
&= F(w_0)+Ef'_{w_0}(W)\Big[1-\frac{2G(W-\widetilde{W})}{1+W^2}\Big]\\
&\quad+2EG\Big[\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}-\frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2}+\frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big]\\
&\quad+4E\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}-2E\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}\\
&= F(w_0)+Ef'_{w_0}(W)\Big[1-\frac{2G(W-\widetilde{W})}{1+W^2}\Big]\\
&\quad+2EG\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w\le\widetilde{W})-I(w\le W)]\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)'\,dw\\
&\quad-4EG\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w\le\widetilde{W})-I(w\le W)]\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'\,dw\\
&\quad+4E\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}-2E\frac{(E^{\mathcal{C}}G)f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} \qquad (3.4)
\end{aligned}$$

where we have use (3.3) in the last equality. □

The following lemma is the properties of f_{w_0} which we need in the proof of Theorem 2.1.

Lemma 3.3. *For any real numbers w_0 and w,*
1. $|f_{w_0}(w)/(1+w^2)|\le \pi F(w_0)(1-F(w_0))$
2. $|f'_{w_0}(w)|\le 3$
3. $|f''_{w_0}(w)|\le 3+2\pi$
4. $|(f'_{w_0}(w)/(1+w^2))'|\le 6+2\pi$
5. $|(wf_{w_0}(w)/(1+w^2)^2)'|\le 3+5\pi.$

Proof. See [9]. □

3.1 Proof of Theorem 2.1.

Let I be a random variable, uniformly distributed over $\{1,2,\dots,n\}$ independent of $\{X_1,X_2,\dots,X_n\}$. Let $\widetilde{\mathcal{B}}$ be a σ-algebra in which the random variables

I and $\{X_1, X_2, \ldots, X_n\}$ are measurable, $\mathcal{B}$ the sub-σ-algebra of $\widetilde{\mathcal{B}}$ generated by $\{X_1, X_2, \ldots, X_n\}$ and $\mathcal{C}$ the σ-algebra generated by I and $\{X_j \mid j \notin S_I\}$. Let

$$G = nX_I.$$

Then

$$E^{\mathcal{B}}G = E^{\mathcal{B}}nX_I = \sum_{i=1}^{n} X_i = W.$$

So the basic assumption is satisfied. Let

$$\widetilde{W} = W - \sum_{j \in S_I} X_j.$$

To prove the theorem, let $w_0 \in \mathbb{R}$. By Lemma 3.2, we obtain

$$\begin{aligned}
&|P(W \le w_0) - F(w_0)| \\
&\le \sup_{w \in \mathbb{R}} |f'_{w_0}(w)| E\Big|1 - 2\frac{G(W-\widetilde{W})}{1+W^2}\Big| + 4E\Big|\frac{G(W-\widetilde{W})Wf_{w_0}(W)}{(1+W^2)^2}\Big| + 2\sup_{w\in\mathbb{R}}\Big|\frac{f_{w_0}(w)}{1+w^2}\Big| E|E^{\mathcal{C}}G| \\
&+2\sup_{w\in\mathbb{R}}\Big|\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)'\Big| E|G|\int_{-\infty}^{\infty} |\widetilde{W}-w||[I(w\le\widetilde{W}) - I(w\le W)]|dw \\
&+4\sup_{w\in\mathbb{R}}\Big|\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'\Big| E|G|\int_{-\infty}^{\infty} |\widetilde{W}-w||[I(w\le\widetilde{W}) - I(w\le W)]|dw \\
&\le \sup_{w\in\mathbb{R}} |f'_{w_0}(w)| \sqrt{E\Big[1 - \frac{2}{1+W^2}E^{\mathcal{B}}G(W-\widetilde{W})\Big]^2} + 4E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| \\
&+2\sup_{w\in\mathbb{R}}\Big|\frac{f_{w_0}(w)}{1+w^2}\Big| E|E^{\mathcal{C}}G| + \Big(\sup_{w\in\mathbb{R}}\Big|\Big(\frac{f'_{w_0}(w)}{1+w^2}\Big)'\Big| + 2\sup_{w\in\mathbb{R}}\Big|\Big(\frac{wf_{w_0}(w)}{(1+w^2)^2}\Big)'\Big|\Big) E|G|(\widetilde{W}-W)^2.
\end{aligned}$$

Hence, by Lemma 3.3,

$$\begin{aligned}
|P(W \le w_0) - F(w_0)| \le& 3\sqrt{E\Big[1 - \frac{2}{1+W^2}E^{\mathcal{B}}G(W-\widetilde{W})\Big]^2} + 4E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| \\
&+ 2\pi E|E^{\mathcal{C}}G| + 12(\pi+1)E|G|(\widetilde{W}-W)^2.
\end{aligned}$$

We see that

$$E|E^{\mathcal{C}}G| = E|E^{\mathcal{C}}nX_I| = E\Big|E^{\mathcal{C}}\sum_{i=1}^{n} X_i\Big| \le E\sum_{i=1}^{n}\Big|E^{\{X_j \mid j\notin S_i\}}X_i\Big|,$$

$$E|G|(\widetilde{W}-W)^2 = E\Big|\sum_{i=1}^{n} X_i\Big(\sum_{j\in S_i} X_j\Big)^2\Big| \le E\sum_{i=1}^{n} |X_i|\Big(\sum_{j\in S_i} X_j\Big)^2,$$

$$\begin{aligned}\sqrt{E\Big[1-\frac{2}{1+W^2}E^{\mathcal{B}}G(W-\widetilde{W})\Big]^2} &= \sqrt{E\Big[1-\frac{2}{1+W^2}E^{\mathcal{B}}nX_I\sum_{j\in S_I}X_j\Big]^2}\\ &= \sqrt{E\Big[1-\frac{2}{1+W^2}\sum_{i=1}^{n}E^{\mathcal{B}}X_i\sum_{j\in S_i}X_j\Big]^2}\\ &= \sqrt{E\Big[1-\frac{2}{1+W^2}\sum_{i=1}^{n}\sum_{j\in S_i}X_iX_j\Big]^2}.\end{aligned}$$

By Lemma 3.3(1), we have

$$\begin{aligned}E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| &\le \pi F(w_0)(1-F(w_0))E|G||W-\widetilde{W}|\\ &= \pi F(w_0)(1-F(w_0))E\Big|\sum_{i=1}^{n}X_i\Big(\sum_{j\in S_i}X_j\Big)\Big|\\ &\le \pi F(w_0)(1-F(w_0))E\sum_{i=1}^{n}|X_i|\Big|\Big(\sum_{j\in S_i}X_j\Big)\Big|\end{aligned}$$

and

$$\begin{aligned}E\Big|\frac{G(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\Big| &\le \pi F(w_0)(1-F(w_0))E|G||W-\widetilde{W}||W|\\ &\le \pi F(w_0)(1-F(w_0))\sqrt{E|G|^2|W-\widetilde{W}|^2}\sqrt{EW^2}\\ &= \pi F(w_0)(1-F(w_0))\sqrt{E\Big|\sum_{i=1}^{n}X_i\sum_{j\in S_i}X_j\Big|^2E\sum_{i=1}^{n}X_i^2}.\end{aligned}$$

This complete the proof. □

3.2 Proof of Corollary 2.2.

Follows from Theorem 2.1 by choosing $S_i = \{i\}$, so

$$E|\sum_{i=1}^{n}X_i\sum_{j\in S_i}X_j|^2 = E|\sum_{i=1}^{n}X_i^2|^2 \le n\sum_{i=1}^{n}EX_i^4,$$

and the assumption that $X_1, X_2, \ldots, X_n$ are independent implies

$$|E^{\{X_j|j\notin S_i\}}X_i| = |E^{\{X_j|j\neq i\}}X_i| = |EX_i| = 0.$$

□

3.3 Proof of Example 2.3.

See[9].

3.4 Proof of Theorem 2.4.

To prove the necessity, we assume that W has a Cauchy distribution and let $f:\mathbb{R}\to\mathbb{R}$ be such that f' exists a.e., continuous a.e. and $E|f'(W)|<\infty$. Then

$$
\begin{aligned}
&2E\frac{Wf(W)}{1+W^2}\\
&=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{wf(w)}{(1+w^2)^2}dw\\
&=\frac{2}{\pi}\Big\{\int_{-\infty}^{0}\frac{wf(w)}{(1+w^2)^2}dw+\int_{0}^{\infty}\frac{wf(w)}{(1+w^2)^2}dw\Big\}\\
&=\frac{2}{\pi}\Big\{\int_{-\infty}^{0}\frac{w}{(1+w^2)^2}(f(0)-\int_{w}^{0}f'(t)dt)dw+\int_{0}^{\infty}\frac{w}{(1+w^2)^2}(f(0)+\int_{0}^{w}f'(t)dt)dw\Big\}\\
&=\frac{2}{\pi}\Big\{\int_{-\infty}^{0}\int_{w}^{0}\frac{(-w)f'(t)}{(1+w^2)^2}dtdw+\int_{0}^{\infty}\int_{0}^{w}\frac{wf'(t)}{(1+w^2)^2}dtdw+f(0)\int_{-\infty}^{\infty}\frac{w}{(1+w^2)^2}dw\Big\}\\
&=\frac{2}{\pi}\Big\{\int_{-\infty}^{0}f'(t)\int_{\infty}^{t}\frac{(-w)}{(1+w^2)^2}dwdt+\int_{0}^{\infty}f'(t)\int_{t}^{\infty}\frac{w}{(1+w^2)^2}dwdt\Big\}\\
&=\frac{1}{\pi}\Big\{\int_{-\infty}^{0}\frac{f'(t)}{1+t^2}dt+\int_{0}^{\infty}\frac{f'(t)}{1+t^2}dt\Big\}\\
&=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f'(t)}{1+t^2}dt\\
&=Ef'(W).
\end{aligned}
$$

Conversely, let $w_0\in\mathbb{R}$. By Lemma 3.3(2), we see that $E|f'_{w_0}(W)|<\infty$. Then

$$
\begin{aligned}
0&=E\Big[f'_{w_0}(W)-2\frac{Wf_{w_0}(W)}{1+W^2}\Big]\\
&=E[I_{(-\infty,w_0]}(W)-Cau(I_{(-\infty,w_0]})]\\
&=P(W\le w_0)-F(w_0)
\end{aligned}
$$

where we have used Stein's equation (2.3) in the first equality. Hence W has a Cauchy distribution. □

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