Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 305–319
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Weak and Strong Convergence Theorems for Zero Points of Inverse Strongly Monotone Mapping and Fixed Points of Quasi-nonexpansive Mappings in Hilbert Space

Buris Tongnoi[†] **and Suthep Suantai**[‡,1]

[†]Graduate M.Sc. Degree Program in Mathematics
Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : buris_t@cmu.ac.th

[‡]Data Science Research Center, Department of Mathematics
Faculty of Science, Chiang Mai University
Chiang Mai 50200, Thailand
e-mail : suthep.s@cmu.ac.th

Abstract : In this paper, we propose a new algorithm for zero points of inverse strongly monotone mapping and fixed points of a finite of quasi-nonexpansive mappings in Hilbert space and prove weak and strong convergence theorems for the proposed methods under some conditions. Moreover, we also show that the sequence generated by our algorithm converges to a solution of some variational inequality problem.

Keywords : variational inequality problem; zero point; fixed point; inverse strongly monotone; quasi-nonexpansive mappings.

2010 Mathematics Subject Classification : 47H05; 47H10; 47J20.

---

[1]Corresponding author.

1 Introduction

In 1966, Hartman-Stampacchia [1] were interested in studying the variational inequalites. After that it have been widely studied, since it covered diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance (see [2–7]). So, in 1990s the variational inequality problem became more and more important in nonlinear analysis and optimization. Several methods for solving variational inequalities, fixed point problems and zeros of monotone operators were proposed by many authors, see [8–11].

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and real numbers, respectively. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. The letter I stands for the identity mapping on H. For a given sequence $\{x_n\}$ in H, we denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote $\omega_w(x_n)$ is the set of all weak cluster point of $\{x_n\}$.

A mapping $F : H \to H$ is called *strongly monotone* if there exists $\eta \in \mathbb{R}$ with $\eta > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in H.$$

Such F is called a *η-strongly monotone mapping.*

A mapping $F : H \to H$ is called *inverse strongly monotone continuous* if there exists $\alpha \in \mathbb{R}$ with $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \alpha \|Fx - Fy\|^2, \quad \forall x, y \in H.$$

Such F is called *α-inverse strongly monotne.*

A mapping $F : H \to H$ is called *L-Lipschitz continuous* if there exsits a positive number L such that

$$\|Fx - Fy\| \leq L \|x - y\|, \quad \forall x, y \in H.$$

We can esaily see that if F is an α-inverse strongly monotone mapping, then F is $\frac{1}{\alpha}$-Lipschitz continuous. If $L = 1$, such F is called a nonexpansive mapping. A fixed point of a mapping $T : H \to H$ is a point $x \in H$ such that $Tx = x$. The set of all fixed points of T is denoted by $Fix(T)$. It is well known that if $T : H \to H$ is a nonexpansive mapping, then $Fix(T)$ is closed convex subset of H.

A self-mapping T of a subset C of a normed linear space is said to be *quasi-nonexpansive* provided T has at least one fixed point in C, and if $p \in C$ is any fixed point of T then $\|Tx - p\| \leq \|x - p\|$ holds for all $x \in C$.

The *Varational Inequality Problem* (VIP) is to find $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of H. We denote the set of solutions of variational inequality problem (1.1) by $VI(C, F)$.

It is known that if F is a strongly monotone and Lipschitzian mapping on C, then $VI(C,F)$ has a unique solution. It is well known that a variational inequality with respect to a closed convex subset in a Hilbert space $VI(C,F)$ is equivalent to a fixed-point equation

$$u^* = P_C(u^* - \mu F(u^*)) \tag{1.2}$$

where P_C is the (nearest point) projection from H onto C; i.e.,

$$P_C x = \underset{y\in C}{arg\,min}\|x-y\|, \quad \text{for } x, y \in H,$$

and where $\mu \geq 0$ is an arbitrarily fixed constant. Then it can be sloved by using the fixed-point method. Nevertheless, the formulation (1.2) concerns about the projection P_C, which is really hard to calculate because of the complication of the convex set C.

In order to overcome this problem caused by the metric projection, Yamada [12] made a hybrid steepest-descent method in 2001, it is new iterative algorithm for solving (1.1). His algorithm state as follows.

Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that a mapping $F : H \to H$ is L-Lipschitzian and η-strongly inverse strongly monotone over $T(H)$. Then with any $x_0 \in H$, any $\mu \in (0, \frac{2\eta}{L^2})$ and any sequence $\{\alpha_n\}_{n\geq 1} \subset (0,1]$ satisfying the condition below:

(L1) $\lim\limits_{n\to\infty} \alpha_n = 0$,

(L2) $\sum_{n\geq 1} \alpha_n = \infty$,

(L3) $\lim\limits_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0$,

generate a sequence $\{x_n\}$ by the following:

$$x_{n+1} := Tx_n - \alpha_{n+1}\mu F(Tx_n), \quad n \geq 0. \tag{1.3}$$

Then, he showed that $\{x_n\}$ converges strongly to the unique solution of $VI(C,F)$. In the case where $C = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ with $N \in \mathbb{N}$ and $T_i : H \to H$ is nonexpansive. He also offered similarly algorithm,

$$x_{n+1} := T_{[n+1]}x_n - \alpha_{n+1}\mu F(T_{[n+1]}x_n), \quad n \geq 0, \tag{1.4}$$

where $T_{[n]} = T_{n \mod N}$ with the mod function taking values in $\{1,2,\dots,N\}$, $\mu \in (0, \frac{2\eta}{L^2})$ and the sequence $\{\alpha_n\}$ satisfies condition (L1),(L2),(L4) and (L5), where

(L4) $\sum_{n=1}^{\infty} \|\alpha_n - \alpha_{n+N}\| < \infty$,

(L5) $C = Fix(T_1T_2\cdots T_{N-1}T_N) = Fix(T_NT_1\cdots T_{N-2}T_{N-1}) = \dots$
$= Fix(T_2T_3\cdots T_NT_1)$.

Yamada [12] proved that the sequence $\{x_n\}_{n\geq 0}$ defined by (1.4) converges strongly to the unique solution x^* of (1.1).

Later, in 2003, Xu and Kim [13] could replaced condition (L3) by the condition

(L3)$'$ $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, or equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$.

Clearly that condition (L3)$'$ is strictly weaker than that of (L3), including with conditions (L1) and (L2). In 2005, Yamada and Ogura [14] proposed Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings.

Afterward, in 2010, Liu and Cui [15] employed weak condition (L4) used in [12] and [13] by showing that if C is nonempty, then

$$C = \bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1 T_2 \cdots T_{N-1} T_N). \tag{1.5}$$

In 2011, Boung and Duong [16] mixed the concept of the hybrid steepest-descent method for variational inequalities with the Krasnosel'skii-Mann type algorithm for fixed-point problems together turn into a new itertive algorithm to solve (1.1).

In 2014, motivated by the [16], Zhou and Wang [17] developed an explicit iterative algorithm which is simpler than Boung and Duong's algorithm to solve (1.1), where the feasible set $C = \bigcap_{i=1}^{N} Fix(T_i)$ and prove a strong convergence theorem in the absence of conditions (1.5), (L3), (L3)$'$, (L4) and (L5). To be precise, let $\{T_i\}_{i=1}^{N}$ be nonexpansive self-mappings of H such that $C = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\rho_n^i\}$ be the same as before. For any point $x_0 \in H$, define a sequence $\{x_n\}_n \geq 0$ in the following:

$$x_{n+1} = (I - \alpha_n \mu F) T_N^n T_{N-1}^n \dots T_1^n x_n, \quad n \geq 0, \tag{1.6}$$

where $\mu \in (0, 2\eta/L^2)$ and $T_i^n := (1 - \rho_n^i) I + \rho_n^i T_i$, for $i = 1, \dots N$.

Recently, in 2017, Tian and Jiang [18] interested in the mapping F which is inverse strongly monotone mapping based on Zhou and Wang's algorithm for case $N = 1$. i.e.,

$$x_{n+1} = (I - \alpha_n \mu F)[(1 - \rho_n) I + \rho_n T](x_n), \quad \text{for each } n \in \mathbb{N}. \tag{1.7}$$

They modified the conditions of parameters in [17] and weaken the condition of F. They got a weak convergence theorem result for zero points of inverse strongly monotone mapping and fixed points of nonexpansive mapping. The weak limit of their algorithm is also a solution of the variational inequality problem (1.1).

In this paper, we extend a new iterative algorithm in [18] for a finite qusi-nonexpansive mapping $T_i (i = 1, \dots, n)$. In case of T_i is a nonexpansive mapping for $i = 1, \dots, n$, it will be our corollary.

2 Preliminaries

In this section, we introduce some useful lemmas which can be used for the main result.

Lemma 2.1. *[19] Let H be a real Hilbert space. Let $\{x_i, i = 1, 2, ..., n\} \subset H$. For $\alpha_i \in (0, 1), i = 1, 2, ..., n$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Then the following identity holds:*

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 = \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i \neq j}^{n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Definition 2.2. Let C be a closed convex subset of a metric space (X, d). A mapping $T : C \to C$ is said to be *semi-compact* if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{x_j} \to p \in C$

Definition 2.3. A point $x \in C$ is said to be a *zero point of* T if $Tx = 0$. We denote $T^{-1}0 = \{x \in C : Tx = 0\}$. We say that $T^{-1}0$ the set of zero point of T.

It is easy to see that $T^{-1}0 = Fix(I - \mu T)$ for all $\mu > 0$ where I is the identity mapping.

Lemma 2.4. *[20] Let H be a real Hilbert space. Let $T : H \to H$ be a mapping.*

1. *T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-inverse strongly monotone.*
2. *If T is ν-inverse strongly monotone, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-inverse strongly monotone,*
3. *For $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-inverae strongly monotone.*

Lemma 2.5. *[21] Let H be a real Hilbert space and $T : H \to H$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.6. *[22] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H satisfying the properties:*

1. *$\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in C$;*
2. *$\omega_w(x_n) \subset C$.*

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.7. *[23] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H. Suppose that,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall u \in C,$$

for every $n = 1, 2, \ldots$. Then, the sequence $\{P_C x_n\}$ converges strongly to a point in C.

3 Main Results

In this section, we demonstrate our main results. First of all, we need to propose the remark as the following:

Remark 3.1. *If $z \in \bigcap_{i=1}^N Fix(T_i) \cap F^{-1}0$ for $N \in \mathbb{N}$, then $z \in VI(\bigcap_{i=1}^N Fix(T_i), F)$.*

Proof. Let $z \in \bigcap_{i=1}^N Fix(T_i) \cap F^{-1}0$. Then we have $z = T_i z$ for all i and $0 = Fz$. It follows that $\langle Fz, x - z\rangle = \langle 0, x - z\rangle = 0$ for all $x \in \bigcap_{i=1}^N Fix(T_i)$. □

This remark shows that if we can show that $z \in \bigcap_{i=1}^N Fix(T_i) \cap F^{-1}0$ for some $z \in C$ where C is a nonempty closed convex subset of a Hilbert space H, then z is a solution of variational inequalities on common fixed points of nonexpansive mappings with respect to mapping F, i.e., $z \in VI(\bigcap_{i=1}^N Fix(T_i), F)$.

Now, we prove the first main result.

Theorem 3.2. *Let H be a real Hilbert space and $\{T_i\}_{i=1}^M$ be a finite family of quasi-nonepansive mappings of H into itself such that each $I - T_i$ is a demiclosed at zero. Let F be a k-inverse strongly monotone mapping of H into itself. Assume that $\Omega := \bigcap_{i=1}^M Fix(T_i) \cap F^{-1}0 \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 = x \in H$ and*

$$\begin{cases} y_n = \lambda_n x_n + \sum_{i=1}^M \lambda_{n,i} T_i x_n \\ x_{n+1} = (I - \mu\alpha_n F) y_n \end{cases} \tag{3.1}$$

for each $n \in \mathbb{N}$ where $\liminf_{n\to\infty} \lambda_{n,i} > 0$, $\lambda_n + \sum_{i=1}^M \lambda_{n,i} = 1$, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{\mu\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 2k)$. Then

(i) $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in \Omega$.

(ii) $\|x_n - T_i x_n\| \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$.

(iii) $\omega_w(x_n) \subset F^{-1}0$.

(iv) the sequence $\{x_n\}$ converges weakly to a point $z \in \Omega$ where $z = \lim_{n\to\infty} P_\Omega x_n$ and z is also a special point in $VI(\bigcap_{i=1}^M Fix(T_i), F)$.

Proof. Let $u \in \bigcap_{i=1}^M Fix(T_i) \cap F^{-1}0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|y_n - u\| &= \|\lambda_n x_n + \sum_{i=1}^M \lambda_{n,i} T_i x_n - u\| \\ &\leq \lambda_n \|x_n - u\| + \sum_{i=1}^M \lambda_{n,i} \|T_i x_n - u\| \\ &\leq \lambda_n \|x_n - u\| + \sum_{i=1}^M \lambda_{n,i} \|x_n - u\| \\ &= \|x_n - u\|. \end{aligned}$$

From (3.1) and above inequality, we have

$$\begin{aligned}\|x_{n+1}-u\|^2 =&\|(I-\mu\alpha_n F)y_n - u\|^2\\ =&\|y_n-\mu\alpha_n Fy_n - u+\mu\alpha_n Fu\|^2\\ \leq&\|y_n-u\|^2+\mu^2\alpha_n^2\|Fy_n-Fu\|^2-2\mu\alpha_n < y_n-u, Fy_n-Fu >\\ \leq&\|y_n-u\|^2+\mu^2\alpha_n^2\|Fy_n-Fu\|^2-2\mu\alpha_n k\|Fy_n-Fu\|^2\\ =&\|y_n-u\|^2+\mu\alpha_n(\mu\alpha_n-2k)\|Fy_n-Fu\|^2\\ \leq&\|y_n-u\|^2\leq\|x_n-u\|^2.\end{aligned}$$

So (i) had been proved. This implies that there exists $c\in\mathbb{R}$ be such that

$$c=\lim_{n\to\infty}\|x_n-u\|=\lim_{n\to\infty}\|y_n-u\|,$$

and the sequence $\{x_n\}$ and $\{y_n\}$ are bounded. On the other hand, by Lemma 2.1 we have

$$\begin{aligned}\|y_n-u\|^2 =&\|\lambda_n x_n+\sum_{i=1}^{M}\lambda_{n,i}T_i x_n-u\|^2\\ =&\|\lambda_n(x_n-u)+\sum_{i=1}^{M}\lambda_{n,i}(T_i x_n-u)\|^2\\ \leq&\lambda_n\|(x_n-u)\|^2+\sum_{i=1}^{M}\lambda_{n,i}\|T_i x_n-u\|^2-\lambda_n\lambda_{n,l}\|x_n-T_l x_n\|^2\\ \leq&(\lambda_n+\sum_{i=1}^{N}\lambda_{n,i})\|(x_n-u)\|^2-\lambda_n\lambda_{n,l}\|x_n-T_l x_n\|^2,\ \text{for } l=1,2,\ldots,M.\end{aligned}$$

For $l=1,2,\ldots,M$, we get

$$\lambda_n\lambda_{n,l}\|x_n-T_l x_n\|^2\leq\|x_n-u\|^2-\|y_n-u\|^2.$$

It follows that

$$x_n-T_i x_n\to 0,\quad n\to\infty. \tag{3.2}$$

So (ii) is proved.

Next we prove (iii), by making some tools as follows. Let $x\in\omega_w(x_n)$. Then there exits a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}\rightharpoonup x$. Since $I-T_i$ is demiclosed at zero, we obtain that $x\in Fix(T_i)$ for all $i=1,2,\ldots,n$. So $x\in\bigcap_{i=1}^{M}Fix(T_i)$. Hence $\omega_w(x_n)\subset\bigcap_{i=1}^{M}Fix(T_i)$. Now we show that $\omega_w(x_n)\subset F^{-1}0$. Setting $\beta_n=\frac{\mu\alpha_n}{2k}$ and $V=I-2kF$. Then V is nonexpansive (By Lemma 2.4) and

$$\begin{aligned}x_{n+1} =&(I-\mu\alpha_n F)y_n+\beta_n y_n-\beta_n y_n\\ =&y_n-2k\beta_n Fy_n+\beta_n y_n-\beta_n y_n\\ =&(I-\beta_n)y_n+\beta_n(I-2kF)y_n\\ =&(1-\beta_n)y_n+\beta_n Vy_n.\end{aligned} \tag{3.3}$$

Then, we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|(1-\beta_n)y_n+\beta_n Vy_n-u\|^2\\ &= \|(1-\beta_n)(y_n-u)+\beta_n(Vy_n-u)\|^2\\ &\le (1-\beta_n)\|y_n-u\|^2+\beta_n\|Vy_n-Vu\|^2-\beta_n(1-\beta_n)\|y_n-Vy_n\|^2\\ &\le \|y_n-u\|^2-\beta_n(1-\beta_n)\|y_n-Vy_n\|^2,\end{aligned}$$

which implies

$$\beta_n(1-\beta_n)\|y_n-Vy_n\|^2 \le \|y_n-u\|^2-\|x_{n+1}-u\|^2.$$

It follows that

$$y_n - Vy_n \to 0, \quad n\to\infty. \tag{3.4}$$

From (3.2) and Lemma 2.1, we have

$$\begin{aligned}\|x_n-y_n\|^2 &= \|x_n-(\lambda_n x_n+\sum_{i=1}^{M}\lambda_{n,i}T_ix_n)\|^2\\ &= \|\sum_{i=1}^{M}\lambda_{n,i}x_n-\sum_{i=1}^{M}\lambda_{n,i}T_ix_n\|^2\\ &= \|\sum_{i=1}^{M}\lambda_{n,i}(x_n-T_ix_n)\|^2\\ &\le \sum_{i=1}^{M}\lambda_{n,i}\|x_n-T_ix_n\|^2-\sum_{i,l=1,i\neq l}^{M}\lambda_{n,i}\lambda_{n,l}\|x_n-T_ix_n\|^2\\ &\le \sum_{i=1}^{M}\lambda_{n,i}\|x_n-T_ix_n\|^2 \quad \to 0, \text{ as } n\to\infty.\end{aligned}$$

Hence

$$x_n - y_n \to 0, \quad n\to\infty. \tag{3.5}$$

Let $z\in\omega_w(x_n)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to z. We may assume that $\alpha_{n_j}\to\alpha$ for some α such that $\mu\alpha\in[c,d]$. Since $x_{n_j}\rightharpoonup z$ and (3.5), we have $y_{n_j}\rightharpoonup z$. Thus we can show that $y_{n_{j_p}}\rightharpoonup z$.

Next we show that $z\in F^{-1}0$. Since $\{y_{n_j}\}$ is bounded sequence and F is $\frac{1}{k}$-continuous Lipschiz, there is $K>0$ such that $\|y_{n_j}\|\le K$ for all $j\in\mathbb{N}$ and

$$\begin{aligned}\|Fy_{n_j}\| &\le \|Fy_{n_j}-Fx_1\|+\|Fx_1\|\\ &\le \frac{1}{k}\|y_{n_j}-x_1\|+\|Fx_1\|\\ &\le \frac{1}{k}K+\frac{1}{k}\|x_1\|+\|Fx_1\|.\end{aligned}$$

Hence $\{Fy_{n_j}\}$ is bounded. It is noted that

$$\|(I-\mu\alpha_{n_j}F)y_{n_j}-(I-\mu\alpha F)y_{n_j}\|\le|\mu\alpha_{n_j}-\mu\alpha|\|Fy_{n_j}\|$$

From $\alpha_{n_j}\to\alpha$, we obtain

$$\|(I-\mu\alpha_{n_j}F)y_{n_j}-(I-\mu\alpha F)y_{n_j}\|\to 0,\quad j\to\infty. \tag{3.6}$$

From (3.3) and (3.4), we get

$$\|y_n-x_{n+1}\|=\|y_n-[(1-\beta_n)y_n+\beta_nVy_n]\|=\beta_n\|Vy_n-y_n\|\ \to 0,\ n\to\infty.$$

Hence $y_n-x_{n+1}\to 0,$ as $n\to\infty$. From (3.1), we get

$$\|y_{n_j}-(I-\mu\alpha_{n_j}F)y_{n_j}\|\ \to 0,\ j\to\infty. \tag{3.7}$$

Since

$$\begin{aligned}\|y_{n_j}-(I-\mu\alpha F)y_{n_j}\|\le{}&\|y_{n_j}-(I-\mu\alpha_{n_j}F)y_{n_j}\|\\&+\|(I-\mu\alpha_{n_j}F)y_{n_j}-(I-\mu\alpha F)y_{n_j}\|,\end{aligned}$$

by (3.6) and (3.7), we get

$$\|y_{n_j}-(I-\mu\alpha F)y_{n_j}\|\to 0,\quad j\to\infty.$$

It follows that $[I-(I-\mu\alpha F)]z=0$ and so $z=(I-\mu\alpha F)z$, which implies

$$z\in Fix(I-\mu\alpha F)=F^{-1}0.$$

Thus

$$\omega_w(x_n)\subset\bigcap_{i=1}^{M}Fix(T_i)\bigcap F^{-1}0.$$

From Lemma 2.6, we obtain that

$$x_n\rightharpoonup z\in\bigcap_{i=1}^{M}Fix(T_i)\bigcap F^{-1}0,$$

and Lemma 2.7 assures that

$$z=\lim_{n\to\infty}P_{\bigcap_{i=1}^{M}Fix(T_i)\bigcap F^{-1}0}x_n.$$

From Remark 3.1, we also obtain that $z\in VI(\bigcap_{i=1}^{M}Fix(T_i),F)$. Now we have been already proved (iii) and (iv). □

Theorem 3.3. *Let H, F, $\{T_i\}_{i=1}^M$, Ω, $\{\lambda_n\}$, $\{\lambda_{n,i}\}$ be the same as Theorem 3.2. Let $\{x_n\}$ be sequence generated by* (3.1)*. Then*

(i) $\|x_{n+1} - u\| \leq \|x_n - u\|$ *for all* $n \in \mathbb{N}$ *and* $u \in \Omega$.

(ii) $\|x_n - T_i x_n\| \to 0$ *as* $n \to \infty$ *for all* $i \in \mathbb{N}$.

(iii) $\omega_w(\{x\}) \subset F^{-1}0$.

(iv) If T_i is semi-compact for some $i \in \mathbb{N}$, then the sequence $\{x_n\}$ converges strongly to a point $z \in \Omega$ and z is also a special point in $VI(\bigcap_{i=1}^M Fix(T_i), F)$.

Proof. Let $u \in \bigcap_{i=1}^M Fix(T_i) \cap F^{-1}0$.
(i) It follows directly from Theorem 3.2(i) that $\|x_{n+1}-u\| \leq \|x_n-u\|$ for all $n \in \mathbb{N}$ and $u \in \Omega$. It follows that $\lim_{n\to\infty} \|x_n - u\|$ exists.
Using the same proof as in Theorem 3.2(ii) and (iii), we obtain (ii) and (iii).
(iv), Now, suppose that T_i is semi-compact for some $i \in \mathbb{N}$. Then there exists subsequence x_{n_j} of $\{x_n\}$ such that $x_{n_j} \to z \in H$. Next, we will show that $z \in \bigcap_{i=1}^M Fix(T_i)$. For $i \in \mathbb{N}$, we have

$$\begin{aligned}\|z - T_i z\| \leq& \|z - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|T_i x_{n_j} - T_i z\| \\ \leq& \|z - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|x_{n_j} - z\| \to 0 \text{ as } j \to \infty,\end{aligned}$$

hence which implies $z = T_i z$, $z \in \bigcap_{i=1}^M Fix(T_i)$. Now, we show that $z \in \bigcap_{i=1}^M Fix(T_i) \cap F^{-1}0$. Using the same proof as in Theorem 3.2, we can show that

$$x_n - y_n \to 0, \quad n \to \infty, \tag{3.8}$$

and there is a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\|y_{n_j} - (I - \mu\alpha F) y_{n_j}\| \to 0, \quad j \to \infty. \tag{3.9}$$

Futhermore, $\alpha_{n_{j_p}} \to \alpha$, for some $\alpha \in [\frac{c}{\mu}, \frac{d}{\mu}]$. It follows from (3.8), (3.9) and nonexpansitivity of $I - \mu\alpha_{n_{j_p}} F$ that

$$\begin{aligned}\|z - (I - \mu\alpha_{n_{j_p}} F) z\| \leq& \|z - x_{n_{j_p}}\| + \|x_{n_{j_p}} - (I - \mu\alpha_{n_{j_p}} F) x_{n_{j_p}}\| \\ &+ \|(I - \mu\alpha_{n_{j_p}} F) x_{n_{j_p}} - (I - \mu\alpha_{n_{j_p}} F) z\| \\ \leq& 2\|z - x_{n_{j_p}}\| + \|x_{n_{j_p}} - (I - \mu\alpha_{n_{j_p}} F) x_{n_{j_p}}\| \to 0 \text{ as } p \to \infty.\end{aligned}$$

Hence $\|z - (I - \mu\alpha_{n_{j_p}} F) z\| \to 0$, as $p \to 0$, which implies $(I - \mu\alpha F) z = z$. So $z \in Fix(I - \mu\alpha_{n_{j_p}} F) = F^{-1}0$. Since $\lim_{n\to\infty} \|x_n - z\|$ exists and $\lim_{n\to\infty} \|x_{n_j} - z\| = 0$, then $\lim_{n\to\infty} \|x_n - z\| = 0$. Moreover, by Remark 3.1 we also get that $z \in VI(\bigcap_{i=1}^M Fix(T_i), F)$. Therefore the proof is complete. □

The following results shows necessary and sufficient conditions for strong convergence of the proposed algorithm.

Theorem 3.4. *Let H,$\{T_i\}_{i=1}^M$, F, Ω, $\{\lambda_n\}$, $\{\lambda_{n,i}\}$ be the same as Theorem 3.3. Let $\{x_n\}$ be a sequence generated by* (3.1)*. Then $\{x_n\}$ converges strongly to $z \in \Omega$ if and only if* $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.

Proof. ($\Rightarrow$) The necessity is obvious.

($\Leftarrow$) Conversely, assume that $\liminf_{n\to\infty} d(x_n, \Omega) = 0$. By Theorem 3.2(i), we have $\|x_{n+1} - p\| \leq \|x_n - p\|, \forall p \in \Omega$. It follows that $d(x_{n+1}, \Omega) \leq d(x_n, \Omega)$. Thus $\lim\limits_{n\to\infty} d(x_n, \Omega)$ exists. By our hypothesis, we get $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Next show that $\{x_n\}$ is Cauchy sequence in H. Let $\epsilon > 0$ be arbitrary. Since $\lim\limits_{n\to\infty} d(x_n, \Omega) = 0$, there exists n_0 such that for all $n \geq n_0$, $d(x_n, \Omega) < \frac{\epsilon}{3}$. Thus $\inf\{\|x_{n_0} - p\| : p \in \Omega\} < \frac{\epsilon}{3}$. Then there exists $p^* \in \Omega$ such that $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$. For $m, n \geq n_0$, we get

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &< \epsilon.\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Hence $\lim\limits_{n\to\infty} x_n = q$ for some $q \in X$.
By Theorem 3.2(ii) we have that $\lim\limits_{n\to\infty} \|x_n - T_i x_n\| = 0, \forall i \in \mathbb{N}$. Since $I - T_i$ is demiclosed at 0, we obtain that $q \in Fix(T_i)$ for all $i \in \mathbb{N}$. Because $x_n \to q$, then we have $x_n \rightharpoonup q$. So using the same prove as in Theorem 3.2(iii), we can show that $q \in F^{-1}0$. Thus $q \in \Omega$. $\square$

Now, we using Theorem 3.4 and add more condition to get the follwing results.

Corollary 3.5. *Let H,$\{T_i\}_{i=1}^M$, F, Ω be the same as Theorem 3.3. Assume that there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $f(r) > 0$ for all $r > 0$ such that*

$$d(x_n, T_i x_n) \geq f(d(x_n, \Omega)) \quad \textit{for some } i \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ defined by (3.1) *converges strongly to a point in Ω.*

Proof. Assume that $d(x_n, T_i x_n) \geq f(d(x_n, \Omega))$ for some $i \in \mathbb{N}$. By Theorem 3.2(ii), we have $x_n - T_i x_n \to 0$, $n \to 0$. It follows that $\lim\limits_{n\to\infty} d(x_n, \Omega) = 0$. By Theorem 3.4, we obtain the desired result. $\square$

4 Numerical Example for Theorem 3.2

We now give some numerical example supporting our main result. Let $H = \mathbb{R}^2$ with the usual norm ($\|\cdot\|_2$). Define the mappings $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x_1, x_2) = (x - 1, x_2), \quad \forall x_1, x_2 \in \mathbb{R}.$$

and define $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ from the example in [24] by

$$T_i(x_1, x_2) = \begin{cases} (x_1, \dfrac{x_2}{3i} \sin \dfrac{1}{x_2}) & \text{if } x_2 \neq 0, \\ (x_1, 0) & \text{if } x_2 = 0, \end{cases} \qquad i \in \mathbb{N},$$

for all $x_1, x_2 \in \mathbb{R}$. Then we have F is 1-inverse strongly monotone mapping and $F^{-1}0 = \{(1,0)\}$, T_i are quasi-nonexpansive mapping for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{M} Fix(T_i) = \mathbb{R} \times \{0\}$. Define the real sequence $\{\lambda_n\}$ and $\{\lambda_{n,i}\}$ as follow:

$$\lambda_n = \frac{1}{3}\left(\frac{n+1}{n+2}\right), \quad n = 1, 2, 3, \ldots$$

and

$$\lambda_{n,i} = \begin{cases} \frac{1}{3^{i+1}}\left(\frac{n+1}{n+2}\right) & n > i; \\ 1 - \sum_{i=1}^{n} \frac{1}{3^i}\left(\frac{n+1}{n+2}\right) & n = i; \\ 0 \quad \text{otherwise}, \end{cases} \qquad i \in \mathbb{N}, n \in \mathbb{N},$$

That is,

$$\lambda_{n,i} = \begin{pmatrix} 1 - \frac{1}{3}\left(\frac{2}{3}\right) & 0 & 0 & 0 & \ldots \\ \frac{1}{3^2}\left(\frac{3}{4}\right) & 1 - \left[\frac{1}{3}\left(\frac{3}{4}\right) + \frac{1}{3^2}\left(\frac{3}{4}\right)\right] & 0 & 0 & \ldots \\ \frac{1}{3^2}\left(\frac{4}{5}\right) & \frac{1}{3^3}\left(\frac{4}{5}\right) & 1 - \left[\frac{1}{3}\left(\frac{4}{5}\right) + \frac{1}{3^2}\left(\frac{4}{5}\right) + \frac{1}{3^3}\left(\frac{4}{5}\right)\right] & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

We see that $\lim\limits_{n\to\infty} \lambda_n = \frac{1}{3}$ and $\lim\limits_{n\to\infty} \lambda_{n,i} = \frac{1}{3^{i+1}}$. Now, let our starting point is $x_1 = (3, 2)$ and let $\{x_n\}$ be the sequence generated by (3.1). Suppose that x_n is in the form $x_n = (x_1^n, x_2^n)$, where $x_1^n, x_2^n \in \mathbb{R}$. The critrion for stopping our testing method is taken as : $\|x_n - (1,0)\|_2 < 10^{-6}$. Choose $\mu\alpha_n = 0.1$. The value of x_n and $\|x_n - (1,0)\|_2$ are shown in Table 1.

Table 1: Numerical result for Theorem 3.2

n	x_1^n	x_2^n	$\|x_n - (1,0)\|_2$
1	3.00000000	2.00000000	2.82842712
2	2.80000000	0.62373191	1.90500433
3	2.62000000	0.21826510	1.63463746
4	2.45800000	0.03234654	1.45835876
5	2.31220000	0.00688208	1.31221804
⋮	⋮	⋮	⋮
137	1.00000119	0.00000000	0.00000011
138	1.00000107	0.00000000	0.00000011
139	1.00000096	0.00000000	0.00000009

The graph below shows the relation between the absolute error ($\|x_n - (0,1)\|_2$) and number of iteration(n).

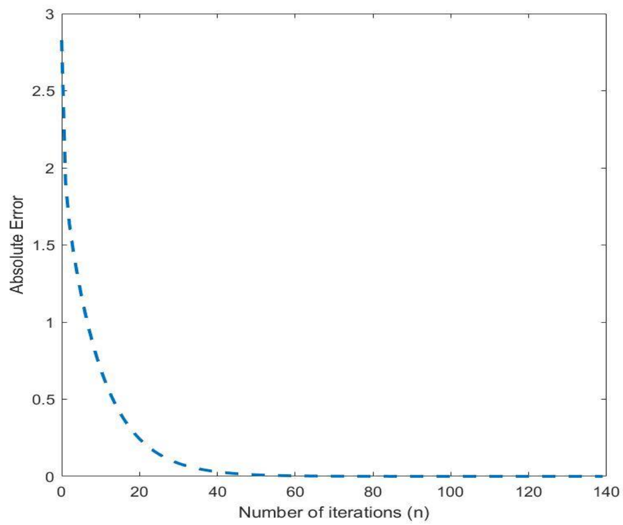

Figure 1: Graph for absolute errors of algorithm (3.1)

We observe from Table 1 that $x_n \to (1,0) \in \Omega$. We also note that the absolute error bounded of $\|x_{13} - (1,0)\|_2 < 10^{-6}$ and we can use

$$x_{139} = (1.00000096, 0.00000000)$$

to approximate the solution of (3.2) with accuracy at least 6 D.P.

Acknowledgements : The authors would like to thank the referees for valueable comments and suggestions for improving this work. This work was supported by the Development and Promotion of Science and Technology Talents Project (DPST), and Chiang Mai University.

References

[1] P. Hartamn, G. Stampaccha, On some non-linear elliptic differential-functional equations, Acta Math. 115 (1) (1966) 271-310.

[2] G. Duvaut, J.L. Lions, Inequalities in Mechanics and Physics, Springer, Berlin, 1976.

[3] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[4] I. Hlavacek, J. Haslinger, J.J. Necas, Lovicek, Solution of Variational Inequalities in Mechanics, Springer, New York, 1982.

[5] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer, New York, 1984.

[6] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications, Birkhuser, Boston, 1985.

[7] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Springer, New York, 1985.

[8] L.-C. Ceng, J.-C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, Applied Mathematics and Computation 190 (1) (2007) 205-215.

[9] S. Suantai, W. Cholamjiak, P. Cholamjiak, An implicit iteration process for solving a fixed point problem of a finite family of multi-valued mappings in Banach spaces, Applied Mathematics Letters 25 (11) (2012) 1656-1660.

[10] P. Cholamjiak, W. Cholamjiak, S. Suantai, A modified regularization method for finding zeros of monotone operators in Hilbert spaces, Journal of Inequalities and Applications (2015) https://doi.org/10.1186/s13660-015-0739-8.

[11] Y. Shehu, P. Cholamjiak, Iterative method with inertial for variational inequalities in Hilbert spaces, Calcolo 56 (1) (2019) 1-21.

[12] Y. Yamada, The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings, In: Inherently parallel algorithms in feasibility and optimization and there applications, (2001) 473-504.

[13] H.K. Xu, T.H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, J. Optim. Theory Appl. 119 (2003) 185-201.

[14] I. Yanada, N. Ogura, Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, Numer. Funct. Anal. Optim. 25 (2005) 619-655.

[15] X. Liu, Y. Cui, The common minimal-norm fixed point of a finite family of nonexpansive mappings, Nonlinear Anal. 73 (2010) 76-83.

[16] D. Buong, L.T. Duong, An explicit iterative algorithm for a class of variational inequalities in Hilbert spaces, J. Optim. Theory Appl. 151 (2011) 513-524.

[17] H. Zhou, P. Wang, A simpler explicit iterative algorithm for a class of variational inequalities in Hilbert spaces, J. Optim. Theory Appl. 161 (2014) 716-727.

[18] M. Tian, B.N. Jiang, Weak convergence theorem for zero points of inverse strongly monotone mapping and fixed points of nonexpansive mapping in Hilbert space, Optimization (2017) 1689-1698.

[19] C.E. Chidume, J.N. Ezeora, Krasnoselskii-type Algorithm for family of multi-valued strictly pseudo-conttractive mappings, Fixed Point Theory Appl. (2014) https://doi.org/10.1186/1687-1812-2014-111.

[20] L.C. Ceng, Q.H. Ansari, J.C. Yao, Some iterative method for finding fixed points and solving constrained convex minimization problem, Number Algebra, Control Optim. 1 (3) (2011) 341-359.

[21] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J Math Anal Appl. 298 (2004) 279-291.

[22] H.K. Xu, Averaged mappings and the gradient-projection algorithm, J. Optim. Theory Appl. 150 (2011) 360-378.

[23] W. Takahashi, M. Toyoda, Weak convergence theorem for nonexpensive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417-428.

[24] A. Hanjing, S. Suantai, The split common fixed point problem for infinite families of demicontractive mappings, Fixed Point Theory and Applications (2018) https://doi.org/10.1186/s13663-018-0639-y.

(Received 20 May 2019)
(Accepted 18 July 2019)