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On n-Tupled Coincidence and Fixed Point Results in Partially Ordered G-Metric Spaces

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Abstract : The notion of n-tupled fixed point is inaugurated by Imdad et al. [1] in 2013. In this paper, some n-tupled coincidence and common fixed point theorems (for even n) are established in partially ordered complete G-metric spaces. Presented theorems can not be obtained from the existing theorems in the frame of reference of allied metric spaces and do not reconcile with the remarks of Samet et al. [2] and Jleli et al. [3]. In fact in a note Agarwal et al. [4] and Asadi et al. [5], recommended new statements to which the technique used in [2,3] were not applicable. Our results, unify, generalize and extend various known results from the current literature. Also, an example is presented to show the validity of the hypotheses of our results and to distinguish them from the existing ones.

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1 Introduction

In the last few decades, fixed point theory has been one of the most interesting field in nonlinear functional analysis. Fixed point of mappings in ordered metric space are of great use in many mathematical problems in applied and pure mathematics. It can be applied in various areas for instance approximation theory, optimization and variational inequalities. One of the newest branches of this theory is devoted to the study of G- metric spaces. The notion of G- metric space was introduced by Mustafa in collaboration with Sims [6]. This was a generalization of metric spaces in which a non-negative real number was assigned to every triplet of an arbitrary set. Mustafa et al. studied many fixed point results for a self mappings in G-metric space under certain conditions (see [6–8]).
On the other hand, Bhaskar and Lakshmikantham [9] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and establish some coupled fixed point theorems in partially ordered complete metric space. After that, Lakshmikantham and Ciric [10] introduced the notion of mixed g-monotone mapping and coupled coincidence point and proved some coupled coincidence point and coupled common fixed point theorems in partially ordered metric space. Afterwords, Brinde and Borcut [11] introduced the concept of tripled fixed point and proved some related theorems. In this continuation, Karapinar et al. [12] inaugurated the notion of quadruple fixed point and established some results on the existence and uniqueness of quadruple fixed points. Most recently, Imdad et al. [1] launched the concept of n-tupled coincidence as well as n-tupled fixed point (for even n) and utilize these two definitions to obtain n-tupled coincidence as well as n-tupled common fixed point theorems for nonlinear ϕ-contraction mappings in partially ordered complete metric spaces.

In this manuscript, we furnish n-tupled fixed point results for a pair of weakly compatible mapping with mixed g-monotone property in generalized ordered metric spaces for nonlinear contractive condition related to an alternating distance function, which generalize result of Y.J. Cho [13], B.S. Choudhury and P. Maity [14], H. Aydi et al. [15], H. Nashine [16], Z. Mustafa [17] and H. Lee [18].

The following definitions and results will be needed in the sequel.

2 Preliminaries

Definition 2.1. [19] A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an alternating distance function if the following properties are satisfied:

(i) ψ is continuous and monotonically non-decreasing;

(ii) ψ (t)=0 if and only if $t = 0$.

We borrow the definition of n-tupled fixed point and n-tupled coincidence point from Imdad et al. [1].

Throughout the paper, we consider n to be an even integer.

Definition 2.2. [1] An element $(x^1, x^2, \dots, x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = x^1, \\ F(x^2, x^3, \dots, x^n, x^1) = x^2, \\ F(x^3, \dots, x^n, x^1, x^2) = x^3, \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = x^n. \end{cases}$$

In the following example we establish n-tupled fixed point.

Example 2.3. Let $X = \mathbb{R}$, then $(X, \leq)$ be a partially ordered set with usual ordering. Let $F : X^n \to X$ be a mapping defined by
$F(x^1, x^2, x^3, \dots, x^n) = \frac{x^1+x^2+x^3+\dots+x^n}{n}$, for all $(x^1, x^2, x^3, \dots, x^n) \in X$
Then $(n, n, n, \dots, n)$ is an n-tupled fixed point of F.

Definition 2.4. [1] An element $(x^1, x^2, \dots, x^n) \in X^n$ is called an n-tupled coincidence point of the mapping $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = gx^1, \\ F(x^2, x^3, \dots, x^n, x^1) = gx^2, \\ F(x^3, \dots, x^n, x^1, x^2) = gx^3, \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = gx^n. \end{cases}$$

Following example establishes n-tupled coincidence point.

Example 2.5. Let $X = \mathbb{R}$, then $(X, \leq)$ be a partially ordered set with usual ordering. Let $F : X^n \to X$ and $g : X \to X$ be two mappings defined by
$F(x^1, x^2, x^3, \dots, x^n) = \frac{x^1+x^2+x^3+\dots+x^n}{n}$, for all $(x^1, x^2, x^3, \dots, x^n) \in X$ and $gx = x$.
Then $(n, n, n, \dots, n)$ is an n-tupled coincidence point of F.

Definition 2.6. Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space then $(X, G, \preceq)$ is called regular if the following conditions hold:

*(i)*If a non-decreasing sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ then $x_n \preceq x, \forall n \in \mathbb{N}$;

*(ii)*If a non-increasing sequence $\{y_n\} \subseteq X$ such that $y_n \to y$ then $y \preceq y_n, \forall n \in \mathbb{N}$.

For the rest of the definitions and other notions utilized in our paper one can refer to M. Imdad [1] and Mustafa et al. [6].

3 Main Results

3.1 n-Tupled Coincidence Point Theorems

Our main theorem runs as follows:

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings on X. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an alternating distance function such that for,*
$(x^1, x^2, \dots, x^n), (y^1, y^2, \dots, y^n), (z^1, z^2, \dots, z^n) \in X$, we have

$$\begin{aligned}
&\psi(G(F(x^1, x^2, \dots, x^n), F(y^1, y^2, \dots, y^n), F(z^1, z^2, \dots, z^n))) \\
&\leq \psi\Big(\max\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), \dots, G(gx^n, gy^n, gz^n)\}\Big) \\
&\quad - \phi\Big(\max G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), \dots, G(gx^n, gy^n, gz^n)\Big),
\end{aligned} \tag{3.1}$$

with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gv^2 \preceq gz^2, \dots, gx^n \preceq gv^n \preceq gz^n$. Suppose that F has the mixed g-monotone property, $F(X^n) \subseteq g(X)$, g is continuous and commutes with F. Also, assume that, either

(a) F is continuous, or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n;

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all n.
If there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ such that

$$\begin{cases}
gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n), \\
F(x_0^2, x_0^3, \dots, x_0^n, x_0^1) \preceq gx_0^2, \\
\vdots \\
F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}) \preceq gx_0^n.
\end{cases}$$

Then F and g have an n*-tupled coincidence point in* X.

Proof. Let $x_0^1, x_0^2, \dots, x_0^n \in X$ such that

$$\begin{cases}
gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n), \\
F(x_0^2, x_0^3, \dots, x_0^n, x_0^1) \preceq gx_0^2, \\
\vdots \\
F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}) \preceq gx_0^n.
\end{cases} \tag{3.2}$$

We choose $x_1^1, x_1^2, \dots, x_1^n \in X$ such that

$$\begin{cases}
gx_1^1 = F(x_0^1, x_0^2, x_0^3, \dots, x_0^n), \\
gx_1^2 = F(x_0^2, x_0^3, \dots, x_0^n, x_0^1), \\
\vdots \\
gx_1^n = F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}),
\end{cases} \tag{3.3}$$

this can be done in view of $F(X^n) \subseteq g(X)$. Similarly, we can choose $x_2^1, x_2^2, \ldots, x_2^n \in X$ such that

$$\begin{cases} gx_2^1 = F(x_1^1, x_1^2, x_1^3, \ldots, x_1^n), \\ gx_2^2 = F(x_1^2, x_1^3, \ldots, x_1^n, x_1^1), \\ \quad\vdots \\ gx_2^n = F(x_1^n, x_1^1, x_1^2, \ldots, x_1^{n-1}). \end{cases}$$

Continuing this process, one can construct n sequences $\{x_m^1\}$, $\{x_m^2\}$, $\ldots$, $\{x_m^n\}$ $(m \geq 0)$ in X such that

$$\begin{cases} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n), \\ gx_{m+1}^2 = F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1), \\ \quad\vdots \\ gx_{m+1}^n = F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}). \end{cases} \tag{3.4}$$

Now, with the help of mathematical induction method, we shall show that for all $m \geq 0$,

$$gx_m^1 \preceq gx_{m+1}^1, \;\; gx_{m+1}^2 \preceq gx_m^2, \;\; gx_m^3 \preceq gx_{m+1}^3, \;\; \ldots \;\; , \;\; gx_{m+1}^n \preceq gx_m^n. \tag{3.5}$$

By (3.2) and (3.3), we get

$$gx_0^1 \preceq gx_1^1, gx_1^2 \preceq gx_0^2, gx_0^3 \preceq gx_1^3, \ldots, gx_1^n \preceq gx_0^n,$$

i.e. (3.5) holds for $m = 0$. Assume that, (3.5) holds for some $m > 0$. From the mixed g-monotone property of F and in account of (3.4), we have

$$\begin{aligned} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n) &\preceq F(x_{m+1}^1, x_m^2, x_m^3, \ldots, x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_m^3, \ldots, x_m^n) \\ &\vdots \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, \ldots, x_{m+1}^n) \\ &= gx_{m+2}^1. \end{aligned}$$

$$\begin{aligned} gx_{m+2}^2 = F(x_{m+1}^2, x_{m+1}^3, \ldots, x_{m+1}^n, x_{m+1}^1) &\preceq F(x_{m+1}^2, x_{m+1}^3, \ldots, x_{m+1}^n, x_m^1) \\ &\preceq F(x_{m+1}^2, x_{m+1}^3, \ldots, x_m^n, x_m^1) \\ &\vdots \\ &\preceq F(x_{m+1}^2, x_m^3, \ldots, x_m^n, x_m^1) \\ &\preceq F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1) \\ &= gx_{m+1}^2. \end{aligned}$$

Analogously, it can be proved that

$$\begin{aligned} gx_{m+2}^n = F(x_{m+1}^n, x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^{n-1}) &\preceq F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}) \\ &= gx_{m+1}^n. \end{aligned}$$

Therefore, it follows from the method of induction that, inequality (3.5) holds, for all $m \geq 0$. Hence

$$\begin{cases} gx_0^1 \preceq gx_1^1 \preceq gx_2^1 \preceq \ldots \preceq gx_m^1 \preceq gx_{m+1}^1 \preceq \ldots, \\ \ldots \preceq gx_{m+1}^2 \preceq gx_m^2 \preceq \ldots \preceq gx_2^2 \preceq gx_1^2 \preceq gx_0^2, \\ gx_0^3 \preceq gx_1^3 \preceq gx_2^3 \preceq \ldots \preceq gx_m^3 \preceq gx_{m+1}^3 \preceq \ldots, \\ \vdots \\ \ldots \preceq gx_{m+1}^n \preceq gx_m^n \preceq \ldots \preceq gx_2^n \preceq gx_1^n \preceq gx_0^n. \end{cases} \tag{3.6}$$

Then from (3.1) and (3.5), we have

$$\begin{aligned} &\psi\Big(G(gx_{m+1}^1, gx_m^1, gx_m^1) \Big) \\ &= \psi\Bigg(G\Big(F(x_m^1, x_m^2, \ldots, x_m^n), F(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n), \\ &\qquad F(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n)\Big)\Bigg) \\ &\leq \psi\Big(\max\Big\{ G(gx_m^1, gx_{m-1}^1, gx_{m-1}^1), G(gx_m^2, gx_{m-1}^2, gx_{m-1}^2), \\ &\qquad G(gx_m^3, gx_{m-1}^3, gx_{m-1}^3), \ldots, G(gx_m^n, gx_{m-1}^n, gx_{m-1}^n)\Big\}\Big) \\ &\quad - \phi\Big(\max\Big\{ G(gx_m^1, gx_{m-1}^1, gx_{m-1}^1), G(gx_m^2, gx_{m-1}^2, gx_{m-1}^2), \\ &\qquad G(gx_m^3, gx_{m-1}^3, gx_{m-1}^3), \ldots, G(gx_m^n, gx_{m-1}^n, gx_{m-1}^n)\Big\}\Big). \end{aligned} \tag{3.7}$$

Again, from (3.1) and (3.5), we get

$$\begin{aligned} &\psi\Big(G(gx_{m+1}^2, gx_m^2, gx_m^2) \Big) \\ &= \psi\Bigg(G\Big(F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1), \\ &\qquad F(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^n, x_{m-1}^1), F(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^n, x_{m-1}^1)\Big)\Bigg) \\ &\leq \psi\Big(\max\Big\{ G(gx_m^2, gx_{m-1}^2, gx_{m-1}^2), G(gx_m^3, gx_{m-1}^3, gx_{m-1}^3), \\ &\qquad \ldots, G(gx_m^n, gx_{m-1}^n, gx_{m-1}^n), G(gx_m^1, gx_{m-1}^1, gx_{m-1}^1)\Big\}\Big) \\ &\quad - \phi\Big(\max\Big\{ G(gx_m^2, gx_{m-1}^2, gx_{m-1}^2), G(gx_m^3, gx_{m-1}^3, gx_{m-1}^3), \\ &\qquad \ldots, G(gx_m^n, gx_{m-1}^n, gx_{m-1}^n), G(gx_m^1, gx_{m-1}^1, gx_{m-1}^1)\Big\}\Big). \end{aligned} \tag{3.8}$$

In a similar way, we obtain

$$\begin{aligned}
&\psi\Big(\ G(\ gx^n_{m+1}, gx^n_m, gx^n_m\ )\ \Big)\\
&= \psi\Bigg(G\Big(F(x^n_m, x^1_m, x^2_m, \ldots, x^{n-1}_m),\\
&\qquad F(x^n_{m-1}, x^1_{m-1}, x^2_{m-1}, \ldots, x^{n-1}_{m-1}), F(x^n_{m-1}, x^1_{m-1}, x^2_{m-1}, \ldots, x^{n-1}_{m-1})\Big)\Bigg)\\
&\le \psi\Big(\ \max\Big\{G(gx^n_m, gx^n_{m-1}, gx^n_{m-1}), G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}),\\
&\qquad G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}), \ldots, G(gx^{n-1}_m, gx^{n-1}_{m-1}, gx^{n-1}_{m-1})\Big\}\Big)\\
&\quad - \phi\Big(\ \max\Big\{G(gx^n_m, gx^n_{m-1}, gx^n_{m-1}), G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}),\\
&\qquad G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}), \ldots, G(gx^{n-1}_m, gx^{n-1}_{m-1}, gx^{n-1}_{m-1})\Big\}\Big).
\end{aligned} \tag{3.9}$$

As ψ is a non-decreasing function, for $a_1, a_2, a_3, \ldots a_n \in [0, +\infty)$, we have

$$\psi(\max\{a_1, a_2, a_3, \ldots a_n\}) = \max\{\psi(a_1), \psi(a_2), \psi(a_3), \ldots, \psi(a_n)\}.$$

Due to (3.7), (3.8) and (3.9), it follows that

$$\begin{aligned}
&\psi\Big(\max\Big\{G(gx^1_{m+1}, gx^1_m, gx^1_m), G(gx^2_{m+1}, gx^2_m, gx^2_m), ..., G(gx^n_{m+1}, gx^n_m, gx^n_m)\Big\}\Big)\\
&\le \psi\Big(\ \max\Big\{G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}), G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}),\\
&\qquad G(gx^3_m, gx^3_{m-1}, gx^3_{m-1}),\ \ldots, G(gx^n_m, gx^n_{m-1}, gx^n_{m-1})\Big\}\Big)\\
&\quad - \phi\Big(\ \max\Big\{G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}), G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}),\\
&\qquad G(gx^3_m, gx^3_{m-1}, gx^3_{m-1}),\ \ldots, G(gx^n_m, gx^n_{m-1}, gx^n_{m-1})\Big\}\Big).
\end{aligned}$$

$$\begin{aligned}
\text{Let, } A_m = \max\Big\{&G(gx^1_m, gx^1_{m-1}, gx^1_{m-1}), G(gx^2_m, gx^2_{m-1}, gx^2_{m-1}),\\
&G(gx^3_m, gx^3_{m-1}, gx^3_{m-1}), \ldots, G(gx^n_m, gx^n_{m-1}, gx^n_{m-1})\Big\}.
\end{aligned} \tag{3.10}$$

From the above inequality, we arrive at

$$\psi(A_m) \le \psi(A_{m-1}) - \phi(A_{m-1}). \tag{3.11}$$

As the function ϕ is non negative, we obtain

$$\psi(A_m) \le \psi(A_{m-1}) \Rightarrow A_m \le A_{m-1}.$$

Thus, $\{A_m\}$ is a positive non increasing sequence. Hence there exists $r \ge 0$ such that $A_m \to r$ as $m \to \infty$. Letting the limit as $m \to \infty$ in (3.11) and using the continuity of ψ and ϕ, we get

$$\psi(r) \leq \psi(r) - \phi(r).$$

Hence, $\phi(r) = 0$ and by the property of ϕ, we find $r = 0$. Therefore

$$\lim_{m\to\infty} A_m = 0. \tag{3.12}$$

Our next step is to show that $\{gx_m^1\}, \{gx_m^2\}, \{gx_m^3\}, \ldots, \{gx_m^n\}$ are G-Cauchy sequences. On the contrary assume that, at least one of $\{gx_m^1\}, \{gx_m^2\}, \{gx_m^3\}, \ldots, \{gx_m^n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ for which we can find sequences of positive integer $\{m(k)\}$ and $\{l(k)\}$ with $l(k) > m(k) \geq k$, such that

$$\begin{aligned} B_k = \max\Big\{&G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big), G\big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\big), \\ &\ldots, G\big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\big)\Big\} \geq \epsilon. \end{aligned} \tag{3.13}$$

Moreover, corresponding to $m(k)$ we can choose $l(k)$ in such a way that it is the smallest integer with $l(k) > m(k)$ and satisfying (3.13). Then

$$\begin{aligned} \max\Big\{&G\big(gx^1_{m(k)}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big), G\big(gx^2_{m(k)}, gx^2_{l(k)-1}, gx^2_{l(k)-1}\big), \\ &\ldots, G\big(gx^n_{m(k)}, gx^n_{l(k)-1}, gx^n_{l(k)-1}\big)\Big\} < \epsilon. \end{aligned} \tag{3.14}$$

By the rectangle inequality of G-metric and from (3.14), we arrive at

$$\begin{aligned} G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big) &\leq G\big(gx^1_{m(k)}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big) \\ &\quad + G\big(gx^1_{l(k)-1}, gx^1_{l(k)}, gx^1_{l(k)}\big) \\ &< \epsilon + G\big(gx^1_{l(k)-1}, gx^1_{l(k)}, gx^1_{l(k)}\big). \end{aligned} \tag{3.15}$$

Thus, from (3.12), we acquire

$$\lim_{k\to\infty} G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big) \leq \lim_{k\to\infty} G\big(gx^1_{m(k)}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big) \leq \epsilon. \tag{3.16}$$

In a same manner, we get

$$\lim_{k\to\infty} G\big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\big) \leq \lim_{k\to\infty} G\big(gx^2_{m(k)}, gx^2_{l(k)-1}, gx^2_{l(k)-1}\big) \leq \epsilon. \tag{3.17}$$

Similarly, we have

$$\lim_{k\to\infty} G\big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\big) \leq \lim_{k\to\infty} G\big(gx^n_{m(k)}, gx^n_{l(k)-1}, gx^n_{l(k)-1}\big) \leq \epsilon. \tag{3.18}$$

Again from rectangle inequality of G-metric and from (3.12), we obtain

$$\begin{aligned}
G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big) &\leq G\big(gx^1_{m(k)}, gx^1_{m(k)-1}, gx^1_{m(k)-1}\big) \\
&\quad + G\big(gx^1_{m(k)-1}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big) \\
&\quad + G\big(gx^1_{l(k)-1}, gx^1_{l(k)}, gx^1_{l(k)}\big) \\
&\leq G\big(gx^1_{m(k)}, gx^1_{m(k)-1}, gx^1_{m(k)-1}\big) \\
&\quad + G\big(gx^1_{m(k)-1}, gx^1_{m(k)}, gx^1_{m(k)}\big) \\
&\quad + G\big(gx^1_{m(k)}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big) \\
&\quad + G\big(gx^1_{l(k)-1}, gx^1_{l(k)}, gx^1_{l(k)}\big) \\
&< G\big(gx^1_{m(k)}, gx^1_{m(k)-1}, gx^1_{m(k)-1}\big) \\
&\quad + G\big(gx^1_{m(k)-1}, gx^1_{m(k)}, gx^1_{m(k)}\big) \\
&\quad + \epsilon + G\big(gx^1_{l(k)-1}, gx^1_{l(k)}, gx^1_{l(k)}\big).
\end{aligned}$$

Taking the limit when $k \to \infty$ in above inequality and using (3.12), we acquire

$$\lim_{k\to\infty} G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big) \leq \lim_{k\to\infty} G\big(gx^1_{m(k)-1}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big) \leq \epsilon. \tag{3.19}$$

Similarly, we arrive at

$$\lim_{k\to\infty} G\big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\big) \leq \lim_{k\to\infty} G\big(gx^2_{m(k)-1}, gx^2_{l(k)-1}, gx^2_{l(k)-1}\big) \leq \epsilon \tag{3.20}$$

and

$$\lim_{k\to\infty} G\big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\big) \leq \lim_{k\to\infty} G\big(gx^n_{m(k)-1}, gx^n_{l(k)-1}, gx^n_{l(k)-1}\big) \leq \epsilon. \tag{3.21}$$

From (3.13) and in account of inequalities (3.19)-(3.21), we have

$$\begin{aligned}
&\lim_{k\to\infty} \max\Big\{G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big), G\big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\big), \ldots, \\
&\qquad\qquad G\big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\big)\Big\} \\
&= \lim_{k\to\infty} \max\Big\{G\big(gx^1_{m(k)-1}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big), G\big(gx^2_{m(k)-1}, gx^2_{l(k)-1}, gx^2_{l(k)-1}\big), \\
&\qquad\qquad \ldots, G\big(gx^n_{m(k)-1}, gx^n_{l(k)-1}, gx^n_{l(k)-1}\big)\Big\} \\
&= \epsilon.
\end{aligned} \tag{3.22}$$

Therefore from inequality (3.1) and (3.4), we obtain

$$\begin{aligned}
&\psi\Bigg(G\Big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\Big)\Bigg)\\
&=\psi\Bigg(G\Big(F(x^1_{m(k)-1}, x^2_{m(k)-1}, \dots, x^n_{m(k)-1}), F(x^1_{l(k)-1}, x^2_{l(k)-1}, \dots, x^n_{l(k)-1}),\\
&\qquad\qquad F(x^1_{l(k)-1}, x^2_{l(k)-1}, \dots, x^n_{l(k)-1})\Big)\Bigg)\\
&\leq \psi\Bigg(\max\Big\{G\big(gx^1_{m(k)-1}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big), G\big(gx^2_{m(k)-1}, gx^2_{l(k)-1}, gx^2_{l(k)-1}\big), ...,\\
&\qquad\qquad G\big(gx^n_{m(k)-1}, gx^n_{l(k)-1}, gx^n_{l(k)-1}\big)\Big\}\Bigg)\\
&-\phi\Bigg(\max\Big\{G\big(gx^1_{m(k)-1}, gx^1_{l(k)-1}, gx^1_{l(k)-1}\big), G\big(gx^2_{m(k)-1}, gx^2_{l(k)-1}, gx^2_{l(k)-1}\big), ...,\\
&\qquad\qquad G\big(gx^n_{m(k)-1}, gx^n_{l(k)-1}, gx^n_{l(k)-1}\big)\Big\}\Bigg).
\end{aligned}$$

Which gives

$$\psi\Bigg(G\Big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\Big)\Bigg) \leq \psi(B_{k-1}) - \phi(B_{k-1}). \tag{3.23}$$

Analogously, one can show that

$$\psi\Bigg(G\Big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\Big)\Bigg) \leq \psi(B_{k-1}) - \phi(B_{k-1}). \tag{3.24}$$

and

$$\psi\Bigg(G\Big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\Big)\Bigg) \leq \psi(B_{k-1}) - \phi(B_{k-1}). \tag{3.25}$$

By the monotone property of ψ and in account of inequalities (3.23)-(3.25), we have

$$\psi(B_k) = \psi\Bigg(\max\Big\{ G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big), G\big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\big), \\ \ldots, G\big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\big)\Big\}\Bigg) \\ = \max\Bigg\{ \psi\Big(G\big(gx^1_{m(k)}, gx^1_{l(k)}, gx^1_{l(k)}\big)\Big), \psi\Big(G\big(gx^2_{m(k)}, gx^2_{l(k)}, gx^2_{l(k)}\big)\Big), \\ \ldots, \psi\Big(G\big(gx^n_{m(k)}, gx^n_{l(k)}, gx^n_{l(k)}\big)\Big)\Bigg\},$$

that is,

$$\psi(B_k) \le \psi(B_{k-1}) - \phi(B_{k-1}).$$

Taking The limit as $k \to \infty$ in the above inequality, using (3.22), we obtain

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon).$$

Which gives $\phi(\epsilon) = 0 \Longrightarrow \epsilon = 0$, a contradiction. We conclude that $\{gx^1_m\}$, $\{gx^2_m\}$, $\{gx^3_m\}$, ..., $\{gx^n_m\}$ are G-Cauchy sequences in the G-metric space (X, G), which is G-complete. Then there exist $x^1, x^2, \ldots, x^n \in X$ such that

$$\begin{cases} \lim\limits_{m\to\infty} G(gx^1_m, gx^1_m, x^1) = \lim\limits_{m\to\infty} G(gx^1_m, x^1, x^1) = 0, \\ \lim\limits_{m\to\infty} G(gx^2_m, gx^2_m, x^2) = \lim\limits_{m\to\infty} G(gx^2_m, x^2, x^2) = 0, \\ \vdots \\ \lim\limits_{m\to\infty} G(gx^n_m, gx^n_m, x^n) = \lim\limits_{m\to\infty} G(gx^n_m, x^n, x^n) = 0. \end{cases} \tag{3.26}$$

From (3.26) and by the continuity of g, we arrive at

$$\begin{cases} \lim\limits_{m\to\infty} G(g(gx^1_m), g(gx^1_m), gx^1) = \lim\limits_{m\to\infty} G(g(gx^1_m), gx^1, gx^1) = 0, \\ \lim\limits_{m\to\infty} G(g(gx^2_m), g(gx^2_m), gx^2) = \lim\limits_{m\to\infty} G(g(gx^2_m), gx^2, gx^2) = 0, \\ \vdots \\ \lim\limits_{m\to\infty} G(g(gx^n_m), g(gx^n_m), gx^n) = \lim\limits_{m\to\infty} G(g(gx^n_m), gx^n, gx^n) = 0. \end{cases} \tag{3.27}$$

Thus, $g(gx^1_m)$ is convergent to $gx^1, g(gx^2_m)$ is convergent to $gx^2, \ldots, g(gx^n_m)$ is convergent to gx^n. Since,

$$\begin{cases} gx^1_{m+1} = F(x^1_m, x^2_m, x^3_m, \ldots, x^n_m), \\ gx^2_{m+1} = F(x^2_m, x^3_m, \ldots, x^n_m, x^1_m), \\ gx^3_{m+1} = F(x^3_m, \ldots, x^n_m, x^1_m, x^2_m), \\ \vdots \\ gx^n_{m+1} = F(x^n_m, x^1_m, x^2_m, \ldots, x^{n-1}_m). \end{cases}$$

Therefore, from the commutativity of F and g, we have

$$\begin{cases} g(gx^1_{m+1}) = g(F(x^1_m, x^2_m, x^3_m, \ldots, x^n_m)) = F(gx^1_m, gx^2_m, gx^3_m, \ldots, gx^n_m), \\ g(gx^2_{m+1}) = g(F(x^2_m, x^3_m, \ldots, x^n_m, x^1_m)) = F(gx^2_m, gx^3_m, \ldots, gx^n_m, gx^1_m), \\ \vdots \\ g(gx^n_{m+1}) = g(F(x^n_m, x^1_m, x^2_m, \ldots, x^{n-1}_m)) = F(gx^n_m, gx^1_m, gx^2_m, \ldots, gx^{n-1}_m). \end{cases} \tag{3.28}$$

Let the condition (a) holds, i.e., F is continuous. Taking limit as $m \to \infty$ in (3.28), utilizing the continuity of F and in account of (3.26) and (3.27), we get

$$\begin{cases} F(x^1_m, x^2_m, x^3_m, \ldots, x^n_m) = gx^1, \\ F(x^2_m, x^3_m, \ldots, x^n_m, x^1_m) = gx^2, \\ F(x^3_m, \ldots, x^n_m, x^1_m, x^2_m) = gx^3, \\ \vdots \\ F(x^n_m, x^1_m, x^2_m, \ldots, x^{n-1}_m) = gx^n. \end{cases}$$

Hence the element $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ is an n-tupled coincidence point of the mapping $F : X^n \to X$ and $g : X \to X$.
Let us assume that, the condition (b) holds, that is, (X, G) is regular. Since, $\{gx^1_m\}, \{gx^3_m\}, \ldots, \{gx^{n-1}_m\}$ are non decreasing and $\{gx^2_m\}, \{gx^4_m\}, \ldots, \{gx^n_m\}$ are non increasing sequences. Due to the regularity of $(X, G, \preceq)$, we have

$$\begin{cases} gx^1_m \preceq gx^1, & gx^2_m \succeq gx^2, \\ gx^3_m \preceq gx^3, & gx^4_m \succeq gx^4, \\ \vdots & \vdots \\ gx^{n-1}_m \preceq gx^{n-1}, & gx^n_m \succeq gx^n. \end{cases}$$

Now, using inequality (3.1) and from the rectangular inequality of G-metric, we acquire

$$\begin{aligned} &\psi\Bigg(G\Big(F(x^1, x^2, x^3, \ldots, x^n), gx^1_{m+1}, gx^1_{m+1}\Big)\Bigg) \\ &= \psi\Bigg(G\Big(F(x^1, x^2, x^3, \ldots, x^n), F(x^1_m, x^2_m, x^3_m, \ldots, x^n_m), F(x^1_m, x^2_m, x^3_m, \ldots, x^n_m)\Big)\Bigg) \\ &\le \psi\Bigg(\max\Big\{G(gx^1, gx^1_m, gx^1_m), G(gx^2, gx^2_m, gx^2_m), \ldots, G(gx^n, gx^n_m, gx^n_m)\Big\}\Bigg) \\ &- \phi\Bigg(\max\Big\{G(gx^1, gx^1_m, gx^1_m), G(gx^2, gx^2_m, gx^2_m), \ldots, G(gx^n, gx^n_m, gx^n_m)\Big\}\Bigg). \end{aligned}$$

Making the limit as $m \to \infty$ in above inequality give rises to

$$\psi\Bigg(G\Big(F(x^1, x^2, x^3, \ldots, x^n), gx^1_{m+1}, gx^1_{m+1}\Big)\Bigg) = 0.$$

Which yields

$$F(x^1, x^2, x^3, \ldots, x^n) = gx^1.$$

Repeating the same technique, one can show that

$$\begin{cases} F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1) = gx^2, \\ F(x_m^3, \ldots, x_m^n, x_m^1, x_m^2) = gx^3, \\ \vdots \\ F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}) = gx^n. \end{cases}$$

Thus, we proved $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ is an n-tupled coincidence point of the mapping $F : X^n \to X$ and $g : X \to X$.
This conclude the Theorem. □

3.2 n-Tupled Common Fixed Point Theorems

Theorem 3.2. *In addition to the hypotheses of Theorem* 3.1*, suppose that for every* $(x^1, x^2, x^3, \ldots, x^n)$, $(y^1, y^2, y^3, \ldots, y^n) \in X^n$ *there exists* $(z^1, z^2, z^3, \ldots, z^n) \in X^n$ *such that* $(F(z^1, z^2, \ldots, z^n), \ldots, F(z^n, z^1, \ldots, z^{n-1})) \in X^n$ *is comparable with* $(F(x^1, x^2, \ldots, x^n), \ldots, F(x^n, x^1, \ldots, x^{n-1}))$ *and* $(F(y^1, y^2, \ldots, y^n), \ldots, F(y^n, y^1, \ldots, y^{n-1}))$.
Then F *and* g *have a unique n-tupled common fixed point.*

Proof. From Theorem 3.1, the set of n-tupled coincidence points of F and g is non empty. Assume that $(x^1, x^2, x^3, \ldots, x^n), (y^1, y^2, y^3, \ldots, y^n)$ are two n-tupled coincidence points of F and g, that is

$$\begin{cases} F(x^1, x^2, x^3, \ldots, x^n) = gx^1; & F(y^1, y^2, y^3, \ldots, y^n) = gy^1, \\ F(x^2, x^3, \ldots, x^n, x^1) = gx^2; & F(y^2, y^3, \ldots, y^n, y^1) = gy^2, \\ \vdots & \vdots \\ F(x^n, x^1, x^2, \ldots, x^{n-1}) = gx^n; & F(y^n, y^1, y^2, \ldots, y^{n-1}) = gy^n. \end{cases}$$

Now, we shall prove that $gx^1 = gy^1, gx^2 = gy^2, \ldots, gx^n = gy^n$.
By supposition, there exists $(z^1, z^2, z^3, ..., z^n) \in X^n$ such that $(F(z^1, z^2, z^3, ..., z^n)$, $F(z^2, z^3, \ldots, z^n, z^1), \ldots, F(z^n, z^1, z^2, \ldots, z^{n-1}))$ is comparable with $(F(x^1, x^2, x^3, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1}))$ and $(F(y^1, y^2, y^3, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, y^2, \ldots, y^{n-1}))$.
Put, $z_0^1 = z^1, z_0^2 = z^2, \ldots, z_0^n = z^n$ and choose $z^1, z^2, z^3, \ldots, z^n \in X$ such that

$$\begin{cases} gz_1^1 = F(z_0^1, z_0^2, z_0^3, \ldots, z_0^n), \\ gz_1^2 = F(z_0^2, z_0^3, \ldots, z_0^n, z_0^1), \\ \vdots \\ gz_1^n = F(z_0^n, z_0^1, z_0^2, \ldots, z_0^{n-1}). \end{cases}$$

Then similarly, as in the proof of Theorem 3.1 one can inductively define sequences $\{gz_m^1\}, \{gz_m^2\}, \{gz_m^3\}, \ldots, \{gz_m^n\}$ in X such that

$$\begin{cases} gz_{m+1}^1 = F(z_m^1, z_m^2, z_m^3, \ldots, z_m^n), \\ gz_{m+1}^2 = F(z_m^2, z_m^3, \ldots, z_m^n, z_m^1), \\ \vdots \\ gz_{m+1}^n = F(z_m^n, z_m^1, z_m^2, \ldots, z_m^{n-1}). \end{cases}$$

Moreover, set $x_0^1 = x^1, x_0^2 = x^2, \ldots, x_0^n = x^n$ and $y_0^1 = y^1, y_0^2 = y^2, \ldots, y_0^n = y^n$ and on the same way define the sequences $\{gx_m^1\}, \{gx_m^2\}, \{gx_m^3\}, \ldots, \{gx_m^n\}$ and $\{gy_m^1\}, \{gy_m^2\}, \{gy_m^3\}, \ldots, \{gy_m^n\}$. Then we can easily show that

$$\begin{cases} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n); \quad gy_{m+1}^1 = F(y_m^1, y_m^2, y_m^3, \ldots, y_m^n), \\ gx_{m+1}^2 = F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1); \quad gy_{m+1}^2 = F(y_m^2, y_m^3, \ldots, y_m^n, y_m^1), \\ \qquad\vdots \qquad\qquad\qquad\qquad \vdots \\ gx_{m+1}^n = F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}); \quad gy_{m+1}^n = F(y_m^n, y_m^1, y_m^2, \ldots, y_m^{n-1}). \end{cases}$$

As

$$\begin{aligned} (F(x^1, x^2, x^3, \ldots, x^n), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})) &= (gx_1^1, gx_1^2, \ldots, gx_1^n) \\ &= (gx^1, gx^2, \ldots, gx^n) \end{aligned}$$

and

$$(F(z^1, z^2, z^3, \ldots, z^n), \ldots, F(z^n, z^1, z^2, \ldots, z^{n-1})) = (gz_1^1, gz_1^2, \ldots, gz_1^n)$$

are comparable, then

$$gx^1 \preceq gz_1^1, gz_1^2 \preceq gx^2, gx^3 \preceq gz_1^3, \ldots, gz_1^n \preceq gx^n.$$

Similarly, we can show that for all $m \geq 1$,

$$gx^1 \preceq gz_m^1, gz_m^2 \preceq gx^2, gx^3 \preceq gz_m^3, \ldots, gz_m^n \preceq gx^n.$$

Thus, from (3.1), we obtain

$$\begin{aligned} &\psi\Big(G(gx^1, gx^1, gz_{m+1}^1)\Big) \\ &= \psi\Big(G(F(x^1, x^2, x^3, \ldots, x^n), F(x^1, x^2, x^3, \ldots, x^n), F(z_m^1, z_m^2, z_m^3, \ldots, z_m^n)\Big) \\ &\leq \psi\Bigg(\max\Big\{G(gx^1, gx^1, gz_m^1), G(gx^2, gx^2, gz_m^2), G(gx^3, gx^3, gz_m^3), \\ &\qquad \ldots, G(gx^n, gx^n, gz_m^n)\Big\}\Bigg) \\ &- \phi\Bigg(\max\Big\{G(gx^1, gx^1, gz_m^1), G(gx^2, gx^2, gz_m^2), G(gx^3, gx^3, gz_m^3), \\ &\qquad \ldots, G(gx^n, gx^n, gz_m^n)\Big\}\Bigg). \end{aligned}$$

Also

$$
\begin{aligned}
&\psi\Big(G(gx^2, gx^2, gz^2_{m+1})\Big)\\
&= \psi\Big(G(F(x^2, x^3, \ldots, x^n, x^1), F(x^2, x^3, \ldots, x^n, x^1), F(z^2_m, z^3_m, \ldots, z^n_m, z^1_m)\Big)\\
&\leq \psi\Bigg(\max\Big\{G(gx^2, gx^2, gz^2_m), G(gx^3, gx^3, gz^3_m), \ldots,\\
&\qquad\qquad G(gx^n, gx^n, gz^n_m), G(gx^1, gx^1, gz^1_m)\Big\}\Bigg)\\
&- \phi\Bigg(\max\Big\{G(gx^2, gx^2, gz^2_m), G(gx^3, gx^3, gz^3_m), \ldots,\\
&\qquad\qquad G(gx^n, gx^n, gz^n_m), G(gx^1, gx^1, gz^1_m)\Big\}\Bigg).
\end{aligned}
$$

In a similar way, one can show that

$$
\begin{aligned}
&\psi\Big(G(gx^n, gx^n, gz^n_{m+1})\Big)\\
&= \psi\Big(G(F(x^n, x^1, x^2, ..., x^{n-1}), F(x^n, x^1, x^2, ..., x^{n-1}), F(z^n_m, z^1_m, z^2_m, ..., z^{n-1}_m)\Big)\\
&\leq \psi\Bigg(\max\Big\{G(gx^n, gx^n, gz^n_m), G(gx^1, gx^1, gz^1_m), G(gx^2, gx^2, gz^2_m),\\
&\qquad\qquad \ldots, G(gx^{n-1}, gx^{n-1}, gz^{n-1}_m)\Big\}\Bigg)\\
&- \phi\Bigg(\max\Big\{G(gx^n, gx^n, gz^n_m), G(gx^1, gx^1, gz^1_m), G(gx^2, gx^2, gz^2_m),\\
&\qquad\qquad \ldots, G(gx^{n-1}, gx^{n-1}, gz^{n-1}_m)\Big\}\Bigg).
\end{aligned}
$$

Using the monotone property of ψ and from the above inequalities, we arrive at

$$
\begin{aligned}
&\psi\Bigg(\max\Big\{G(gx^1, gx^1, gz^1_{m+1}), G(gx^2, gx^2, gz^2_{m+1}), \ldots, G(gx^n, gx^n, gz^n_{m+1})\Big\}\Bigg)\\
&= \max\Big\{\psi\Big(G(gx^1, gx^1, gz^1_{m+1})\Big), \psi\Big(G(gx^2, gx^2, gz^2_{m+1})\Big),\\
&\qquad\qquad \ldots, \psi\Big(G(gx^n, gx^n, gz^n_{m+1})\Big)\Big\}
\end{aligned}
$$

$$\begin{aligned}
&\le \psi\Bigg(\max\Big\{G(gx^1,gx^1,gz^1_m),G(gx^2,gx^2,gz^2_m),G(gx^3,gx^3,gz^3_m),\\
&\qquad\qquad \ldots,G(gx^n,gx^n,gz^n_m)\Big\}\Bigg)\\
&-\phi\Bigg(\max\Big\{G(gx^1,gx^1,gz^1_m),G(gx^2,gx^2,gz^2_m),G(gx^3,gx^3,gz^3_m),\\
&\qquad\qquad \ldots,G(gx^n,gx^n,gz^n_m)\Big\}\Bigg).
\end{aligned} \tag{3.29}$$

Set,
$\alpha_m = \max\Big\{G(gx^1,gx^1,gz^1_{m+1}),G(gx^2,gx^2,gz^2_{m+1}),G(gx^3,gx^3,gz^3_{m+1}),$
$\ldots,G(gx^n,gx^n,gz^n_{m+1})\Big\}.$
Then, from (3.29), we get

$$\psi(\alpha_m) \le \psi(\alpha_{m-1}) - \phi(\alpha_{m-1}). \tag{3.30}$$

Since, ϕ is non negative, therefore we obtain $\psi(\alpha_m) \le \psi(\alpha_{m-1}) \Longrightarrow \alpha_m \le \alpha_{m-1}$. Therefore $\{\alpha_m\}$ is a monotonically decreasing sequence of non-negative real numbers. So, there exists $\alpha \ge 0$ such that $\alpha_m \to \alpha$ as $m \to \infty$.
Letting the limit as $m \to \infty$ in (3.30), one can get

$$\psi(\alpha) \le \psi(\alpha) - \phi(\alpha).$$

Which gives, $\phi(\alpha) = 0$ and by the property of ϕ we get $\alpha = 0$. Hence

$$\lim_{m\to\infty} \alpha_m = 0.$$

Which yields

$$\lim_{m\to\infty} G(gx^1,gx^1,gz^1_{m+1}) = 0,\ \lim_{m\to\infty} G(gx^2,gx^2,gz^2_{m+1}) = 0,\ \ldots,\ \lim_{m\to\infty} G(gx^n,gx^n,gz^n_{m+1}) = 0.$$

Similarly, one can show that

$$\lim_{m\to\infty} G(gy^1,gy^1,gz^1_{m+1}) = 0,\ \lim_{m\to\infty} G(gy^2,gy^2,gz^2_{m+1}) = 0,\ \ldots,\ \lim_{m\to\infty} G(gy^n,gy^n,gz^n_{m+1}) = 0.$$

Now, by the rectangle inequality of G-metric, we have

$$\begin{cases}
G(gx^1,gx^1,gy^1) \le G(gx^1,gx^1,gz^1_{m+1}) + G(gz^1_{m+1},gz^1_{m+1},gy^1) \to 0,\\
G(gx^2,gx^2,gy^2) \le G(gx^2,gx^2,gz^2_{m+1}) + G(gz^2_{m+1},gz^2_{m+1},gy^2) \to 0,\\
\qquad\vdots\\
G(gx^n,gx^n,gy^n) \le G(gx^n,gx^n,gz^n_{m+1}) + G(gz^n_{m+1},gz^n_{m+1},gy^n) \to 0,
\end{cases}$$

as $m \to \infty$. From the above inequality, we obtain

$$gx^1 = gy^1, \quad gx^2 = gy^2, \quad \dots \quad , \quad gx^n = gy^n. \tag{3.31}$$

Since,

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = gx^1, \\ F(x^2, x^3, \dots, x^n, x^1) = gx^2, \\ \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = gx^n. \end{cases}$$

And by the commutativity of F and g, we have

$$\begin{cases} g(F(x^1, x^2, x^3, \dots, x^n)) = F(gx^1, gx^2, gx^3, \dots, gx^n) = ggx^1, \\ g(F(x^2, x^3, \dots, x^n, x^1)) = F(gx^2, gx^3, \dots, gx^n, gx^1) = ggx^2, \\ \vdots \\ g(F(x^n, x^1, x^2, \dots, x^{n-1})) = F(gx^n, gx^1, gx^2, \dots, gx^{n-1}) = ggx^n. \end{cases}$$

Now, put $gx^1 = u^1, gx^2 = u^2, \dots, gx^n = u^n$, then above inequality turns into

$$\begin{cases} F(u^1, u^2, u^3, \dots, u^n) = gu^1, \\ F(u^2, u^3, \dots, u^n, u^1) = gu^2, \\ \vdots \\ F(u^n, u^1, u^2, \dots, u^{n-1}) = gu^n. \end{cases} \tag{3.32}$$

Hence, $(u^1, u^2, u^3, \dots, u^n)$ is an n-tupled coincidence point of F and g. Now, put $y^1 = u^1, y^2 = u^2, \dots, y^n = u^n$ in (3.31), we get

$$gx^1 = gu^1, \quad gx^2 = gu^2, \quad \dots \quad , \quad gx^n = gu^n.$$

This gives,

$$gu^1 = u^1, \quad gu^2 = u^2, \quad \dots \quad , \quad gu^n = u^n. \tag{3.33}$$

From (3.32) and (3.33), we get

$$\begin{cases} F(u^1, u^2, u^3, \dots, u^n) = gu^1 = u^1, \\ F(u^2, u^3, \dots, u^n, u^1) = gu^2 = u^2, \\ \vdots \\ F(u^n, u^1, u^2, \dots, u^{n-1}) = gu^n = u^n. \end{cases}$$

Thus, $(u^1, u^2, u^3, \dots, u^n)$ is n-tupled common fixed point of F and g. To prove the uniqueness, assume that $(v^1, v^2, v^3, \dots, v^n)$ is an another n-tupled common fixed point of F and g. From (3.31), we obtain

$$\begin{cases} gu^1 = u^1 = gv^1 = v^1, \\ gu^2 = u^2 = gv^2 = v^2, \\ \vdots \\ gu^n = u^n = gv^n = v^n. \end{cases}$$

And this makes end to the proof. □

Following example establishes the usability of Theorem 3.1.

Example 3.3. Let $X = \mathbb{R}$ be equipped with the G-metric defined by
$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$,
for all $x, y, z \in X$ and the order $\preceq$ defined by $x \preceq y \Leftrightarrow x \leq y$.
Then $(X, G, \leq)$ is a complete partially ordered G-metric space.
Take $\psi = \frac{22t}{23}$ for all $t \in [0, \infty)$.
Consider the (continuous) mapping $F : X^n \to X$ given by

$$F(x^1, x^2, x^3, \ldots, x^n) = \frac{x^1 - 2x^2 + 3x^3 - \ \ldots - nx^n}{2n(n+1)}$$

for all $(x^1, x^2, x^3, \ldots, x^n) \in X$ and $n \geq 1$. And the mapping $g : X \to X$ given by $gx = \frac{x}{2}$.
Clearly F has the mixed g-monotone property, g is continuous and commutes with F and $F(X^n) \subseteq g(X)$.
Then for $(x^1, x^2, \ldots, x^n)$, $(y^1, y^2, \ldots, y^n)$, $(z^1, z^2, \ldots, z^n) \in X$,
with $gx^1 \geq gy^1 \geq gz^1$, $gx^2 \leq gy^2 \leq gz^2$, $\ldots$, $gx^n \leq gy^n \leq gz^n$, we have

$$\begin{aligned}
&\psi\Big(G\Big(F(x^1, x^2, x^3, \ldots, x^n), F(y^1, y^2, y^3, \ldots, y^n), F(z^1, z^2, z^3, \ldots, z^n)\Big)\Big)\\
&= \psi\Big(G\Big(\frac{x^1 - 2x^2 + 3x^3 - \ \ldots - nx^n}{2n(n+1)}, \frac{y^1 - 2y^2 + 3y^3 - \ \ldots - ny^n}{2n(n+1)},\\
&\qquad \frac{z^1 - 2z^2 + 3z^3 - \ \ldots - nz^n}{2n(n+1)}\Big)\Big)\\
&= \psi\Big(\Big|\frac{(x^1 - 2x^2 + 3x^3 - \ \ldots - nx^n) - (z^1 - 2z^2 + 3z^3 - \ \ldots - nz^n)}{2n(n+1)}\Big|\Big)\\
&= \frac{22}{23}\Big(\Big|\frac{(gx^1 - gz^1) - 2(gx^2 - gz^2) + 3(gx^3 - gz^3) - \ \ldots - n(gx^n - gz^n)}{n(n+1)}\Big|\Big)\\
&\leq \frac{22}{23\, n(n+1)}\Big(|gx^1 - gz^1| + 2|gx^2 - gz^2| + 3|gx^3 - gz^3| + \ ... + n|gx^n - gz^n|\Big)\\
&\leq \frac{11}{23}\Big(\max\{|gx^1 - gz^1|, |gx^2 - gz^2|, |gx^3 - gz^3|, \ldots, |gx^n - gz^n|\}\Big)\\
&= \frac{11}{23}\Big(\max\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), G(gx^3, gy^3, gz^3),\\
&\qquad \ldots, G(gx^n, gy^n, gz^n)\}\Big)\\
&\leq \psi\Big(\max\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), G(gx^3, gy^3, gz^3),\\
&\qquad \ldots, G(gx^n, gy^n, gz^n)\}\Big)
\end{aligned}$$

$$- \phi\Big(\max\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), G(gx^3, gy^3, gz^3), \\ \ldots, G(gx^n, gy^n, gz^n)\}\Big).$$

Hence, inequality (3.1) of Theorem 3.1 is satisfied with the choice of $\phi(t) = \frac{11t}{23}$ and $\phi(t) = \frac{10t}{23}$. Finally, we assert that all the conditions of Theorem 3.1 are satisfied. Clearly, $(0, 0, 0, \ldots, 0)$ is an n-tupled unique common fixed point of F and g.

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings on X. Assume that there exists $k \in [0, 1)$ such that for,*
$(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n), (z^1, z^2, \ldots, z^n) \in X$, we have

$$G(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n), F(z^1, z^2, \ldots, z^n)) \\ \leq k. \max\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), \ldots, G(gx^n, gy^n, gz^n)\},$$

with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gv^2 \preceq gz^2$,...,$gx^n \preceq gv^n \preceq gz^n$. Suppose that F has the mixed g-monotone property, $F(X^n) \subseteq g(X)$, g is continuous and commutes with F. Also, assume that, either

(a) F is continuous, or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n;

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all n.
If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that

$$\begin{cases} gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n), \\ F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \preceq gx_0^2, \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) \preceq gx_0^n. \end{cases}$$

Then F and g have an n*-tupled coincidence point in* X.

Proof. It is sufficient to take $\psi(t) = t$ and $\phi(t) = (1 - k)t$ for all $t \geq 0$ in Theorem 3.1. □

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings on X. Assume that there exists $k \in [0, 1)$ such that for,*
$(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n), (z^1, z^2, \ldots, z^n) \in X$, we have

$$G(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n), F(z^1, z^2, \ldots, z^n)) \\ \leq \frac{k}{n}\Big(G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \ldots + G(gx^n, gy^n, gz^n)\Big),$$

with $gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gv^2 \preceq gz^2, \ldots, gx^n \preceq gv^n \preceq gz^n$. *Suppose that* F *has the mixed* g*-monotone property,* $F(X^n) \subseteq g(X)$, g *is continuous and commutes with* F. *Also, assume that, either*

(a) F *is continuous, or*

(b) X *has the following properties:*

(i) if a non-decreasing sequence $\{x_n\}$ *is such that* $x_n \to x$, *then* $x_n \preceq x$ *for all* n;

(ii) if a non-increasing sequence $\{y_n\}$ *is such that* $y_n \to y$, *then* $y \preceq y_n$ *for all* n.
If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ *such that*

$$\begin{cases} gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n), \\ F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \preceq gx_0^2, \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) \preceq gx_0^n. \end{cases}$$

Then F *and* g *have an* n*-tupled coincidence point in* X.

Proof. We have

$$\begin{aligned} &\max\Big\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), \ldots, G(gx^n, gy^n, gz^n)\Big\} \\ &\le G(gx^1, gy^1, gz^1) + G(gx^2, gy^2, gz^2) + \ldots + G(gx^n, gy^n, gz^n) \\ &\le n \max\{G(gx^1, gy^1, gz^1), G(gx^2, gy^2, gz^2), \ldots, G(gx^n, gy^n, gz^n)\}. \end{aligned}$$

$gx^1 \succeq gy^1 \succeq gz^1$, $gx^2 \preceq gv^2 \preceq gz^2, \ldots, gx^n \preceq gv^n \preceq gz^n$.
Therefore, Corollary 3.5 follows from Corollary 3.4. □

Remark 3.6. In Theorem 3.1, if we restrict $F : X \times X \to X$, then we obtain Theorem 3.1 and 3.2 by Yeol Je Cho et al. in [13].

Remark 3.7. Theorem 3.1 and 3.2 of B.S. Choudhury and P. Maity [14] are particular case of Corollary 3.5 by taking $n = 2$ and $g = I$.

Remark 3.8. Corollary 3.1 and 3.2 of H. Aydi et al. [15] are particular case of Corollary 3.5 by taking $n = 2$.

Remark 3.9. Theorem 3.1 and 3.2 of H. Nashine [16] are particular case of Corollary 3.5 by taking $n = 2$ and restricting $k \in [0, \frac{1}{2})$.

Remark 3.10. Corollary 2.2 of Z. Mustafa [17] is a particular case of Corollary 3.5 by taking $n = 4$.

Remark 3.11. In Theorem 3.1, if we take $n = 2$ and $g = I$ then we obtain Theorem 2.1 and 2.2 of H. Lee [18].

References

[1] M. Imdad, A. Sharma, K.P.R. Rao, On n-tupled coincidence and common fixed points results for weakly contractive mappings in complete metric spaces, Bulletin of Mathematical Analysis and applications 5 (4) (2013) 19-39.

[2] B. Samet, C. Vetro, F. Vetro, Remarks on G-metric spaces, Int. J. Anal. 2013 (2013) Article ID 917158.

[3] M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl. 2012 (2012) https://doi.org/10.1186/1687-1812-2012-210.

[4] R. Agarwal, E. Karapinar, Remarks on some coupled fixed point theorems in G-metric spaces, Fixed Point Theory Appl. 2013 (2013) https://doi.org/10.1186/1687-1812-2013-2.

[5] M. Asadi, E. Karapinar, P. Salimi, A new approach to G-metric and related fixed point theorems, J. Inequal. Appl. 2013 (2013) https://doi.org/10.1186/1029-242X-2013-454.

[6] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2) (2006) 289-297.

[7] Z. Mustafa, H. Obiedat, F. Awawdeh, Some of fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory Appl. 2008 (2008) https://doi.org/10.1155/2008/189870.

[8] Z. Mustafa, W. Shatanawi, M. Bataineh, Fixed point theorems on complete G-metric spaces, J. Math. Stat. 4 (4) (2008) 196-201.

[9] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379-1393.

[10] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341-4349.

[11] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric space, Nonlinear Anal. 74 (15) (2011) 4889-4897.

[12] E. Karapinar, N.V. Luong, Quadrupled fixed point theorems for nonlinear contractions, Comp. Math. Appl. 64 (2012) 1839-1848.

[13] Y. Cho, B. Rhoades, R. Saadati, B. Samet, W. Shatanwi, Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type, Fixed point Theory and Appl. 2012 (2012) https://doi.org/10.1186/1687-1812-2012-8.

[14] B.S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Model 54 (2011) 73-79.

[15] H. Aydi, B. Damjanovic, B. Samet, W. Shatanwi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Math. Comput. Model 54 (2011) 2443-2450.

[16] H. Nashine, Coupled common fixed point results in ordered G-metric spaces, J. Nonlinear Sci. Appl. 2012 (2012) 1-13.

[17] Z. Mustafa, Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using $(\phi - \psi)$ contractions, Fixed Point Theory and Appl. 2012 (2012) https://doi.org/10.1186/1687-1812-2012-199.

[18] H. Lee, A coupled fixed point theorems for mixed monotone mappings on partial ordered G-metric spaces, KYUNGPOOK Math. J. 54 (2014) 485-500.

[19] M.S. Khan, M. Swalesh, S. Sessa, Fixed point theorems by altering distances between the points., Bull. Aust. Math. Soc. 30 (1) (1984) 1-9.

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