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On φ-$\mathcal{T}$-Symmetric (ε)-Para Sasakian Manifolds

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Abstract : The purpose of the present paper is to study the globally and locally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold. The globally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold is either Einstein manifold or has a constant scalar curvature. The necessary and sufficient condition for Einstein manifold to be globally φ-$\mathcal{T}$ -symmetric is given. A 3-dimensional (ε) -para Sasakian manifold is locally φ-$\mathcal{T}$-symmetric if and only if the scalar curvature r is constant. A 3-dimensional (ε)-para Sasakian manifold with η-parallel Ricci tensor is locally φ-$\mathcal{T}$-symmetric. In the last, an example of 3-dimensional locally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold is given.

Keywords : $\mathcal{T}$-curvature tensor; (ε)-para Sasakian manifold; globally and locally φ-$\mathcal{T}$-symmetric manifold; η-parallel Ricci tensor.
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1 Introduction

Let M be an m-dimensional semi-Riemannian manifold and ∇ the Levi-Civita connection on M. A semi-Riemannian manifold M is said to recurrent [1] if the Riemann curvature tensor R satisfies the relation

$$(\nabla_U R)(X,Y,Z,V) = \alpha(U) R(X,Y,Z,V), \quad X,Y,Z,V,U \in TM,$$

where α is 1-form. If $\alpha = 0$, then M is called symmetric in the sense of Cartan [2].

In 1977, Takahashi [3] introduced the notion of locally φ-symmetry on a Sasakian manifold, which is weaker than the local symmetry. A Sasakian manifold is said to have locally φ-symmetry if it satisfies

$$\varphi^2\left((\nabla_U R)(X,Y)Z\right)=0,$$

where X, Y, Z, U are horizontal vector fields. If X, Y, Z, U are arbitrary vector fields, then it is known as globally φ-symmetric Sasakian manifold. A φ-symmetric space condition is weak condition for a Sasakian manifold in comparision to the symmetric space condition. Local symmetry is a very strong condition for the class of K-contact or Sasakian manifolds. Indeed, such spaces must have constant curvature equal to 1 ([4, 5]). On the other hand, local symmetry is also a very strong condition for the class of (ε)-para Sasakian manifold. Such spaces must have constant curvature equal to $-\varepsilon$ [6]. In 2010, Tripathi et al. [6] proved that the condition of semi-symmetry ($R\cdot R=0$), symmetry and have a constant curvature $-\varepsilon$ is equivalent for (ε)-para Sasakian manifold.

Three-dimensional locally φ-symmetric Sasakian manifold is studied by Watanabe [7]. Many authors like De [8], De et al. [9], De and Pathak [10], Shaikh and De [11] have extended this notion to 3-dimensional Kenmotsu, trans-Sasakian and LP-Sasakian manifolds. Yildiz et al. [12] studied the case for 3-dimensional α-Sasakian manifolds and gave the example for locally φ-symmetric 3-dimensional α-Sasakian manifolds. De and De [13] studied the φ-concircularly symmetric Kenmotsu manifold and gave the example of such manifold in dimension 3. De et al. [14] studied the 3-dimensional globally and locallly φ-quasiconformally symmetric Sasakian manifolds and also gave the example.

In the present work, globally and locally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold are studied. The paper is organized as follows: Section 2 and 3 is devoted to the study of $\mathcal{T}$-curvature tensor and (ε)-para Sasakian manifold, respectively. In section 4, the definition of globally and locally φ-$\mathcal{T}$-symmetric manifold are given. Globally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold is either Einstein or has a constant scalar curvature under some condition. The necessary and sufficient condition for locally φ-$\mathcal{T}$-symmetric 3-dimensional (ε)-para Sasakian manifold to be locally φ-symmetric is given. In section 5, the definition of η-parallel (ε)-para Sasakian manifold is given. A 3-dimensional (ε)-para Sasakian manifold with η-parallel Ricci tensor is locally φ-$\mathcal{T}$-symmetric. In the last section, the example of a locally φ-$\mathcal{T}$-symmetric in 3-dimensional (ε)-para Sasakian manifold is given.

2 $\mathcal{T}$-Curvature Tensor

The definition of $\mathcal{T}$-curvature tensor [15] is given by

Definition 2.1. In an m-dimensional semi-Riemannian manifold (M,g), the $\mathcal{T}$-

curvature tensor of type $(1,3)$ defined by

$$\begin{aligned}\mathcal{T}(X,Y)Z &= a_0 R(X,Y)Z \\ &\quad + a_1 S(Y,Z)X + a_2 S(X,Z)Y + a_3 S(X,Y)Z \\ &\quad + a_4 g(Y,Z)QX + a_5 g(X,Z)QY + a_6 g(X,Y)QZ \\ &\quad + a_7 r\left(g(Y,Z)X - g(X,Z)Y\right), \end{aligned} \tag{2.1}$$

for all $X, Y, Z \in TM$, where $a_0, \dots, a_7$ are some constants; and R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator of type $(1,1)$ and the scalar curvature respectively.

In particular, the $\mathcal{T}$-curvature tensor is reduced to

1. the *Riemann curvature tensor* R if

$$a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

2. the *quasiconformal curvature tensor* $\mathcal{C}_*$ [16] if

$$a_1 = -a_2 = a_4 = -a_5, \quad a_3 = a_6 = 0, \quad a_7 = -\frac{1}{m}\left(\frac{a_0}{m-1} + 2a_1\right),$$

3. the *conformal curvature tensor* $\mathcal{C}$ [17, p. 90] if

$$a_0 = 1, \quad a_1 = -\,a_2 = a_4 = -a_5 = -\frac{1}{m-2}, \quad a_3 = a_6 = 0,$$
$$a_7 = \frac{1}{(m-1)(m-2)},$$

4. the *conharmonic curvature tensor* $\mathcal{L}$ [18] if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{m-2}, \quad a_3 = a_6 = 0, \quad a_7 = 0,$$

5. the *concircular curvature tensor* $\mathcal{V}$ ([19, 20, p. 87]) if

$$a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{m(m-1)},$$

6. the *pseudo-projective curvature tensor* $\mathcal{P}_*$ [21] if

$$a_1 = -a_2, \quad a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{m}\left(\frac{a_0}{m-1} + a_1\right),$$

7. the *projective curvature tensor* $\mathcal{P}$ [20, p. 84] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(m-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

8. the *M-projective curvature tensor* [22] if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(m-1)}, \quad a_3 = a_6 = a_7 = 0,$$

9. the W_0-*curvature tensor* [22, eq (1.4)] if

$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{(m-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

10. the W_0^*-*curvature tensor* [22, eq (1.4)] if

$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{(m-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

11. the W_1-*curvature tensor* [22] if

$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{(m-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

12. the W_1^*-*curvature tensor* [22] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(m-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

13. the W_2-*curvature tensor* [23] if

$$a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{(m-1)}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

14. the W_3-*curvature tensor* [22] if

$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{(m-1)}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

15. the W_4-*curvature tensor* [22] if

$$a_0 = 1, \quad a_5 = -a_6 = \frac{1}{(m-1)}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

16. the W_5-*curvature tensor* [24] if

$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{(m-1)}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

17. the W_6-*curvature tensor* [24] if

$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{(m-1)}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

18. the W_7-*curvature tensor* [24] if

$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{(m-1)}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

19. the W_8-*curvature tensor* [24] if

$$a_0 = 1, \quad a_1 = -a_3 = -\frac{1}{(m-1)}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

20. the W_9-*curvature tensor* [24] if

$$a_0 = 1, \quad a_3 = -a_4 = \frac{1}{(m-1)}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

3 (ε)-Para Sasakian Manifold

A manifold M is said to admit an almost paracontact structure if it admit a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{3.1}$$

Let g be a semi-Riemannian metric with index$(g) = \nu$ such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \qquad X, Y \in TM, \tag{3.2}$$

where $\varepsilon = \pm 1$. Then M is called an (ε)-*almost paracontact metric manifold* equipped with an (ε)-*almost paracontact metric structure* $(\varphi, \xi, \eta, g, \varepsilon)$. In particular, if index$(g) = 1$, then an (ε)-almost paracontact metric manifold is said to be a *Lorentzian almost paracontact manifold*. In particular, if the metric g is positive definite, then an (ε)-almost paracontact metric manifold is the usual *almost paracontact metric manifold* [25].

The equation (3.2) is equivalent to

$$g(X, \varphi Y) = g(\varphi X, Y) \tag{3.3}$$

along with

$$g(X, \xi) = \varepsilon\eta(X). \tag{3.4}$$

From (3.1) and (3.4) it follows that

$$g(\xi, \xi) = \varepsilon. \tag{3.5}$$

Definition 3.1. An (ε)-almost paracontact metric structure is called an (ε)-*para Sasakian structure* if

$$(\nabla_X \varphi) Y = -g(\varphi X, \varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X, \qquad X, Y \in TM, \tag{3.6}$$

where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with an (ε)-para Sasakian structure is called an (ε)-*para Sasakian manifold* [6].

For $\varepsilon = 1$ and g Riemannian, M is the usual para Sasakian manifold [26, 27]. For $\varepsilon = -1$, g Lorentzian and ξ replaced by $-\xi$, M becomes a Lorentzian para Sasakian manifold [28].

For (ε)-para Sasakian manifold, it is easy to prove that

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.7}$$

$$R(\xi,X)Y = \eta(Y)X - \varepsilon g(X,Y)\xi, \tag{3.8}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{3.9}$$

$$R\left(X,Y,Z,\xi\right) = \eta\left(Y\right)g\left(X,Z\right) - \eta\left(X\right)g\left(Y,Z\right), \tag{3.10}$$

$$\eta\left(R\left(X,Y\right)Z\right) = \varepsilon(\eta\left(Y\right)g\left(X,Z\right) - \eta\left(X\right)g\left(Y,Z\right)), \tag{3.11}$$

$$S(X,\xi) = -(m-1)\eta(X), \tag{3.12}$$

$$Q\xi = -\,\varepsilon(m-1)\xi, \tag{3.13}$$

$$S(\xi,\xi) = -(m-1), \tag{3.14}$$

$$S(\varphi X,\varphi Y) = S(Y,Z) + (m-1)\eta\left(X\right)\eta\left(Y\right), \tag{3.15}$$

$$\nabla_X\xi = \varepsilon\varphi X. \tag{3.16}$$

For detail study of (ε)-para Sasakian manifold, see [6].

4 φ-$\mathcal{T}$-Symmetric (ε)-Para Sasakian Manifold

We begin with the following definition.

Definition 4.1. An (ε)-para Sasakian manifold is said to be locally φ-$\mathcal{T}$-symmetric manifold if

$$\varphi^2((\nabla_W\mathcal{T})(X,Y)Z) = 0, \tag{4.1}$$

for arbitrary vector fields X,Y,Z,W orthogonal to ξ. If X, Y, Z, W are arbitrary vector fields, then it is known as globally φ-$\mathcal{T}$-symmetric manifold.

This notion of locally φ-symmetric was introduced by Takahashi for Sasakian manifolds [3].

Theorem 4.2. *Let M be a m-dimensional globally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold. Then*

(i) *M is Einstein manifold if $a_0 + (m-1)a_1 + a_2 + a_6 \neq 0$.*

(ii) *M has constant scalar curvature if $a_0 + (m-1)a_1 + a_2 + a_6 = 0$ and $a_4 + (m-1)a_7 \neq 0$.*

Proof. Let M be a m-dimensional globally φ-$\mathcal{T}$-symmetric (ε)-para Sasakian manifold. Then by using (3.1) and (4.1), we have

$$(\nabla_W \mathcal{T})(X,Y)Z - \eta\left((\nabla_W \mathcal{T})(X,Y)Z\right)\xi = 0,$$

from which it follows that

$$g((\nabla_W \mathcal{T})(X,Y)Z,U) - \eta\left((\nabla_W \mathcal{T})(X,Y)Z\right)g(\xi,U) = 0. \tag{4.2}$$

Using (2.1) in (4.2), we obtain

$$\begin{aligned} 0 = {} & a_0\,(\nabla_W R)(X,Y,Z,U) + a_1\,(\nabla_W S)\,(Y,Z)\,g(X,U) + a_2\,(\nabla_W S)\,(X,Z)\,g(Y,U) \\ & + a_3\,(\nabla_W S)(X,Y)g(Z,U) + a_4\,(\nabla_W S)(X,U)g\,(Y,Z) + a_5\,(\nabla_W S)(Y,U)g\,(X,Z) \\ & + a_6\,(\nabla_W S)(Z,U)g(X,Y) + a_7\,(\nabla_W r)\,(g\,(Y,Z)\,g(X,U) - g\,(X,Z)\,g(Y,U)) \\ & + \eta(U)\,(a_0\,(\nabla_W R)(X,Y,Z,\xi) + a_1(\nabla_W S)\,(Y,Z)\,g(X,\xi) + a_2\,(\nabla_W S)\,(X,Z)\,g(Y,\xi) \\ & + a_3\,(\nabla_W S)(X,Y)g(Z,\xi) + a_4\,g\,(Y,Z)\,(\nabla_W S)(X,\xi) + a_5\,g\,(X,Z)\,(\nabla_W S)(Y,\xi) \\ & + a_6\,g(X,Y)(\nabla_W S)(Z,\xi) + a_7\,(\nabla_W r)\,(g\,(Y,Z)\,g(X,\xi) - g\,(X,Z)\,g(Y,\xi)))\,. \end{aligned} \tag{4.3}$$

Let $\{e_i\}$, $i = 1,\ldots,m$ be an orthonormal basis of tangent space at any point of the manifold. Taking $X = U = e_i$ in (4.3), we get

$$\begin{aligned} 0 \;=\; & (a_0 + (m-1)a_1 + a_2 + a_3 + a_5 + a_6)\;(\nabla_W S)(Y,Z) \\ & - a_0\varepsilon \sum_{i=1}^{m} (\nabla_W R)\,(e_i,Y,Z,\xi)\,g(e_i,\xi) \\ & + (a_4 + (m-1)a_7)\;(\nabla_W r)g\,(Y,Z) + a_7\,(\nabla_W r)\,(g\,(Y,Z) - \varepsilon\eta(Y)\eta(Z)) \\ & - (a_2 + a_6)\;(\nabla_W S)(Z,\xi)\eta(Y) - (a_3 + a_5)\;(\nabla_W S)(Y,\xi)\eta(Z). \end{aligned} \tag{4.4}$$

Putting $Z = \xi$ in (4.4), we have

$$\begin{aligned} 0 \;=\; & (a_0 + (m-1)a_1 + a_2 + a_6)\;(\nabla_W S)(Y,\xi) \\ & - a_0\varepsilon \sum_{i=1}^{m} (\nabla_W R)\,(e_i,Y,\xi,\xi)\,g(e_i,\xi) \\ & + (a_4 + (m-1)a_7)\;(\nabla_W r)g\,(Y,\xi) \\ & - (a_2 + a_6)\;(\nabla_W S)(\xi,\xi)\eta(Y). \end{aligned} \tag{4.5}$$

Since, we have

$$\begin{aligned} (\nabla_W R)\,(e_i,Y,\xi,\xi) \;&=\; g((\nabla_W R)\,(e_i,Y)\xi,\xi) \\ &=\; g(\nabla_W R\,(e_i,Y)\xi,\xi) - g(R\,(\nabla_W e_i,Y)\xi,\xi) \\ &\quad\; - g(R\,(e_i,\nabla_W Y)\xi,\xi) - g(R\,(e_i,Y)\nabla_W \xi,\xi) \end{aligned} \tag{4.6}$$

at any point $p \in M$. We know that $\{e_i\}$ is an orthonormal basis, therefore $\nabla_W e_i = 0$ at p. Using (3.4) and (3.7) in (4.6), we have

$$(\nabla_W R)\,(e_i,Y,\xi,\xi) = g(\nabla_W R\,(e_i,Y)\xi,\xi) - g(R\,(e_i,Y)\nabla_W \xi,\xi)\,. \tag{4.7}$$

By using the property of curvature tensor

$$g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0,$$

we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0. \tag{4.8}$$

By (4.7) and (4.8), we get

$$(\nabla_W R)\,(e_i, Y, \xi, \xi) = 0. \tag{4.9}$$

We know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \tag{4.10}$$

Using (3.12), (3.16) in (4.10), we get

$$\begin{aligned} (\nabla_W S)(Y, \xi) &= \nabla_W(-(m-1)\eta(Y)) + (m-1)\eta\,(\nabla_W Y) - S(Y, \varepsilon\varphi W) \\ &= -(m-1)\varepsilon g(Y, \varepsilon\varphi W) - \varepsilon S(Y, \varphi W) \\ &= -(m-1)g(Y, \varphi W) - \varepsilon S(Y, \varphi W). \end{aligned} \tag{4.11}$$

By (4.11), we have

$$(\nabla_W S)(\xi, \xi) = 0. \tag{4.12}$$

Using (4.9), (4.11), (4.12) in (4.5), we have

$$\begin{aligned} 0 &= (a_0 + (m-1)a_1 + a_2 + a_6)\,(-(m-1)g(Y, \varphi W) - \varepsilon S(Y, \varphi W)) \\ &\quad +\varepsilon\,(a_4 + (m-1)a_7)\;(\nabla_W r)\eta(Y). \end{aligned} \tag{4.13}$$

Replacing Y by φY in (4.13) and using (3.2), (3.15), we get

$$S(Y, W) = -\varepsilon(m-1)g(Y, W), \quad a_0 + (m-1)a_1 + a_2 + a_6 \neq 0.$$

If $a_0 + (m-1)a_1 + a_2 + a_6 = 0$ and $a_4 + (m-1)a_7 \neq 0$, then by (4.5), we have $\nabla_W r = 0$, that is, $r =$ constant. □

Remark 4.3. *The first condition of Theorem 4.2 is satisfied if* $\mathcal{T} \in \{R, \mathcal{C}_*, \mathcal{V}, \mathcal{P}_*, \mathcal{P}, \mathcal{M}, \mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_2, \ldots, \mathcal{W}_6, \mathcal{W}_9\}$ *and second condition is satisfied if* $\mathcal{T} \in \{\mathcal{L}, \mathcal{W}_7\}$. *If* $\mathcal{T} \in \{\mathcal{C}, \mathcal{W}_0, \mathcal{W}_8\}$ *none of the condition is satisfied.*

Theorem 4.4. *An Einstein manifold is globally* φ-$\mathcal{T}$*-symmetric iff it is globally* φ*-symmetric and* $a_0 \neq 0$.

Proof. By using (2.1) and (4.1), we have the result. □

Remark 4.5. *For all known curvature tensors* $a_0 \neq 0$.

5 3-Dimensional Locally φ-$\mathcal{T}$-Symmetric (ε)-Para Sasakian Manifold

It is well known that in a 3-dimensional semi-Riemannian manifold the conformal curvature tensor $\mathcal{C}$ vanishes, therefore

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{5.1}$$

Take $Z=\xi$ in (5.1) and using (3.4), (3.7), (3.12), we get

$$\left(\frac{\varepsilon r}{2}+1\right)(\eta(Y)X-\eta(X)Y) = \varepsilon\,(\eta(Y)QX-\eta(X)QY)\,. \tag{5.2}$$

Putting $Y=\xi$ in (5.2) and using (3.13), we get

$$QX = \left(\frac{r}{2}+\varepsilon\right)X - \left(\frac{r}{2}+3\varepsilon\right)\eta(X)\xi. \tag{5.3}$$

Then by (5.3), we easily obtain

$$S(X,Y) = \left(\frac{r}{2}+\varepsilon\right)g(X,Y) - \left(\frac{\varepsilon r}{2}+3\right)\eta(X)\eta(Y) \tag{5.4}$$

and

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2}+2\varepsilon\right)(g(Y,Z)X-g(X,Z)Y) \\ &\quad +\left(\frac{\varepsilon r}{2}+3\right)(\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X) \\ &\quad +\left(\frac{r}{2}+3\varepsilon\right)(g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi). \end{aligned} \tag{5.5}$$

Lemma 5.1. *A 3-dimensional (ε)-para Sasakian manifold is a manifold of constant curvature $-\varepsilon$ if and only if $r=-6\varepsilon$.*

Corollary 5.2. *Let M be a 3-dimensional (ε)-para Sasakian manifold. Then*

$$\begin{aligned} \mathcal{T}(X,Y)Z &= \left(\left(\frac{r}{2}+\varepsilon\right)(a_0+a_1+a_4)+a_7r+\varepsilon a_0\right)g(Y,Z)X \\ &\quad -\left(\left(\frac{r}{2}+\varepsilon\right)(a_0-a_2-a_5)+a_7r+\varepsilon a_0\right)g(X,Z)Y \\ &\quad +\left(\frac{r}{2}+\varepsilon\right)(a_3+a_6)\,g(X,Y)Z-\left(\frac{\varepsilon r}{2}+3\right)a_3\eta(X)\eta(Y)Z \\ &\quad -\left(\frac{\varepsilon r}{2}+3\right)(a_0+a_1)\,\eta(Y)\eta(Z)X+\left(\frac{\varepsilon r}{2}+3\right)(a_0-a_2)\,\eta(X)\eta(Z)Y \\ &\quad +\left(\frac{r}{2}+3\varepsilon\right)(a_0-a_5)\,g(X,Z)\eta(Y)\xi-\left(\frac{r}{2}+3\varepsilon\right)a_6g(X,Y)\eta(Z)\xi \\ &\quad -\left(\frac{r}{2}+3\varepsilon\right)(a_0+a_4)\,g(Y,Z)\eta(X)\xi. \end{aligned} \tag{5.6}$$

Theorem 5.3. *Let M be a 3-dimensional (ε)-para Sasakian manifold. M is locally φ-$\mathcal{T}$-symmetric manifold if and only if the scalar curvature r is constant.*

Proof. Let M be a 3-dimensional (ε)-para Sasakian manifold. Differentiate covariantly on both sides of (5.6), we have

$$\begin{aligned}
(\nabla_W \mathcal{T})(X,Y)Z \;=\; & \frac{\nabla_W r}{2}\left(a_0+a_1+a_4+2a_7\right) g(Y,Z)X \\
& -\frac{\nabla_W r}{2}\left(a_0-a_2-a_5+2a_7\right) g(X,Z)Y \\
& +\frac{\nabla_W r}{2}\left(a_3+a_6\right) g(X,Y)Z \\
& -\frac{\nabla_W r}{2} a_3\eta(X)\eta(Y)Z \\
& -\left(\frac{\varepsilon r}{2}+3\right) a_3\left(\nabla_W \eta\right)(X)\eta(Y)Z \\
& -\left(\frac{\varepsilon r}{2}+3\right) a_3\eta(X)\left(\nabla_W \eta\right)(Y)Z \\
& -\frac{\nabla_W r}{2}\left(a_0+a_1\right)\eta(Y)\eta(Z)X \\
& -\left(\frac{\varepsilon r}{2}+3\right)\left(a_0+a_1\right)\left(\nabla_W \eta\right)(Y)\eta(Z)X \\
& -\left(\frac{\varepsilon r}{2}+3\right)\left(a_0+a_1\right)\eta(Y)\left(\nabla_W \eta\right)(Z)X \\
& +\frac{\nabla_W r}{2}\left(a_0-a_2\right)\eta(X)\eta(Z)Y \\
& +\left(\frac{\varepsilon r}{2}+3\right)\left(a_0-a_2\right)\left(\nabla_W \eta\right)(X)\eta(Z)Y \\
& +\left(\frac{\varepsilon r}{2}+3\right)\left(a_0-a_2\right)\eta(X)\left(\nabla_W \eta\right)(Z)Y \\
& +\frac{\nabla_W r}{2}\left(a_0-a_5\right) g(X,Z)\eta(Y)\xi \\
& +\left(\frac{r}{2}+3\varepsilon\right)\left(a_0-a_5\right) g(X,Z)\left(\nabla_W \eta\right)(Y)\xi \\
& +\left(\frac{r}{2}+3\varepsilon\right)\left(a_0-a_5\right) g(X,Z)\eta(Y)\nabla_W \xi \\
& -\left(\frac{r}{2}+3\varepsilon\right)\left(a_0+a_4\right) g(Y,Z)\left(\nabla_W \eta\right)(X)\xi \\
& -\left(\frac{r}{2}+3\varepsilon\right)\left(a_0+a_4\right) g(Y,Z)\eta(X)\nabla_W \xi \\
& -\left(\frac{r}{2}+3\varepsilon\right) a_6 g(X,Y)\eta(Z)\nabla_W \xi \\
& -\frac{\nabla_W r}{2}\left(a_0+a_4\right) g(Y,Z)\eta(X)\xi \\
& -\frac{\nabla_W r}{2} a_6 g(X,Y)\eta(Z)\xi \\
& -\left(\frac{r}{2}+3\varepsilon\right) a_6 g(X,Y)\left(\nabla_W \eta\right)(Z)\xi. \qquad (5.7)
\end{aligned}$$

Applying φ^2 on both sides of (5.7), we have

$$\begin{aligned}
\varphi^2(\nabla_W \mathcal{T})(X,Y)Z &= \frac{\nabla_W r}{2}(a_0+a_1+a_4+2a_7)\,g(Y,Z)(X-\eta(X)\xi) \\
&-\frac{\nabla_W r}{2}(a_0-a_2-a_5+2a_7)\,g(X,Z)(Y-\eta(Y)\xi) \\
&+\frac{\nabla_W r}{2}(a_3+a_6)\,g(X,Y)(Z-\eta(Z)\xi) \\
&-\frac{\nabla_W r}{2}a_3\eta(X)\eta(Y)(Z-\eta(Z)\xi) \\
&-\left(\frac{\varepsilon r}{2}+3\right)a_3\left(\nabla_W\eta\right)(X)\eta(Y)(Z-\eta(Z)\xi) \\
&-\left(\frac{\varepsilon r}{2}+3\right)a_3\eta(X)\left(\nabla_W\eta\right)(Y)(Z-\eta(Z)\xi) \\
&-\frac{\nabla_W r}{2}(a_0+a_1)\,\eta(Y)\eta(Z)(X-\eta(X)\xi) \\
&-\left(\frac{\varepsilon r}{2}+3\right)(a_0+a_1)\left(\nabla_W\eta\right)(Y)\eta(Z)(X-\eta(X)\xi) \\
&-\left(\frac{\varepsilon r}{2}+3\right)(a_0+a_1)\,\eta(Y)\left(\nabla_W\eta\right)(Z)(X-\eta(X)\xi) \\
&+\frac{\nabla_W r}{2}(a_0-a_2)\,\eta(X)\eta(Z)(Y-\eta(Y)\xi) \\
&+\left(\frac{\varepsilon r}{2}+3\right)(a_0-a_2)\left(\nabla_W\eta\right)(X)\eta(Z)(Y-\eta(Y)\xi) \\
&+\left(\frac{\varepsilon r}{2}+3\right)(a_0-a_2)\,\eta(X)\left(\nabla_W\eta\right)(Z)(Y-\eta(Y)\xi) \\
&+\left(\frac{r}{2}+3\varepsilon\right)(a_0-a_5)\,g(X,Z)\eta(Y)\varphi^2\nabla_W\xi \\
&-\left(\frac{r}{2}+3\varepsilon\right)a_6 g(X,Y)\eta(Z)\varphi^2\nabla_W\xi \\
&-\left(\frac{r}{2}+3\varepsilon\right)(a_0+a_4)\,g(Y,Z)\eta(X)\varphi^2\nabla_W\xi. \qquad (5.8)
\end{aligned}$$

Using the fact that X, Y, Z are horizontal vector fields in (5.8), we get

$$\begin{aligned}
\varphi^2(\nabla_W \mathcal{T})(X,Y)Z &= \frac{\nabla_W r}{2}(a_0+a_1+a_4+2a_7)\,g(Y,Z)X \\
&-\frac{\nabla_W r}{2}(a_0-a_2-a_5+2a_7)\,g(X,Z)Y \\
&+\frac{\nabla_W r}{2}(a_3+a_6)\,g(X,Y)Z. \qquad (5.9)
\end{aligned}$$

If one of them $a_0+a_1+a_4+2a_7$, $a_0-a_2-a_5+2a_7$ and a_3+a_6 is not equal to zero, then by using (4.1), we get the result. □

Remark 5.4. *One of them $a_0+a_1+a_4+2a_7$, $a_0-a_2-a_5+2a_7$ and a_3+a_6 is not equal to zero, for all the known curvature tensors.*

6 η-Parallel Ricci Tensor

Definition 6.1. The Ricci tensor S of an (ε)-para-Sasakian manifold is called η-parallel Ricci tensor if it satisfies

$$(\nabla_X S)(\varphi Y, \varphi Z) = 0$$

for all vector fields X, Y and Z.

Theorem 6.2. *In a 3-dimensional (ε)-para Sasakian manifold with η-parallel Ricci tensor, the scalar curvature r is constant.*

Proof. By equation (5.4), we get

$$S(\varphi Y, \varphi Z) = \left(\frac{r}{2} + \varepsilon\right)(g(Y,Z) - \varepsilon\eta(Y)\eta(Z)). \tag{6.1}$$

Differentiating (6.1) covariantly with respect to X, we get

$$\begin{aligned}(\nabla_X S)(\varphi Y, \varphi Z) &= \frac{\nabla_X r}{2}(g(Y,Z) - \varepsilon\eta(Y)\eta(Z)) - \varepsilon\left(\frac{r}{2} + \varepsilon\right)((\nabla_X \eta)(Y)\eta(Z) \\ &\quad + \eta(Y)(\nabla_X \eta)(Z)).\end{aligned}$$

Suppose the Ricci tensor is η-parallel. Then from the above, we obtain

$$\frac{\nabla_X r}{2}(g(Y,Z) - \varepsilon\eta(Y)\eta(Z)) = \varepsilon\left(\frac{r}{2} + \varepsilon\right)((\nabla_X \eta)(Y)\eta(Z) + \eta(Y)(\nabla_X \eta)(Z)). \tag{6.2}$$

Let $\{e_i\}$, $i = 1, 2, 3$ be the orthonormal basis of tangent space at each point of the manifold. Taking $Y = e_i = Z$ in (6.2), we have $\nabla_X r = 0$. Hence scalar curvature r is constant. □

From Theorems 5.3 and 6.2, we can state the following:

Corollary 6.3. *A 3-dimensional (ε)-para Sasakian manifold with η-parallel Ricci tensor is locally φ-$\mathcal{T}$-symmetric.*

7 Example of a Locally φ-$\mathcal{T}$-Symmetric (ε)-Para Sasakian Manifold of Dimension 3

Consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbf{R}^3, z \neq 0\}$, where (x,y,z) are the standard coordinates of $\mathbf{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = -\,z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the semi-Riemannian metric defined by

$$\begin{aligned} &g(e_1,e_3) = 0, \qquad g(e_1,e_2) = 0, \qquad g(e_2,e_3) = 0, \\ &g(e_1,e_1) = 1, \qquad g(e_2,e_2) = 1, \qquad g(e_3,e_3) = \varepsilon, \end{aligned}$$

where $\varepsilon = \pm 1$. Let η be the 1-form defined by $\eta(Z) = \varepsilon g(Z, e_3)$ for any $Z \in TM$. Let φ be the $(1,1)$-tensor field defined by

$$\varphi e_1 = \varepsilon e_1, \qquad \varphi e_2 = \varepsilon e_2, \qquad \varphi e_3 = 0.$$

Using the linearity of φ and g, we have

$$\varphi^2 X = X - \eta(X) e_3,$$

$$\eta(e_3) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y),$$
$$g(X, e_3) = \varepsilon \eta(X),$$
$$(\nabla_X \varphi)Y = -g(\varphi X, \varphi Y)e_3 - \varepsilon \eta(Y)\varphi^2 X,$$

for any $X, Y \in TM$. Then for $\xi = e_3$, the structure $(\varphi, \xi, \eta, g, \varepsilon)$ defines an (ε)-para Sasakian structure on M. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = e_1, \qquad [e_1, e_2] = e_2.$$

The Koszul's formula for the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By using Koszul's formula, we have

$$\begin{array}{ccc} \nabla_{e_1} e_1 = -\varepsilon e_3, & \nabla_{e_2} e_1 = 0, & \nabla_{e_3} e_1 = -e_1, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_2} e_2 = -\varepsilon e_3, & \nabla_{e_3} e_2 = -e_2, \\ \nabla_{e_1} e_3 = e_1, & \nabla_{e_2} e_3 = e_2, & \nabla_{e_3} e_3 = 0. \end{array}$$

From the above results, it is easy to check that equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) hold. Hence the manifold is an (ε)-para Sasakian manifold.

Using the above results, it is easy to find out the following results

$$\begin{array}{ccc} R(e_1, e_2)e_1 = \varepsilon e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = 2\varepsilon e_3, \\ R(e_1, e_2)e_2 = -\varepsilon e_1, & R(e_2, e_3)e_2 = 2\varepsilon e_3, & R(e_1, e_3)e_2 = 0, \\ R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = 0, & R(e_1, e_3)e_3 = 0. \end{array}$$

Then

$$S(e_1, e_1) = -(\varepsilon + 2), \quad S(e_2, e_2) = -(\varepsilon + 2), \quad S(e_3, e_3) = 0,$$

and

$$r = -2(\varepsilon + 2).$$

Hence the scalar curvature r is constant. From Theorem 5.3, M is a 3-dimensional locally φ-$\mathcal{T}$-symmetric (ε) -para Sasakian manifold.

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