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Study of Prime Graph of a Ring

Kishor F. Pawar[†,1] **and Sandeep S. Joshi**[‡]

[†]Department of Mathematics, School of Mathematical Sciences
Kavayitri Bahinabai Chaudhari North Maharashtra
University, Jalgaon-425 001, (M.S.) India
e-mail : kfpawar@gmail.com

[‡]Department of Mathematics, D.N.C.V.P's Shirish Madhukarrao
Chaudhari College, Jalgaon-425 001, (M.S.) India
e-mail : sandeep.s.joshi07@gmail.com

Abstract : The notion of a prime graph of a ring R, $(PG(R))$ was first introduced by S. Bhavanari and his coauthors in [1]. In this paper, we introduce the notion of 'Complement of a Prime Graph of a Ring R', denote it by $(PG(R))^c$ and find the degree of vertices in $PG(R)$ and $(PG(R))^c$ for the ring $\mathbb{Z}_n$ and the number of triangles in $PG(R)$ and $(PG(R))^c$. It is proved that for any $n \geq 6$ which not a prime then $gr(PG(\mathbb{Z}_n)) = 3$. If n is any prime number or $n = 4$ then $gr(PG(\mathbb{Z}_n)) = \infty$.

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1 Introduction

This paper builds a simple bridge between graph theory and the ring $\mathbb{Z}_n$ and simplified many concepts of [1]. In this paper we have given simple formulation for finding the degrees of vertices of prime graph $PG(R)$ as well as it's complement $(PG(R))^c$. Also the number of triangles in $PG(R)$ and $(PG(R))^c$ have been calculated using simple combinatorial approach. The present paper is divided into six sections. In Section 2, we collect some preliminary definitions from [1, 2]. In

[1]Corresponding author.

Section 3, we introduce the notion of complement of the prime graph of a ring R and denote it by $(PG(R))^c$ and is the (undirected) graph with vertices $V = R$ and two distinct vertices are adjacent if and only if $xRy \neq 0$ or $yRx \neq 0$. In the next Section 4, we have given some basic results on the degrees of prime graph of a ring and its complement. In Section 5, we define $Z(PG(R))$ and $Reg(PG(R))$ as the subgraphs of $PG(R)$, where $Reg(PG(R))$ be the subgraph of $PG(R)$ with vertex set as $Reg(R)$ and $Z(PG(R))$ be the subgraph of $PG(R)$ with vertex set as $Z(R)$. In Section 6, we have given some interesting results to find the number of triangles in $PG(R)$ and $(PG(R))^c$ and in the last section we compute the girth of the prime graph of a ring and proved that for any $n \geq 6$ which is not a prime then $gr(PG(\mathbb{Z}_n)) = 3$. If n is any prime number or $n = 4$ then $gr(PG(Z_n)) = \infty$.

2 Preliminaries

Definition 2.1. [2] A ring R is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following axioms:

(i) $(R, +)$ is an abelian group,

(ii) $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in R$,

(iii) The distributive law holds in R: for all $a, b, c \in R$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$, for all $a, b, c \in R$.

Moreover, if a ring R satisfies the condition $a \cdot b = b \cdot a$, for all $a, b \in R$, then we say that R is a commutative ring. If R contains the multiplicative identity i.e. $(1 \cdot a = a \cdot 1 = a$, for all $a \in R)$ then we say that R is a ring with identity.

Definition 2.2. [2] Let R be a ring. A nonzero element a of R is called a zero-divisor if there is a nonzero element b in R such that $ab = 0$ or $ba = 0$. The set of all zero-divisors in a ring R is denoted by $Z(R)$.

Definition 2.3. [2] An element of a ring R is said to be a regular element if it is neither a left zero-divisor nor a right zero-divisor. The set of all regular elements in a ring R is denoted by $Reg(R)$.

Here we are listing some preliminary definitions of graph theory and prime graph of a ring, for more details the reader is referred to [1] and [3].

Definition 2.4. A linear graph (or simply a graph) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots\}$ called vertices, and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. v_i and v_j are called the end vertices of e_k. A graph with a finite number of vertices as well as a finite number of edges is called a finite graph. A graph G, containing no edge is called null graph, that is $E = \emptyset$. An edge e between two vertices u and v is denoted by $e = (u, v)$.

Definition 2.5. An edge having the same vertex as both its end vertices is called self-loop and (simply a loop).

Definition 2.6. If two or more edges of graph G have the same end vertices then these edges of G are called as multiple edges or parallel edges.

Definition 2.7. A graph that has neither self-loops nor parallel edges is called a simple graph.

Definition 2.8. The number of edges incident to a vertex v is called the degree of the vertex v, and it is denoted by $d(v)$. The degree of vertex is also known as valancy.

Definition 2.9. A vertex in a graph G, having degree one is called as pendant vertex. A vertex in a graph G, having degree zero is called as an isolated vertex. A vertex v of graph G is called as odd vertex if $d(v)$ is odd. A vertex v of graph G is called as even vertex if $d(v)$ is even.

Definition 2.10. A walk is defined as a finite alternating sequence of vertices and edges, (no repetition of edge allowed) beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. A walk is said to be a closed walk if the terminal vertices are same. A walk that is not closed (i.e. the terminal vertices are distinct) is called an open walk.

Definition 2.11. An open walk in which no vertex appears more than once is called a path (or a simple path or an elementary path). The number of edges in a path is called the length of a path.

Definition 2.12. A graph G is said to be connected if there is at least one path between every pair of vertices in G otherwise graph is said to be disconnected. Moreover, each maximal connected subgraph of a graph G is called a component of the graph G.

Definition 2.13. The length of the shortest path between the vertices is called the distance between the vertices. The distance between the two vertices x and y is denoted by $d(x, y)$.

Definition 2.14. A simple graph in which every pair of vertices is adjacent is called as a complete graph. Complete graph on n vertices is denoted by K_n.

Definition 2.15. A graph G is said to be regular graph if all the vertices of a graph G have same degree.

Definition 2.16. [1] Let $G = (V, E)$ be a graph and $\emptyset \neq X \subseteq V$. Write $E_1 = \{xy \in E \mid x, y \in X\}$. Then $G_1 = (X, E_1)$ is a subgraph of G and it is called as the subgraph generated by X (or the maximal subgraph with vertex set X). If v_1, v_2, v_3 are vertices, and the maximal subgraph with vertex set $\{v_1, v_2, v_3\}$ forms a triangle, then we say that the set $\{v_1, v_2, v_3\}$ forms a triangle.

Definition 2.17. [1] A graph $G = (V, E)$ is said to be a star graph if there exists a fixed vertex v such that $E = \{vu \mid u \in V \text{ and } u \neq v\}$. A star graph is said to be a n-star graph if the number of vertices in the graph is n.

Definition 2.18. A connected graph without circuits is called a tree. It is clear that every star graph is a tree.

Definition 2.19. Let $G = (V, E)$ be a simple graph. The complement of G is denoted by G^c and is defined as the simple graph whose vertex set is same as that of G and two vertices are adjacent in G^c if and only if they are not adjacent in G.

Definition 2.20. The intersection of $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph consisting only those vertices and edges that are both in G_1 and G_2.

Definition 2.21. Two subgraphs G_1 and G_2 of G are edge-disjoint if G_1 and G_2 have no common edges and subgraphs that do not have common vertices are said to be vertex-disjoint subgraphs.

Definition 2.22. [1] Let R be a ring. A graph $G(V, E)$ is said to be a prime graph of R (denoted by $PG(R)$), if $V = R$ and $E = \{\overline{xy} \mid xRy = 0 \text{ or } yRx = 0, \text{ and } x \neq y\}$.

3 Complement of Prime Graph

In [1] Satyanarayana Bhavanari, Syam Prasad Kuncham and Nagaraju Dasari introduced the notion of prime graph of a ring. In this section we define 'Complement of Prime Graph of a Ring'. Some examples and observations are discussed on the line of [1].

Definition 3.1. Let R be a ring. A graph $G(V, E)$ is said to be a complement of prime graph of R(denoted by $(PG(R))^c$) if $V = R$ and $E = \{\overline{xy} \mid xRy \neq 0 \textit{ or } yRx \neq 0, \textit{ and } x \neq y\}$.

Note that complement of a prime graph for any ring R is always a simple graph.

Example 3.2. Consider $\mathbb{Z}_n$, the ring of integers modulo n.

1. Let us construct the prime graph $PG(R)$, where, $R = \mathbb{Z}_2$. We know that $R = \mathbb{Z}_2 = \{0, 1\}$. So, $V(PG(R)) = \{0, 1\}$. Since, $0R1 = 0$, there exists an edge between 0 and 1. There are no other edges. So $E(PG(R)) = \{01\}$. And hence there are no edges in the complement of this prime graph i.e. in $(PG(R))^c$.
2. Now, consider $R = \mathbb{Z}_3$. So, $V(PG(R)) = \{0, 1, 2\}$. Since,$0R1 = 0, 0R2 = 0$, So, $E(PG(R)) = \{01, 02\}$. Hence $E((PG(R))^c) = \{12\}$.
3. Now, consider $R = \mathbb{Z}_4$. So $V(PG(R)) = \{0, 1, 2, 3\}$ and $E(PG(R)) = \{01, 02, 03\}$. Therefore $E(PG(R))^c = \{12, 13, 23\}$.

Note 3.3. *The $(PG(\mathbb{Z}_n))^c$, where $n = 3, 4, 5$ is a disconnected graph with two components in which one component is a null graph and the other is a complete graph.*

Observations 3.4. *Let R be a ring and $(PG(R))^c$ be the complement of a prime graph.*

1. *For any prime p, then $(PG(\mathbb{Z}_p))^c$ is a disconnected graph with two components in which one component is a null graph containing one vertex and other component is always a complete graph with $(p-1)$ vertices.*

2. *In $(PG(R))^c$ if there are three non-zero elements x, y, z in R such that $xRy \neq 0$, $yRz \neq 0$, $xRz \neq 0$, then the subgraph generated by $\{x, y, z\}$ is a triangle graph.*

4 Degree of a Prime Graphs and it's Complement

Definition 4.1. The annihilator of an element a in R is the set of all elements $r \in R$ such that $ra = 0$ and is denoted by $ann(a)$. Therefore, $r = 0$ is contained in every $ann(a)$. Also a is contained in $ann(a)$ if and only if $a^2 = 0$.

Note that when $R = \mathbb{Z}_n$ the ring of integer modulo n then $|ann(v)| = gcd(v, n)$ and $Z(R)$ denotes the set of all zero divisors of a ring R.

Theorem 4.2. *Let $PG(R)$ be the prime graph of a ring R, then the degree of the vertex $v \in V(PG(R))$ is*

$$deg(v) = \begin{cases} |ann(v)|, & \text{if } v \in Z(R) \text{ and } v^2 \neq 0, \\ |ann(v)| - 1, & \text{if } v \in Z(R) \text{ and } v^2 = 0, \\ 1, & \text{if } v \notin Z(R). \end{cases}$$

Proof. Let $PG(R)$ be the prime graph of a ring R and $v \in Z(R)$. Then the vertex v is adjacent to all other vertices $w \in V(PG(R))$ such that $vw = 0$, and no vertex can be adjacent to itself, therefore $w \in ann(v)$. Thus, as $|ann(v)| = gcd(v, n)$ gives us

$$\begin{aligned} deg(v) &= |ann(v)|, && \text{If } v^2 \neq 0 \\ &= |ann(v)| - 1, && \text{If } v^2 = 0. \end{aligned}$$ □

Theorem 4.3. *Let $(PG(R))^c$ be the complement of the prime graph of a ring R and $v \in V(PG(R))^c$ then the degree of the vertex v is*

$$\begin{aligned} deg(v) &= n - gcd(v, n) - 1, && \text{If } v^2 \neq 0 \\ &= n - gcd(v, n), && \text{If } v^2 = 0. \end{aligned}$$

5 Subgraphs of $PG(R)$

In this section we define the subgraphs of the prime graph of R and some results are generalized for $PG(R)$ using [4]. Here we arrived at two general cases whether or not $Z(R)$ is an ideal of R. Let $Reg(R)$ be the set of regular elements. $Z(R)$ the set of zero-divisors. Let $Reg(PG(R))$ be the subgraph of $PG(R)$ with vertex as $Reg(R)$. Let $Z(PG(R))$ be the subgraph of $PG(R)$ with vertex set as $Z(R)$. When $Z(R)$ is not an ideal of R. The subgraph $Z(PG(R))$ of $PG(R)$ is always connected, and $Z(PG(R))$ is complete if and only if $Z(R)$ is an ideal of R where $R = \mathbb{Z}_n, n = p^2$, for any prime p. Moreover, $Z(PG(R))$ and $Reg(PG(R))$ are always an edge disjoint subgraphs of $PG(R)$.

Definition 5.1. Let G be a graph, where the distance between vertices u and v in G, denoted $d(u, v)$, is the minimum length of a (u, v)-path. If no (u, v)-path exists, i.e. if u and v lie in different components, then $d(u, v) = \infty$. Evidently, u and v are adjacent exactly when $d(u, v) = 1$. The diameter of G denoted by $diam(G) = max\ d(u, v)$, where $u \neq v$.

Theorem 5.2. *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then $Z(PG(R))$ is a complete subgraph of $PG(R)$ where $R = \mathbb{Z}_n, n = p^2$, for any prime p and $Z(PG(R))$ is an edge disjoint from $Reg(PG(R))$.*

Proof. Follows from the definitions. □

Theorem 5.3. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R, then*

1. *$Z(PG(R))$ is connected with $diam(Z(PG(R))) = 2$.*
2. *The subgraphs $Z(PG(R))$ and $Reg(PG(R))$ of $PG(R)$ are edge disjoint.*

Proof. (i) Each $x, y \in Z(R), x \neq 0 \neq y$ is always adjacent to 0. Thus $x - 0 - y$ is a path in $Z(PG(R))$ of length two between any two distinct $x, y \in Z(R)$. Moreover, there are non-adjacent $x, y \in Z(R)^* = Z(R) \setminus \{0\}$ since, $Z(R)$ is not an ideal of R. So, $diam(Z(PG(R))) = 2$.

(ii) Obvious and follows from the definitions. □

Observations 5.4.

1. *When $Z(R)$ not an ideal of is R, then $Z(PG(R))$ is a connected subgraph of $PG(R)$ but not complete.*
2. *$Reg(PG(R))$ is always a star graph and hence tree.*
3. *Vertices of triangles in $PG(R)$ are the elements of $Z(R)$.*

6 Number of Triangles in $PG(R)$ and $(PG(R))^c$

Theorem 6.1. *If a ring $R = \mathbb{Z}_n$, where $n = 2p, p \geq 3$ and p prime, then the number of triangles in $PG(R)$ is $(p-1)$.*

Proof. We know that, the vertices of triangles in $PG(R)$ are the elements of $Z(R)$, and the vertex 0 is adjacent to all other vertices of $PG(R)$. For $n = 2p, p \geq 3$ and p is prime, to obtain the number of triangles in $PG(R)$, we consider vertex p and we join the vertex p to those other vertices $q \in PG(R)$ which satisfies $p \cdot 2q = 0$, for $q = 1, 2, 3, \ldots, (p-1)$. Therefore, p is adjacent to $(p-1)$ number of vertices and these all together forms $(p-1)$ number of triangles. Thus we obtain $(p-1)$ number of triangles when $n = 2p, p \geq 3$ and p is prime. □

Theorem 6.2. *If a ring $R = \mathbb{Z}_n$, where $n = 3p, p \geq 5$ and p prime, then the number of triangles in $PG(R)$ is $2(p-1)$.*

Proof. As discussed above, to obtain number of triangles in $PG(R)$ for $n = 3p, p \geq 5$ and p is prime. We consider vertex p and we join the vertex p to those other vertices $q \in PG(R)$ which satisfies $p \cdot 3q = 0$, for $q = 1, 2, 3, \ldots, (p-1)$. Therefore, p is adjacent to $(p-1)$ number of vertices and these all together forms $(p-1)$ number of triangles.

Now we start with the vertex $2p$ and we join it to those other vertices satisfying $2p \cdot 3q = 0$, for $q = 1, 2, 3, \ldots, (p-1)$. Therefore $2p$ is adjacent to $(p-1)$ number of vertices and these all together forms again $(p-1)$ number of triangles. Now adding all these triangles we get,

$$(p-1) + (p-1) = 2(p-1).$$

Therefore, we obtain $2(p-1)$ number of triangles when $n = 3p, p \geq 5$ and p is prime. □

Theorem 6.3. *Let $R = \mathbb{Z}_n$ be a ring and $O(Z_n) = pq$, where p and q are distinct primes then the number of triangles in $PG(R)$ is $(p-1)(q-1)$.*

Theorem 6.4. *The number of triangles in a complete graph of n vertices, K_n is*

$$\triangle(K_n) = \frac{n(n-1)(n-2)}{6}.$$

Theorem 6.5. *The number of triangles in $PG(R), R = \mathbb{Z}_n$ is $\frac{p(p-1)(p-2)}{6}$ if $n = p^2, p$ is prime.*

Proof. We know that, for, $n = p^2$ the elements of $Z(R)$ are the vertices of triangle in $PG(R)$ and these vertices forms a complete graph on p vertices. Hence using Theorem 6.4 the number of triangles in $PG(R)$, is $\frac{p(p-1)(p-2)}{6}$ if $n = p^2, p$ is prime. □

Theorem 6.6. *If p is prime, the number of triangles in $(PG(R))^c, R = \mathbb{Z}_p$ is* $\dfrac{(p-1)(p-2)(p-3)}{6}$.

Proof. We know that for any prime p, the $(PG(Z_p))^c$ is a disconnected graph with two components in which one component is a null graph containing one vertex and other component is always a complete graph with $(p-1)$ vertices. Therefore using Theorem 6.4 and $n = p-1$, the number of triangles in $(PG(R))^c$, is $\dfrac{(p-1)(p-2)(p-3)}{6}$. □

7 Girth of Prime Graph of a Ring

In this section we prove that the prime graph of an integral domain has infinite girth. Also, we show that for $n \geq 6$ and n is not a prime number then $gr(PG(\mathbb{Z}_n)) = 3$ and $gr(PG(\mathbb{Z}_4)) = \infty$.

Definition 7.1. The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle. If no cycle exists then $gr(G) = \infty$. The shortest possible cycle consists of three pairwise adjacent vertices.

Theorem 7.2. *If $\mathbb{Z}_n$ is an integral domain then $gr(PG(\mathbb{Z}_n)) = \infty$.*

Theorem 7.3. *For $n \geq 6$ and n is not a prime number then $gr(PG(\mathbb{Z}_n)) = 3$. Also $gr(PG(\mathbb{Z}_4)) = \infty$.*

Proof. In $(PG(\mathbb{Z}_n))$, the vertices which forms a shortest possible cycle consists of three pair wise adjacent vertices, in which two adjacent vertices are the elements of $Z(R)$ (zero-divisors in $\mathbb{Z}_n$). Therefore, for any $x, y \in V(Z(R)), R = \mathbb{Z}_n, n \geq 6, x \neq y$, and n is not a prime then x and y are adjacent and form a shortest cycle of length 3. Thus, $gr(PG(\mathbb{Z}_n)) = 3$.

For, $n = 4$, the elements in $Z(\mathbb{Z}_4) = \{0, 2\}$ are adjacent but does not form a shortest cycle of length 3. Therefore, $gr(PG(\mathbb{Z}_4)) = \infty$. □

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