Thai Journal of Mathematics
Volume 17 (2019) Number 2 : 293–303

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Quadratic Polynomials with Rational Roots and Integer Coefficients in Arithmetic Progression

Aniruth Phon-On and Rattikan Saelim[1]

Department of Mathematics and Computer Science
Prince of Songkla University, Pattani 94000, Thailand
e-mail : aniruth.p@psu.ac.th (A. Phon-On)
rattikan.s@psu.ac.th (R. Saelim)

Abstract : In this paper, we first give conditions on the integer coefficients a, b, c of the quadratic equations of the form $ax^2+bx+c=0$ and $ax^2+bx-c=0$ so that their solutions are rational. Moreover, the coefficients a, b, c are in an arithmetic progression with a common difference d. Then, some interesting properties of those rational solutions are shown. Finally, the programming codes in *scilab* are given in order to generate those coefficients once the common difference d is given.

Keywords : arithmetic progression; quadratic polynomial; rational root; Fibonacci.
2010 Mathematics Subject Classification : 11B25; 11D09.

---

1 Introduction

It is well known that once we are given a polynomial function, the rational roots theorem can tell how those rational roots, if exist, of the function are like. The problem is we do not know whether the rational roots exist or not until we start solving it. On the other hand, another interesting question arises that how can we give a polynomial function that guarantees the rational roots. This question brought the work of this paper. In particular, under some nice conditions on the coefficients of the quadratic equation, one can guarantee rational roots.

[1]Corresponding author.

Given $a, b, c \in \mathbb{N}$ with $a \neq 0$, the quadratic equations of the form

$$p(x) = \pm ax^2 \pm bx \pm c = 0 \tag{1.1}$$

have different types of roots which are easily determined by the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For example, $p_1(x) = x^2 + 2x + 4 = 0$ has complex roots, while $p_2(x) = x^2 + 2x - 4 = 0$ has real roots but not rational. While $p_3(x) = x^2 + 2x - 3 = 0$ and $p_4(x) = 66c^2 + 68x - 70 = 0$ have rational roots. Surprisingly, the coefficients of both $p_3(x)$ and $p_4(x)$, $(1, 2, 3)$ and $(66, 68, 70)$, are in arithmetic progression with the common difference 1 and 2, respectively. We suddenly wondered if there are other quadratic equations in (1.1) such that the roots are rational and (a, b, c) is in arithmetic progression with the common difference $d \in \mathbb{Z}$. Before searching for the answer, let us notice that the equation (1.1) can be classified into 8 different equations. Assuming $a, b, c \in \mathbb{N} \cup \{0\}$ with $a \neq 0$, one has

$$\begin{aligned} &p_1(x) = ax^2 + bx + c = 0 && p_2(x) = ax^2 + bx - c = 0 \\ &p_3(x) = ax^2 - bx + c = 0 && p_4(x) = ax^2 - bx - c = 0 \\ &p_5(x) = -ax^2 + bx + c = 0 && p_6(x) = -ax^2 + bx - c = 0 \\ &p_7(x) = -ax^2 - bx + c = 0 && p_8(x) = -ax^2 - bx - c = 0 \end{aligned} \tag{1.2}$$

Note that $p_1 = -p_8$, $p_2 = -p_7$, $p_3 = -p_6$, and $p_4 = -p_5$ and also if r is a root of p_1 and s is a root of p_2, then $-r$ and $-s$ are roots of p_3 and p_4, respectively. Moreover, for $p_1^* = cx^2 + bx + a = 0$ and $p_2^* = cx^2 + bx - a = 0$, if the two nonzero rational r and s are the respective roots of p_1 and p_2 then it is easy to show that $-\frac{1}{r}$ and $-\frac{1}{s}$ are a root of p_1^* and p_2^*, respectively.

Now, let $\mathcal{A}$ be the set of all arithmetic progression triples (a, b, c) with $a \in \mathbb{N}$ and the common difference $d \in \mathbb{Z}$.

Thus, it suffices to consider only the equations of the forms:

$$\begin{aligned} p_1(x) = ax^2 + bx + c = 0 \\ p_2(x) = ax^2 + bx - c = 0 \end{aligned} \tag{1.3}$$

where $(a, b, c) \in \mathcal{A}$. In the case of the common difference $d = 1$, Schwartzman [1] has already shown that there are infinitely many quadratic equations of the form p_2 such that the roots are rational. In his paper, Schwartzman did not mention the form p_1 because for this particular case, $d = 1$, all the roots are complex. Also, he claimed without proof that the coefficients b of p_2 are necessarily elements of the Fibonacci numbers. Otherwise, $p_2 = 0$ could not have rational roots. However, the condition is not sufficient. For example, $2x^2 + 3x - 4 = 0$ has no rational root even though 3 is an element of the Fibonacci numbers.

In this paper, based on Schwartzman's conjecture, we proved it in a more general result for the quadratic equation of the form p_2. We also showed that under some conditions on d the quadratic equation of the form p_1 can possibly have rational roots. In fact, we proved that once the common difference d is given, there are only a finite number of elements $(a, b, c) \in \mathcal{A}$ such that the quadratic equation of the form p_1 has rational roots. In other words, the set

$$P_d = \{(a, b, c) \in \mathcal{A} \mid q(x) = ax^2 + bx + c = 0 \text{ has rational roots } \}$$

is finite. In addition, once the common difference d is given, the upper bound of the number $n(P_d)$ can be found. Finally, the algorithms based on the *scilab* code are given for generating all possible quadratic equations of the form p_1 and p_2 for a given d. In particular, in the case of p_1 the least upper bound of $n(P_d)$ is also confirmed by the algorithms.

2 Main Results

This section gives the conditions on which $p_1(x) = 0$ and $p_2(x) = 0$ have rational roots and following by some interesting properties of those solutions.

2.1 Rational Roots of $p_2(x) = 0$

Since $(a, b, c) \in \mathcal{A}$, is in arithmetic progression with the common difference d so that with $a = n \in \mathbb{N}$, p_2 can be written as

$$p_2(x) = nx^2 + (n + d)x - (n + 2d) = 0. \tag{2.1}$$

The roots of p_2 are rational if and only if the discriminant of p_2 is a perfect square. That is, for some $M \in \mathbb{Z}$, one must have

$$(n + d)^2 + 4n(n + 2d) = M^2 \quad \text{or} \quad 5n^2 + 10nd + (d^2 - M^2) = 0 \tag{2.2}$$

Solving equation (2.2) for n yields

$$n = -d \pm \sqrt{\frac{4d^2 + M^2}{5}}. \tag{2.3}$$

In order to get an integer radicand in equation (2.3), it is necessary to have

$$\begin{aligned} 4d^2 + M^2 &\equiv 0 \mod 5 \\ M^2 &\equiv d^2 \mod 5 \\ M &\equiv \pm d \mod 5. \end{aligned} \tag{2.4}$$

Unfortunately, the condition (2.4) on M is necessary to have an integer radicand but not sufficient to guarantee whether the radicand is perfect. Now, for each d, let us consider the set

$$\mathcal{M}_d = \left\{ M \in \mathbb{N} \mid M \equiv \pm d \mod 5 \;\; \& \;\; \frac{4d^2 + M^2}{5} \text{ is a perfect square} \right\}.$$

We claim that $\mathcal{M}_d$ is an infinite set. To prove the claim, we first need the following lemmas.

Lemma 2.1. *For each n, let F_n and L_n be a Fibonacci number and a Lucas number, respectively. Then we have*

$$L_n^2 = 5F_n^2 + 4(-1)^n, \tag{2.5}$$

and hence for each $d \in \mathbb{Z}$ the formula $(dL_n)^2 = 5(dF_n)^2 + 4d^2(-1)^n$ holds.

Proof. By the fundamental identities of Fibonacci and Lucas numbers

$$F_n^2 - F_{n-1}^2 = F_{n-1}F_n + (-1)^n \quad \text{and} \quad L_n = F_{n+1} + F_{n-1},$$

the proof is straightforward. □

Then we have the following Corollary.

Corollary 2.2. *For each $k \in \mathbb{N}$ and $d \in \mathbb{Z}$, we have $dL_{2k+1} \in \mathcal{M}_d$ and hence $\mathcal{M}_d$ is an infinite set.*

Proof. By Lemma 2.1, we have

$$(dL_{2k+1})^2 = 5(dF_{2k+1})^2 + 4d^2(-1)^{2k+1} = 5(dF_{2k+1})^2 - 4d^2.$$

Let $M = dL_{2k+1}$. Then $(dF_{2k+1})^2 = \dfrac{M^2 + 4d^2}{5}$ which implies that $M \in \mathcal{M}_d$. □

To prove the main theorem we need another following lemma.

Lemma 2.3. *Given $d \in \mathbb{Z}$, if $5x^2 - 4d^2$ is a perfect square and x is divisible by d then $x = \pm dF_{2k+1}$ for some k.*

Proof. Let $d \in \mathbb{Z}$. Assume that $5x^2 - 4d^2$ is a perfect square and x is divisible by d. Then there exists an integer k such that $x = dk$. Thus,

$$5x^2 - 4d^2 = 5(dk)^2 - 4d^2 = d^2(5k^2 - 4).$$

Since $d^2(5k^2 - 4)$ is a perfect square, it forces $5k^2 - 4$ to be perfect. So that $5k^2 - 4 = r^2$ for some positive integer r.

Next we will show that $k = \pm F_{2l+1}$ for some l. Consider the Pell's equation

$$x^2 - 5y^2 = -4. \tag{2.6}$$

One can show that, see e.g., [2] and [3], each integer solution of the equation (2.6) is one of the forms

$$\begin{aligned} x_{1n} + y_{1n}\sqrt{5} &= \pm(-1 + \sqrt{5})(9 + 4\sqrt{5})^n \\ x_{2n} + y_{2n}\sqrt{5} &= \pm(1 + \sqrt{5})(9 + 4\sqrt{5})^n \\ x_{3n} + y_{3n}\sqrt{5} &= \pm(4 + 2\sqrt{5})(9 + 4\sqrt{5})^n \end{aligned}$$

where

$$\begin{aligned}
x_{1n} &= \pm\frac{(\sqrt{5}-1)(9+4\sqrt{5})^n-(\sqrt{5}+1)(9-4\sqrt{5})^n}{2}\\
y_{1n} &= \pm\frac{(\sqrt{5}-1)(9+4\sqrt{5})^n+(\sqrt{5}+1)(9-4\sqrt{5})^n}{2\sqrt{5}}\\
x_{2n} &= \pm\frac{(\sqrt{5}+1)(9+4\sqrt{5})^n-(\sqrt{5}-1)(9-4\sqrt{5})^n}{2}\\
y_{2n} &= \pm\frac{(\sqrt{5}+1)(9+4\sqrt{5})^n+(\sqrt{5}-1)(9-4\sqrt{5})^n}{2\sqrt{5}}\\
x_{3n} &= \pm\frac{(4+2\sqrt{5})(9+4\sqrt{5})^n-(-4+2\sqrt{5})(9-4\sqrt{5})^n}{2}\\
y_{3n} &= \pm\frac{(4+2\sqrt{5})(9+4\sqrt{5})^n+(-4+2\sqrt{5})(9-4\sqrt{5})^n}{2\sqrt{5}}.
\end{aligned}$$

We also know that the mth element in the Fibonaci sequence is

$$F_m = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^m - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^m$$

In particular, for $m = 6n-1; n \in \mathbb{N}$, one has

$$\begin{aligned}
\pm F_{6n-1} &= \pm\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{6n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{6n-1}\right)\\
&= \pm\frac{(\sqrt{5}-1)(9+4\sqrt{5})^n+(\sqrt{5}+1)(9-4\sqrt{5})^n}{2\sqrt{5}}\\
&= \pm y_{1n}.
\end{aligned}$$

For $m = 6n+1; n \in \mathbb{N}$, one has

$$\begin{aligned}
\pm F_{6n+1} &= \pm\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{6n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{6n+1}\right)\\
&= \pm\frac{(\sqrt{5}+1)(9+4\sqrt{5})^n+(\sqrt{5}-1)(9-4\sqrt{5})^n}{2\sqrt{5}}\\
&= \pm y_{2n}.
\end{aligned}$$

For $m = 6n+3; n \in \mathbb{N}$, one has

$$\begin{aligned}
\pm F_{6n+3} &= \pm\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{6n+3} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{6n+3}\right)\\
&= \pm\frac{(4+2\sqrt{5})(9+4\sqrt{5})^n+(-4+2\sqrt{5})(9-4\sqrt{5})^n}{2\sqrt{5}}\\
&= \pm y_{3n}.
\end{aligned}$$

Since $5k^2 - 4 = r^2$, it follows that (r, k) is a solution of the equation (2.6) and hence $k = \pm y_{sn}$ for some $s \in \{1, 2, 3\}$. This implies that $x = \pm dF_{2l+1}$ for some l. $\square$

For Lemma 2.3, if x is not divisible by d, then it is not necessary that $x = dF_{2k+1}$. For example if $d = 11$ and $x = 13$, then it is obvious that $5(13)^2 - 4(11)^2 = 19^2$ which is a perfect square but $x \neq 11F_{2k+1}$ for all k.

Now we are ready to prove the main theorem.

Theorem 2.4. *Let $d \in \mathbb{Z}$ and $a, b, c, d \in \mathbb{N}$ be such that (a, b, c) is in arithmetic progression with the common difference d. If $d|\gcd(a, b, c)$ then $p_2(x) = ax^2 + bx - c = 0$ has rational roots if and only if $b = \pm dF_{2k+1}$ for some k where F_n is a Fibonacci number.*

Proof. The converse part is true via Corollary 2.2. Now assume that $p_2(x) = ax^2 + bx - c = 0$ has rational roots. Then the discriminant D is a perfect square. That is, $D = b^2 + 4ac = 5n^2 + 10dn + d^2 = M^2$ for some $M, n \in \mathbb{N}$. From the equation (2.3), we have $5b^2 - 4d^2 = M^2$, a perfect square. Since $d|\gcd(a, b, c)$ it implies that $d|b$. Thus by Lemma 2.3, $b = \pm dF_{2k+1}$ for some k. $\square$

2.2 Rational Roots of $p_1(x) = 0$

Similarly, the roots of the $p_1(x) = 0$ are rational if and only if the discriminant of $p_1(x)$ is a perfect square. In fact,

$$D = (n + d)^2 - 4n(n + 2d) = N^2 \quad \text{or} \quad 3n^2 + 6nd - d^2 + N^2 = 0 \tag{2.7}$$

for some $N \in \mathbb{N}$. Observe that the discriminant D depending on n can be negative. Hence, first we have to figure out the possible value of n for a given d such that D is positive. In this case, it is easy to show that

$$-d - \frac{2}{\sqrt{3}}|d| + 1 \leq n \leq -d + \frac{2}{\sqrt{3}}|d|.$$

Solving equation (2.7) for n yields

$$n = -d \pm \sqrt{\frac{4d^2 - N^2}{3}}. \tag{2.8}$$

In orter to get an integer radicand in equation (2.8), it is necessary to have

$$\begin{aligned} 4d^2 - N^2 &\equiv 0 \mod 3 \\ N^2 &\equiv d^2 \mod 3 \\ N &\equiv \pm d \mod 3. \end{aligned} \tag{2.9}$$

Unfortunately, the condition (2.9) on N is necessary to have an integer radicand but not sufficient to guarantee whether the radicand is perfect. Now, for each d, let us consider the sets

$$\mathcal{N}_d = \left\{ N \in \mathbb{N} \mid N \equiv \pm d \mod 3 \ \& \ \frac{4d^2 - N^2}{3} \text{ is a perfect square} \right\}$$

and

$$P_d = \{(a,b,c) \in \mathcal{A} \mid q(x) = ax^2 + bx + c = 0 \text{ has rational roots}\}.$$

Note that $\mathcal{N}_d \neq \emptyset$ if and only if $P_d \neq \emptyset$. Since

$$-d - \frac{2}{\sqrt{3}}|d| + 1 \leq n \leq -d + \frac{2}{\sqrt{3}}|d|,$$

it follows that $n(\mathcal{N}_d) < \infty$ and $n(P_d) < \infty$. From this fact, we know that for a given d, it is not necessarily that $P_d \neq \emptyset$. For example, $P_d = \emptyset$ if $d = 1, 2, 3, 4, 5, 6$. So, it is quite interesting to find the integer d such that P_d is always non-empty. The following lemma gives such integers d but not all.

Lemma 2.5. *Let $N \in \mathbb{N}$. For each $k \in \mathbb{N}$, define d_k and n_k as follow*

$$\begin{aligned} d_k &= \frac{N}{4}\left((2+\sqrt{3})(7+4\sqrt{3})^k + (2-\sqrt{3})(7-4\sqrt{3})^k\right) \\ n_k &= \frac{N}{4\sqrt{3}}\left((7+4\sqrt{3})^k - (7-4\sqrt{3})^k\right). \end{aligned}$$

Then $p(x) = n_k x^2 + (n_k + d_k)x + (n_k + 2d_k) = 0$ has rational roots.

Proof. It suffices to show that for each k, d_k and n_k are integers and satisfied

$$3n_k^2 + 6n_k d_k - d_k^2 + N^2 = 0.$$

Note that the pair of

$$\frac{d_k - 3n_k}{N} = \frac{(7+4\sqrt{3})^k + (7-4\sqrt{3})^k}{2}$$

and

$$\frac{n_k}{N} = \frac{(7+4\sqrt{3})^k - (7-4\sqrt{3})^k}{4\sqrt{3}}$$

are all integer solutions of the Pell's equation $x^2 - 12y^2 = 1$, [3]. That is, $(d_k - 3n_k)^2 - 12n_k^2 = N^2$ and so it implies that $3n_k^2 + 6n_k d_k - d_k^2 + N^2 = 0$. $\square$

2.3 Interesting Properties

Besides, we have found that with some transformations, the product of the roots of both cases, $p_1(x) = 0$ and $p_2(x) = 0$ are constants, as shown in the following lemmas.

Lemma 2.6. *If α and β are the roots of p_2, then $(\alpha-2)(\beta-2) = 5$. Consequently, if α and β are integers, then $(\alpha, \beta) \in \{(3,7), (7,3), (1,-3), (-3,1)\}$.*

Proof. Assume that α and β are the roots of $p_2(x) = nx^2 + (n+d)x - (n+2d) = 0$ for some d and n. Then $\alpha + \beta = -(\frac{n+d}{n})$ and $\alpha\beta = -\frac{n+2d}{n}$. And so,

$$\begin{aligned}(\alpha - 2)(\beta - 2) &= \alpha\beta - 2(\alpha + \beta) + 4 \\ &= -(\frac{n+2d}{n}) - 2(-\frac{n+d}{n}) + 4 \\ &= 5.\end{aligned}$$

Furthermore, if α and β are integers, then both $\alpha - 2$ and $\beta - 2$ are in $\{\pm 1, \pm 5\}$ and hence $(\alpha, \beta) \in \{(3, 7), (7, 3), (1, -3), (-3, 1)\}$. □

Lemma 2.7. *If α and β are the roots of p_1, then $(\alpha+2)(\beta+2) = 3$. Consequently, if α and β are integers, then $(\alpha, \beta) \in \{(-1, 1), (1, -1), (-3, -5), (-5, -3)\}$.*

Proof. Assume that α and β are the roots of $p_2(x) = nx^2 + (n+d)x + (n+2d) = 0$ for some d and n. Then $\alpha + \beta = -(\frac{n+d}{n})$ and $\alpha\beta = \frac{n+2d}{n}$. And so,

$$\begin{aligned}(\alpha + 2)(\beta + 2) &= \alpha\beta + 2(\alpha + \beta) + 4 \\ &= \frac{n+2d}{n} + 2(-\frac{n+d}{n}) + 4 \\ &= 3.\end{aligned}$$

Furthermore, if α and β are integers, then both $\alpha + 2$ and $\beta + 2$ are in $\{\pm 1, \pm 3\}$ and hence $(\alpha, \beta) \in \{(-1, 1), (1, -1), (-3, -5), (-5, -3)\}$. □

Lemma 2.8. *Let $d \in \mathbb{Z}$. For each $i \in \mathbb{N}$, let n_i be a positive integer such that $p_{2,i}(x) = n_i x^2 + (n_i + d)x - (n_i + 2d) = 0$ has rational roots. If α_i and β_i are such rational roots of $p_{2,i}(x) = 0$ with $\alpha_i < \beta_i$ then $\{\alpha_i\}$ converges to $\frac{-1-\sqrt{5}}{2}$ and $\{\beta_i\}$ converges to $\frac{-1+\sqrt{5}}{2}$.*

Proof. Note that $\alpha_i + \beta_i = -\frac{n_i + d}{n_i}$ and $\alpha_i\beta_i = -\frac{n_i + 2d}{n_i}$ and hence as n_i approaches infinity, both $\alpha_i + \beta_i$ and $\alpha_i\beta_i$ converge to -1. Moreover, α_i and β_i are the two solutions of the quadratic equation $x^2 - (\alpha_i + \beta_i)x + \alpha_i\beta_i = 0$. Since $\alpha_i < \beta_i$, it follows that

$$\beta_i = \frac{(\alpha_i + \beta_i) + \sqrt{(\alpha_i + \beta_i)^2 - 4(\alpha_i\beta_i)}}{2} \to \frac{-1+\sqrt{5}}{2}$$

and

$$\alpha_i = \frac{(\alpha_i + \beta_i) - \sqrt{(\alpha_i + \beta_i)^2 - 4(\alpha_i\beta_i)}}{2} \to \frac{-1-\sqrt{5}}{2}.$$

This completes the proof. □

3 Algorithms

Finally, we complete this study by giving algorithms in *scilab* codes. A simple yet important reason to choose scilab because it is a freeware.

Let us first consider the quadratic equation $p_2(x) = 0$. Recall that once d is chosen we want to fine M such that

$$n = -d \pm \sqrt{\frac{4d^2 + M^2}{5}}$$

is integer and hence the quadratic equation $p_2(x) = nx^2 + (n+d)x - (n+2d) = 0$ has rational roots. For a given d, the function *findm* (Table 1., left) is searching for those M such that $5|(4d^2 + M^2)$. After that the function *getperfect* (Table 1., right) will screen out these M for only $\frac{4d^2+M^2}{5}$ is perfect. Recall that $\mathcal{M}_d$ is an infinite set so that in the input argument, d, of the function *findm* represents the common difference while q restricts the number M we desire. For example, for $d = 1$ and the first 1500 numbers of M ($q = 1500$) there are only seven of M which are the output md including $4, 11, 29, 76, 199, 521$ and 1364 forcing $\frac{4d^2+M^2}{5}$ perfect. Now, in this case, with $d = 1$ and $M \in \{4, 11, 29, 76, 199, 521, 1364\}$, we obtain the coresponding leading coefficients $n \in \{1, 4, 12, 33, 88, 232, 609\}$, respectively. With these leading coefficients n, the equation $p_2(x) = 0$ has rational roots. For examples, for $n = 232$ and $n = 609$, one has $232x^2+233x-234 = (8x-13)(29x+18)$ and $609x^2 + 610x - 611 = (21x - 13)(29x + 47)$, respectively.

function [m]=findm(d, q),
i = 1;
if d <> 5 then
  k = 1;
  while i < q,
    m(i) = 5 * k − d;
    m(i + 1) = 5 * k + d;
    k = k + 1;
    i = i + 2;
  end
else for k = 1 : q,
    m(k) = 5 * k;
  end
end
endfunction
	function [md]=getperfect(m, d)
	n = max(size(m));
	i = 1;
	i = 1;
	md = 0;
	for k = 1 : n,
	  c = (4*d∧2 + m(k)∧2)/5;
	  if (c == (floor(sqrt(c))∧2)) then
	    md(i) = m(k);
	    i = i + 1;
	  end
	end
	endfunction

Table 1: Left: codes for *findm* Right: codes for *getperfect*

It would be useful to note here that for $d > 1$ the coefficient we get are actually the multiple of the case $d = 1$. For example with $d = 2$, the first several leading coefficients n are 2, 8, 24, 66, 176 and 464. For examples, for 176 and 464, one has $176x^2 + 178x - 180 = 2(88x^2 + 89x - 90) = 2(8x + 5)(11x - 18)$ and $464x^2 + 466x - 468 = 2(232x^2 + 233x - 234) = 2(8x - 13)(29x + 8)$, respectively.

Next, let us turn to the quadratic equation $p_1(x)$ of the form $nx^2 + (n+d)x + (n + 2d) = 0$. The first difference made by the condition on n:

$$-d - \frac{2}{\sqrt{3}}|d| + 1 \leq n \leq -d + \frac{2}{\sqrt{3}}|d|.$$

In this study, only the positive leading coefficients are of interest. So that the above condition forces d to be greater than 7. With the quite similar algorithms we can obtain the integer N such that $\frac{4d^2-N^2}{3}$ is integer and perfect. For example, the case $d = 7$ all the possible leading coefficients are shown in Table 2. They are actually within the interval [-16,1] confirmed by

n	$p_1(x)$
1	$x^2 + 8x + 15 = (x + 3)(x + 5)$
-2	$-2x^2 + 5x + 112 = (2x + 3)(-x + 4)$
-4	$-4x^2 + 3x + 10 = (4x + 5)(-x + 2)$
-7	$-7x^2 + 7 = -7(x + 1)(x - 1)$
-10	$-10x^2 - 3x + 4 = (5x + 4)(-2x + 1)$
-12	$-12x^2 - 5x + 2 = (3x + 2)(-4x + 1)$
-15	$-15x^2 - 8x - 1 = -(3x + 1)(5x + 1)$

Table 2: All possible leading coefficients for $d = 7$ so that $p_1(x) = 0$ has rational roots

However, since our interest is restricted only on the positive leading coefficient, so that in this case, $d = 7$, the eligible leading coefficient is only $n = 1$. In particular, we have $x^2 + 8x + 15 = (x + 3)(x + 5)$.

4 Conclusions

In this study, it was shown that the quadratic equation of the form $p_2(x) = ax^2 + bx - c = 0$ has infinitely many $(a, b, c) \in \mathcal{A}$ such that $p_2(x) = 0$ has rational roots. While for the quadratic equation of the form $p_1(x) = ax^2 + bx + c = 0$ has only a finite number of $(a, b, c) \in \mathcal{A}$ such that $p_1(x) = 0$ has rational roots. Furthermore, for $0 < d < 7$ and $n > 0$, we found that there is no rational solution to $p_1(x) = 0$. However, if the condition $n > 0$ is neglected, it turns outs out that the rational solution of $p_1(x) = 0$ does exist! In fact, with the leading coefficient $n = -d$, one always has $p_1(x) = -dx^2 + d = -d(x^2 - 1)$. In which case, the solutions are ± 1.

Acknowledgements : We would like to thank the referees for their comments and suggestions on the manuscript. This work was partially supported by Prince of Songkla University, Pattani Campus, Thailand.

References

[1] S. Schwartzman, Factoring polynomails and Fibonacci, The Mathematics Teacher 79 (1) (1986) 54-57.

[2] E.J. Barbeau, Problem Books in Mathematics: Pell's Equation, Springer-Verlag, New York, 2003.

[3] A.M.S. Ramasamy, Polynomial solutions for the Pell's equation, Indian J. Pure Appl. Math. 25 (6) (1994) 577-581.

(Received 8 August 2015)
(Accepted 22 December 2018)