Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 379–387
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Approximating Fixed Points of Nonlinear Mappings in Convex Metric Space

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Abstract : In this paper, we prove an existence fixed point theorem for a generalized nonlinear mapping in complete convex metric spaces. We introduce a family iterations to approximate fixed points of the generalized nonlinear mapping and discuss the effciency of these iterations. Some examples are also given to illustrate our results.

Keywords : fixed point theorems; convex metric spaces; existence and approximation; rate of convergence.
2010 Mathematics Subject Classification : 47H10; 47H09.

1 Introduction and Preliminaries

In 1922, Banach [1] gave a remarkable fixed point theorem called Banach Contraction Principle: Let (X, d) be a complete metric space and $T : X \to X$ be a contractive mapping, i.e., there exists $h \in [0, 1)$ such that

$$d(Tx, Ty) \leq hd(x, y) \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point. The Principle is an improtant tool in the theory of nonlinear analysis. In the past few decades, as generalization of Banach Contraction Principle, many authors [2-4,6-8] have defined nonexpansive and contractive type mappings in the framework of Banach spaces and metric

[0]This research was supported by the University Science Research Project of Jiangsu Province of China (No. 13KJB110021).

spaces. They all focused on finding suitable conditions for the exitence and convergence of fixed points of given mappings. In uniformly convex Banach spaces, Goeble [9] and Bose [10] considered a generalized nonexpansive mapping: Let $T : K \to K$ (where K is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space) be a mapping such that

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|] \quad (1.2)$$

for all $x, y \in K$, where $a, b, c \geq 0$ and $a+2b+2c \leq 1$. They proved some fixed point theorems for the generalized nonexpansive mappings (1.2). From [9] and [10], we know that the convex structure is important for the existence of fixed point of T. In metric spaces, Takahashi [11] introduced a significant convex structure:

A convex structure in a metric space (X, d) is a mapping $W : X \times X \times [0, 1] \to X$ satisfying, for each $x, y, u \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure W is called a convex metric space (X, d, W). Moreover, a nonempty subset K of X is said to be convex if $W(x, y; \lambda) \in K$ for all $(x, y; \lambda) \in K \times K \times [0, 1]$.

In fact, every linear normed space and its convex subset are special examples of convex metric spaces. Late on, Ciric [5], Wang [12-14] and Moosaei [15] considered some existence and convergence theorems for fixed points of some mappings in convex metric spaces. Such as uniformly convex Banach spaces, Shimizu and Takahashi [16] gave a notion of uniformly convex structure in convex metric spaces, and a convex metric space together with a uniformly convex structure is called a uniformly convex metric space. And then, some known fixed point results in uniformly convex Banach spaces have extended to the case of uniformly convex metric spaces by [17-20].

In this paper, inspired by the above results, we give some fixed point theorems for a generalized nonexpansive type mapping in convex metric spaces. Some iterations (which converge to fixed points of the given mappings) are given and in order to choose the more suitable iteration, the rate of convergence towards fixed points are presented.

Now, we need the following definitions and lemma for the proof of our main results:

Definition 1.1. Let K be a nonempty subset of a metric space (X, d), $T : K \to K$ is said to be a generalized nonexpansive type mapping if for some given number $\lambda \in [0, 1)$,

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \quad (1.3)$$

where $x, y \in K$ and

$$b + |c| \leq a + 2b + c + |c| + (1 - a)\lambda < 1. \quad (1.4)$$

Note that the coefficients (a, b, c) are not necessarily nonnegative in the above mapping. The mappings which considered in [1-11,15,18] can be obtained from the generalized nonexpansive type mapping as special cases by suitably choosing a, b, c and λ.

Definition 1.2. [21] Suppose two sequences $\{x_n\}$ and $\{y_n\}$, $n \in \mathbb{N}$ (the set of natural numbers) converge to the same point x^*, the following error estimates

$$d(x_n, x^*) \leq a_n, \quad d(y_n, x^*) \leq b_n,$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive real numbers (converging to zero). Let $l = \lim_{n\to\infty} \frac{a_n}{b_n}$.

(i) If $l = 0$, then it is said that the sequence $\{x_n\}$ converges to x^* faster than the sequence $\{y_n\}$ to x^* or, simply, that $\{x_n\}$ is better than $\{y_n\}$;

(ii) If $0 < l < \infty$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ have the same rate of convergence.

Lemma 1.3. [11] *Let (x, d, W) be a convex metric space, then the following statements hold:*

(i) $d(x, y) = d(x, W(x, y; \lambda)) + d(y, W(x, y; \lambda))$,

(ii) $d(x, W(x, y; \lambda)) = (1 - \lambda)d(x, y)$, $\quad d(y, W(x, y; \lambda)) = \lambda d(x, y)$

for all $x, y \in X$ and $\lambda \in I = [0, 1]$.

2 Main Results

Theorem 2.1. *Let (X, d, W) be a complete convex metric space, and K be a nonempty, closed and convex subset of X. If $T : K \to K$ be a generalized nonexpansive type mapping defined by Definition 1.1, then T has at least one fixed point in K. Moreover, if $a + 2c < 1$, then T has a unique fixed point in K.*

Proof. For a given number $\lambda \in [0, 1)$, defined a sequence $\{x_n\}_\lambda$ by

$$arbitray \;\; x_0 \in K, \quad x_n = W(x_{n-1}, Tx_{n-1}; \lambda) \tag{2.1}$$

for all $n \in \mathbb{N}$. From Lemma 1.1, we get

$$d(x_n, x_{n+1}) = d(x_n, W(x_n, Tx_n; \lambda)) = (1 - \lambda)d(x_n, Tx_n) \tag{2.2}$$

$$d(x_n, Tx_{n-1}) = d(Tx_{n-1}, W(x_{n-1}, Tx_{n-1}; \lambda)) = \lambda d(x_{n-1}, Tx_{n-1})$$

for all $n \in \mathbb{N}$. Therefore,

$$d(x_n, Tx_n) = \frac{1}{1-\lambda} d(x_n, x_{n+1}), \quad d(x_n, Tx_{n-1}) = \frac{\lambda}{1-\lambda} d(x_n, x_{n-1}) \tag{2.3}$$

for all $n \in \mathbb{N}$. By the triangle inequality and (2.3), we have

$$d(x_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) + d(x_n, Tx_n) = d(x_{n-1}, x_n) + \frac{1}{1-\lambda} d(x_n, x_{n+1})$$

and

$$d(x_{n-1},Tx_n) \geq d(x_{n-1},x_n) - d(x_n,Tx_n) = d(x_{n-1},x_n) - \frac{1}{1-\lambda}d(x_n,x_{n+1})$$

for all $n \in \mathbb{N}$. These imply that

$$cd(x_{n-1},Tx_n) \leq cd(x_{n-1},x_n) + \frac{|c|}{1-\lambda}d(x_n,x_{n+1}) \tag{2.4}$$

for all $n \in \mathbb{N}$. Now, substituting x with x_n and y with x_{n-1} in (1.3), we obtain

$$d(Tx_n,Tx_{n-1}) \leq ad(x_n,x_{n-1}) + b[d(x_n,Tx_n) + d(x_{n-1},Tx_{n-1})] + c[d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n)] \tag{2.5}$$

for all $n \in \mathbb{N}$. By $d(Tx_n,x_n) - d(x_n,Tx_{n-1}) \leq d(Tx_n,Tx_{n-1})$,(2.3), (2.4) and (2.5), it follows that

$$\frac{1}{1-\lambda}d(x_{n+1},x_n) - \frac{\lambda}{1-\lambda}d(x_n,x_{n-1}) \leq ad(x_n,x_{n-1}) + \frac{b}{1-\lambda}d(x_{n+1},x_n) + \frac{b}{1-\lambda}d(x_n,x_{n-1}) + \frac{c\lambda}{1-\lambda}d(x_n,x_{n-1}) + cd(x_n,x_{n-1}) + \frac{|c|}{1-\lambda}d(x_{n+1},x_n).$$

This implies that

$$\frac{1-b-|c|}{1-\lambda}d(x_{n+1},x_n) \leq \frac{a+b+c+(1-a)\lambda}{1-\lambda}d(x_n,x_{n-1})$$

for all $n \in \mathbb{N}$. By (1.4) and $\lambda \in [0,1)$, we get

$$d(x_{n+1},x_n) \leq \theta d(x_n,x_{n-1})$$

for all $n \in \mathbb{N}$, where $\theta = \frac{a+b+c+(1-a)\lambda}{1-b-|c|} \in [0,1)$. Thus,

$$d(x_{n+1},x_n) \leq \theta^n d(x_1,x_0)$$

where $x_1 = W(x_0,Tx_0;\lambda)$. For any $m > n$, we know that

$$d(x_m,x_n) \leq [\theta^n + \theta^{n+1} + \cdots + \theta^{m-1}]d(x_1,x_0) \leq \frac{\theta^n}{1-\theta}d(x_1,x_0). \tag{2.6}$$

$$d(x_m,x_n) \leq \frac{\theta}{1-\theta}d(x_{n-1},x_n). \tag{2.7}$$

Hence, $\{x_n\}$ is a Cauchy sequence in K. Since K is a closed subset in X, there exists $x^* \in K$ such that $\lim_{n\to\infty} x_n = x^*$. Letting $n \to \infty$ in (2.3), we get that $\lim_{n\to\infty} Tx_n = x^*$. By substituting x with x^* and y with x_n in (1.3), we have

$$d(Tx^*,Tx_n) \leq ad(x^*,x_n) + b[d(x^*,Tx^*) + d(x_n,Tx_n)] + c[d(x^*,Tx_n) + d(x_n,Tx^*)]$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we have

$$d(Tx^*, x^*) \le ad(x^*, x^*) + b[d(x^*, Tx^*) + d(x^*, x^*)] + c[d(x^*, x^*) + d(x^*, Tx^*)].$$

This implies that

$$d(Tx^*, x^*) \le (b + c)d(Tx^*, x^*)$$

Since $1 - b - |c|$ is positive, we can get that $d(Tx^*, x^*) = 0$. Thus, $x^* = Tx^*$. It concludes that T has at least one fixed point in K and $\{x_n\}$ converges to a fixed point of T.

If $a + 2c < 1$, assume that there exist $x_1, x_2 \in K$ and $Tx_1 = x_1,\ Tx_2 = x_2$. Substituting x with x_1 and y with x_2 in (1.3), we have

$$d(Tx_1, Tx_2) \le ad(x_1, x_2) + b[d(x_1, Tx_1) + d(x_2, Tx_2)] + c[d(x_1, Tx_2) + d(x_2, Tx_1)].$$

That is

$$d(x_1, x_2) \le (a + 2c)d(x_1, x_2).$$

This implies that $d(x_1, x_2) = 0$. Therefore, $x_1 = x_2$. It can conclude that T has a unique fixed point x^* in K and $\{x_n\}$ converges to x^*. □

Remark 2.2. *Let* $\Gamma = \Gamma(a, b, c, \lambda) = \{\lambda \in [0, 1) : b+|c| \le a+2b+c+|c|+(1-a)\lambda < 1\}$*, Theorem* 2.1 *shows that if* Γ *is nonempty, then the mapping* T *has at least one fixed point. Moreover, if* $a + 2c < 1$*, then* T *has a unique fixed point.*

From the Remark 2.2, if the mapping T has a unique fixed point, there may exist a family iterations $\{x_n\}_\lambda$ (see (2.1)) such that each of them could be used to approximate the unique fixed point x^*. In order to consider which iteration from the above family, i.e., which λ, would be the best one. We shall discuss the rate of convergece for the family iterations.

Theorem 2.3. *Let* T *and* $\{x_n\}_\lambda$ *considered in Theorem* 2.1 *and* $a + 2c < 1$*. For any* $x_0 \in K$*, we have the estimates*

$$d(x_n, x^*) \le \frac{\theta^n}{1-\theta} d(x_1, x_0) = \frac{[a + b + c + (1 - a)\lambda]^n}{(1 - b - c)^{n-1}[1 - a - 2b - c - |c| - (1 - a)\lambda]} d(x_1, x_0),$$
$$d(x_n, x^*) \le \frac{\theta}{1-\theta} d(x_{n-1}, x_n) = \frac{a + b + c + (1 - a)\lambda}{1 - a - 2b - c - |c| - (1 - a)\lambda} d(x_{n-1}, x_n),$$

where x^* *is the unique fixed point of* T*,* $\theta = \frac{a+b+c+(1-a)\lambda}{1-b-|c|} \in [0, 1)$ *and* $x_1 = W(x_0, Tx_0; \lambda)$*.*

Proof. Letting $m \to \infty$ in (2.6) and (2.7), we get the conclusion.

□

Remark 2.4. *In Theorem* 2.3*, suppose a family iterations* $\{x_n\}_\lambda$ *converge to the unique fixed point* x^* *(these* λ *constitute a set* Γ *). According to Definition* 1.2*, we know that* *(i) if $a < 1$ and there exists the minimum number of Γ (let the minimum number be λ_{min}), then $\{x_n\}_{\lambda_{min}}$ converges to x^* faster than the other iterations $\{x_n\}_{\lambda'}$ ($\lambda' \in \Gamma\backslash\{\lambda_{min}\}$) to x^*;*

(ii) if $a = 1$, the family iterations $\{x_n\}_\lambda$ ($\lambda \in \Gamma$) have the same rate of convergence;

(iii) if $a > 1$ and there exists the maximum number of Γ (let the maximum number be λ_{max}), then $\{x_n\}_{\lambda_{max}}$ converges to x^* faster than the other iterations $\{x_n\}_{\lambda''}$ ($\lambda'' \in \Gamma\backslash\{\lambda_{max}\}$) to x^*.

Example 2.5. Let $a = \frac{3}{4}, b = \frac{1}{16}, c = \frac{1}{32}$ and $\Gamma = [0, \frac{1}{4})$ in Theorem 2.2, then T has a unique fixed point x^*. A family iterations $\{x_n\}_\lambda$ ($\lambda \in \Gamma$) converge to the unigue fixed point x^*, and $\{x_n\}_{\lambda=0}$ converges to x^* faster than the other iterations $\{x_n\}_{\lambda'}$ ($0 < \lambda' < \frac{1}{4}$).

Example 2.6. Let $a = 1, b = \frac{-1}{16}, c = \frac{-1}{8}$ and $\Gamma = [0, 1)$ in Theorem 2.2, then T has a unique fixed point x^*. A family iterations $\{x_n\}_\lambda$ ($\lambda \in \Gamma$) converge to the unigue fixed point x^*, and these iterations have the same rate of convergence.

Example 2.7. Let $a = \frac{9}{8}, b = \frac{-1}{32}, c = \frac{-1}{8}$ and $\Gamma = (\frac{1}{2}, 1)$ in Theorem 2.2, then T has a unique fixed point x^*. A family iterations $\{x_n\}_\lambda$ ($\lambda \in \Gamma$) converge to the unigue fixed point x^*, and for any $0 < \varepsilon < \frac{1}{2}$, $\{x_n\}_{\lambda^*=1-\varepsilon}$ converges to x^* faster than the other iterations $\{x_n\}_{\lambda''}$ ($\frac{1}{2} < \lambda'' < 1 - \varepsilon$).

From Theorem 2.1 and Theorem 2.3, we have the following result in Banach spaces.

Corollary 2.8. *Let K be a nonempty, closed and convex subset of a Banach space X. For some given number $\lambda \in [0, 1)$, suppose $T : K \to K$ be a mapping such that*

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|]$$

for all $x, y \in K$, where

$$b + |c| \leq a + 2b + c + |c| + (1 - a)\lambda < 1.$$

Then,

(i) T has at least one fixed point in K. Moreover, if $a + 2c < 1$, then T has a unique fixed point in K;

(ii) if $a + 2c < 1$, defined a sequence $\{x_n\}$ by

$$\textit{arbitray}\ \ x_0 \in K, \quad x_n = \lambda x_{n-1} + (1 - \lambda)Tx_{n-1}$$

for all $n \in \mathbb{N}$, then $\{x_n\}$ converges to the unique fixed point x^*; Moreover, the following estimates hold:

$$d(x_n, x^*) \leq \frac{\theta^n}{1 - \theta} d(x_1, x_0), \quad d(x_n, x^*) \leq \frac{\theta}{1 - \theta} d(x_{n-1}, x_n)$$

where $\theta = \frac{a+b+c+(1-a)\lambda}{1-b-|c|}$ and $x_1 = \lambda x_0 + (1 - \lambda)Tx_0$.

Example 2.9. Let $X = \mathbb{R}$ (the set of real numbers) with the usual metric, $K = [0, \infty)$ and $T : K \to K$ be given by

$$Tx = \begin{cases} \frac{x}{4}, & x \in U = [0, \frac{1}{2}); \\ \frac{x}{8}, & x \in V = [\frac{1}{2}, \infty). \end{cases}$$

Then we have the following statements:

(i) T has a unique fixed point 0;

(ii) T satisfies the conditions of Corrollary 2.8 with $a = \frac{1}{4}, b = \frac{1}{3}, c = 0$ and $\lambda \in [0, \frac{1}{9})$ (which satisfies $b + |c| \le a + 2b + c + |c| + (1 - a)\lambda < 1$ and $a + 2c < 1$). In fact, let $m(x, y) = a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|]$, we know that

Case 1: For $x, y \in U$,

$$\begin{aligned}\|Tx - Ty\| &= |\frac{x}{4} - \frac{y}{4}| \le \frac{1}{4}|x - y| + \frac{1}{3}[|x - \frac{x}{4}| + |y - \frac{y}{4}|] + 0[|x - \frac{y}{4}| + |y - \frac{x}{4}|] \\ &= m(x, y),\end{aligned}$$

Case 2: For $x, y \in V$,

$$\begin{aligned}\|Tx - Ty\| &= |\frac{x}{8} - \frac{y}{8}| \le \frac{1}{4}|x - y| + \frac{1}{3}[|x - \frac{x}{8}| + |y - \frac{y}{8}|] + 0[|x - \frac{y}{8}| + |y - \frac{x}{8}|] \\ &= m(x, y),\end{aligned}$$

Case 3: For $x \in U, y \in V$,

$$\begin{aligned}\|Tx - Ty\| &= |\frac{x}{4} - \frac{y}{8}| \le \frac{x}{4} + \frac{y}{8} \le \frac{1}{4}|x - y| + \frac{1}{3}[\frac{3}{4}x + \frac{7}{8}y] + 0[|x - \frac{y}{8}| + |y - \frac{x}{4}|] \\ &= m(x, y),\end{aligned}$$

Case 4: For $x \in V, y \in U$,

$$\begin{aligned}\|Tx - Ty\| &= |\frac{x}{8} - \frac{y}{4}| \le \frac{x}{8} + \frac{y}{4} \le \frac{1}{4}|x - y| + \frac{1}{3}[\frac{7}{8}x + \frac{3}{4}y] + 0[|x - \frac{y}{4}| + |y - \frac{x}{8}|] \\ &= m(x, y);\end{aligned}$$

(iv) Let $a = \frac{1}{4}, b = \frac{1}{3}, c = 0$, a family iterations

$$x_n = \lambda x_{n-1} + (1 - \lambda)Tx_{n-1} \quad (\lambda \in [0, \frac{1}{9}))$$

converge to the unique fixed point 0, and $x_n = Tx_{n-1}$ converges to 0 faster than the other iterations $x_n = \lambda x_{n-1} + (1 - \lambda)Tx_{n-1}$ $(\lambda \in (0, \frac{1}{9}))$ to 0.

Acknowledgements : I would like to thank the referees for their comments and suggestions on the manuscript.

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(Received 19 March 2015)
(Accepted 6 September 2016)