Thai Journal of Mathematics
Volume 17 (2019) Number 1 : 283–291
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Some Operations Defined on Subspaces via α-Open Sets

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Abstract : The purpose of the present paper is to introduce and study some operations defined on subspaces and subspace-operation by using α-open sets for a given operation. Also, we study some topological properties of such subspaces and investigate some relationships among the families which are obtained by such subspace operation. Finally, we study some types of operation-closures on subsets of subspace-operation by means of α-open sets.

Keywords : operation; α-open set; α_γ-open set; subspace-operation-α-open; operation-closure formula in subspaces.
2010 Mathematics Subject Classification : 54A05; 54A10; 54C05.

1 Introduction and Preliminaries

Njastad [1] initiated the study of α-open sets in a topological space. Further, Ibrahim [2] defined the concept of an operation on the family of α-open sets and he introduced the concept of α_γ-open sets and also investigated the basic properties of this set.

Let A be a subset of a topological space (X, τ). We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively. A subset A of a topological

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space (X, τ) is said to be α-open [1] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open sets in a topological space (X, τ) is denoted by $\alpha O(X, \tau)$. An operation $\gamma : \alpha O(X, \tau) \to P(X)$ [2] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$. A subset A of X is called an α_γ-open set [2] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. The complement of an α_γ-open set is said to be α_γ-closed. We denote the set of all α_γ-open (resp., α_γ-closed) setsof (X, τ) by $\alpha O(X, \tau)_\gamma$ (resp., $\alpha C(X, \tau)_\gamma$). The α_γ-closure [2] of a subset A of X with an operation γ on $\alpha O(X)$ is denoted by $\alpha_\gamma Cl(A)$ and is defined to be the intersection of all α_γ-closed sets containing A. A point $x \in X$ is in αCl_γ-closure [2] of a set $A \subseteq X$, if $U^\gamma \cap A \neq \phi$ for each α-open set U containing x. The αCl_γ-closure of A is denoted by $\alpha Cl_\gamma(A)$. An operation γ on $\alpha O(X, \tau)$ is said to be α-regular [2] if for every α-open sets U and V of each $x \in X$, there exists an α-open set W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. An operation γ on $\alpha O(X, \tau)$ is said to be α-open [2] if for every α-open set U of $x \in X$, there, exists an α_γ-open set V of X such that $x \in V$ and $V \subseteq U^\gamma$. The operation $id : \alpha O(X, \tau) \to P(X)$ is defined by $id(V) = V$ for any set $V \in \alpha O(X, \tau)$ this operation is called the identity operation on $\alpha O(X, \tau)$ [2]. An operation $\gamma : \alpha O(X) \to P(X)$ is said to be α-monotone on $\alpha O(X)$ [3] if for all $A, B \in \alpha O(X)$, $A \subseteq B$ implies $A^\gamma \subseteq B^\gamma$.

2 Some Operations on Subspaces

In the present section, we consider the subspace $(H, \alpha|H)$ of a topological space (X, τ), where H is any non-empty subset of X and $\alpha|H = \{V = U \cap H : U \in \alpha O(X)\}$. For a given operation $\gamma : \alpha O(X) \to P(X)$ we introduce two subspace-operations denoted by γ^α_H and $\gamma_{\alpha H}$. In general we assume that $H \in \alpha O(X)$ whenever $\gamma \neq id$. Also, we introduce the concept of α_γ-open sets relative to a subset H and we investigate general properties of them. In the end of the section, we investigate some forms of operation closures in a subspace.

Definition 2.1. An operation $\gamma^\alpha_H : \alpha|H \to P(H)$ is defined as follows: $\gamma^\alpha_H(V) = V^\gamma \cap H$ for every $V \in \alpha|H$, where $V^\gamma = \gamma(V)$ is the value of γ at $V \in \alpha O(X)$. Whenever $\gamma \neq id$ we have to assume that $H \in \alpha O(X)$ and if $\gamma = id$, then H may not α-open and we have $\gamma^\alpha_H(V) = V$ for every $V \in \alpha|H$.

This operation γ^α_H is said to be the restriction of γ on $\alpha|H$.

Remark 2.2. *When we consider the operation $\gamma^\alpha_H : \alpha|H \to P(H)$, we assume $H \in \alpha O(X)$ if $\gamma \neq id$. And, if $\gamma = id : \alpha O(X) \to P(X)$, then we do not assume that $H \in \alpha O(X)$. Namely, even if $H \notin \alpha O(X)$ by definition, for any subset H of (X, τ), $id^\alpha_H : \alpha|H \to P(H)$ is the identity operation on $\alpha|H$. Indeed, $id^\alpha_H(V) = V$ for any $V \in \alpha|H$.*

Definition 2.3. Let H be any subset of the space X and let $\alpha O(X)_H$ denotes the following family of subsets of H:

$\alpha O(X)_H = \{U : U \subseteq H, U \in \alpha O(X)\}$. An operation $\gamma_{\alpha H} : \alpha O(X)_H \to P(H)$ is defined by $\gamma_{\alpha H}(U) = U^\gamma \cap H \in P(H)$ for every $U \in \alpha O(X)_H$, where $U^\gamma = \gamma(U)$ (the value of γ at $U \in \alpha O(X)_H \subseteq \alpha O(X)$).

Remark 2.4. *The following properties are easy to prove for any subset H of X and any operation $\gamma : \alpha O(X) \to P(X)$.*

1. *$\alpha O(X)_H \subseteq \alpha|H \subseteq P(H)$ and $\alpha O(X)_H = \alpha|H$ is a topology on H whenever H is α-open in (X, τ).*
2. *If H is α-open in (X, τ), then $\gamma_{\alpha H} = \gamma_H^\alpha : \alpha|H \to P(H)$.*
3. *In general if $f : \alpha|H \to P(H)$ defined by $f(W \cap H) = W^\gamma \cap H$, then it is not well defined, where $W \in \alpha O(X)$ even if H is an α-open subset of X.*

Indeed, for some topological space (X, τ) and a subset H of X, we can take two α-open sets W and S of (X, τ) such that $W \cap H = S \cap H$ and $W^\gamma \cap H \neq S^\gamma \cap H$, thus f is not well defined as we can see in the following example:

Example 2.5. Let $X = \{a, b, c\}$, $\tau = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $H = \{a, c\}$. And let $\gamma : \alpha O(X) \to P(X)$ be a given operation defined by

$$U^\gamma = \begin{cases} U & \text{if } b \in U \\ Cl(U) & \text{if } b \notin U. \end{cases}$$

Then, we take $W = \{a, b\} \in \alpha O(X)$ and $S = \{a\} \in \alpha O(X)$, then $W \cap H = \{a\} = S \cap H$ and $W^\gamma \cap H = \{a, b\}^\gamma \cap H = \{a, b\} \cap H = \{a\}$ and $S^\gamma \cap H = \{a\}^\gamma \cap H = Cl(\{a\}) \cap H = \{a, c\} \cap H = \{a, c\}$. Thus, $W^\gamma \cap H \neq S^\gamma \cap H$ and so $f(W \cap H) \neq f(S \cap H)$.

Definition 2.6. Let H be any subset of X and operation γ from $\alpha O(X)$ to $P(X)$ is α-stable with respect to H if γ induces an operation $\gamma_H : \alpha|H \to P(H)$ satisfying the following two properties:

1. $(U \cap H)^{\gamma_H} = U^\gamma \cap H$ for every $U \in \alpha O(X)$ and
2. $W \cap H = S \cap H$ implies that $W^\gamma \cap H = S^\gamma \cap H$ for every $W, S \in \alpha O(X)$.

Proposition 2.7. *Let γ an operation on $\alpha O(X)$ such that $\phi^\gamma = \phi$. If γ is α-stable with respect to all of proper α-closed set of X, then γ is the identity operation.*

Proof. Let $U \in \alpha O(X)$ and $U \neq \phi$. Then $X \setminus U = F$, is a proper α-closed set. By hypothesis γ is α-stable with respect to F, so we have $U^\gamma \cap F = (U \cap F)^{\gamma_F} = \phi^{\gamma_F} = (\phi \cap F)^{\gamma_F} = \phi^\gamma \cap F = \phi \cap F = \phi$. Then we have $U^\gamma \cap (X \setminus U) = \phi$ and hence $U^\gamma \subseteq U$. Therefore, $U = U^\gamma$ for any $U \in \alpha O(X)$. Since $\phi^\gamma = \phi$, hence $U^\gamma = U$ for every $U \in \alpha O(X)$. □

The following example shows that there exists an α-stable operation with respect to a subset H which is not α-identity operation.

Example 2.8. Let H be any subset of the space X. The operations $\gamma : \alpha O(X) \to P(X)$ and $\gamma_H : \alpha|H \to P(H)$ are defined by $U^\gamma = X$ and $(U \cap H)^{\gamma_H} = H$ for every $U \in \alpha O(X)$. Then, it is clear that γ is α-stable with respect to H which is not an α-identity operation.

Definition 2.9. Let H be any subset of the space X and $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$. A nonempty subset A of a subspace H is said to be α_γ-open relative to H, if for each point $x \in A$ there exists a subset $U \in \alpha O(X)$ such that $x \in U$ and $U^\gamma \cap H \subseteq A$. It is assumed that the empty set ϕ is also α_γ-open relative to H.

A subset F of H is said to be α_γ-closed relative to H, if $H \setminus F$ is α_γ-open relative to H. The collection of all α_γ-open sets relative to H is denoted by $\alpha O(X)^\gamma_H$.

Remark 2.10. *Let H be any subset of the space X. Then, we have the following properties:*

1. $\alpha O(X)_{id} = \alpha O(X)$.
2. $\alpha O(X)^{id}_H = \alpha|H$.

Theorem 2.11. *Let $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$ and H a subset of X.*

1. *The union of any family of α_γ-open sets relative to H is an α_γ-open set relative to H.*
2. *If γ is α-regular on $\alpha O(X)$, then the family $\alpha O(X)^\gamma_H$ forms a topology on H.*

Proof. 1. Let $\{A_i|i \in I\}$ be a family of α_γ-open sets relative to H, where I is an index set. Put $A = \cup\{A_i|i \in I\}$. Let $x \in A$. There exists an α_γ-open set A_i relative to H such that $x \in A_i$, where $i \in I$. Then, there exists a subset $U(i) \in \alpha O(X)$ such that $x \in U(i)$ and $U(i)^\gamma \cap H \subseteq A_i \subseteq A$. We prove that A is α_γ-open relative to H.

2. Let B and E be α_γ-open sets relative to H. Let $x \in B \cap E$. There exist two α-open sets U and V of X containing x such that $U^\gamma \cap H \subseteq B$ and $V^\gamma \cap H \subseteq E$. Since γ is α-regular on $\alpha O(X)$, there exists an α-open set W of X containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$, then $W^\gamma \cap H \subseteq (U^\gamma \cap H) \cap (V^\gamma \cap H) \subseteq B \cap E$. Therefore, $B \cap E$ is α_γ-open relative to H. Since for any point $x \in H$, X is α-open set containing x such that $X^\gamma \cap H \subseteq H$, so $H \in \alpha O(X)^\gamma_H$ and also $\phi \in \alpha O(X)^\gamma_H$ by definition. Therefore, $\alpha O(X)^\gamma_H$ is a topology on H. □

Remark 2.12. *Let us consider the following two families of subsets in the subspace H of the topological space X.*

1. *For an operation $\gamma : \alpha O(X) \to P(X)$, let* $\alpha O(X)_\gamma|H = \{V \cap H \in P(H) : V$ *is α_γ-open in* $X\}$.

2. *Suppose that H is α-open in (X,τ) if $\gamma \neq id$. For an operation $\gamma_H^\alpha : \alpha|H \to P(H)$, let*
$(\alpha|H)^{\gamma_H^\alpha} = \{A \in P(H) : A$ *is γ_H^α-open in* $(H, \alpha|H)\}$.

Remark 2.13. *From Remark* 2.12*, we have the following properties:*

1. $\alpha O(X)_{id}|H = \alpha|H$.
2. $(\alpha|H)^{id_H^\alpha} = \{A \in P(H) : A$ *is id_H^α-open in* $(H,\alpha|H)\} = \{A \in P(H) : A$ *is id-open in* $(H,\alpha|H)\} = \alpha|H$.

Hence both the families are the same whenever γ is the identity operation.

More relations and properties among the families $\alpha O(X)_\gamma|H$, $\alpha O(X)_H^\gamma$, $(\alpha|H)^{\gamma_H^\alpha}$ and $\alpha|H$ under some assumptions are given in the next theorems.

Theorem 2.14. *Let $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$ and H be any subset of X.*

1. *If a subset B of (X,τ) is α_γ-open in (X,τ), then $B \cap H$ is α_γ-open relative to H*
2. *Every α_γ-open set relative to H is open in $(H,\alpha|H)$.*
3. $\alpha O(X)_\gamma|H \subseteq \alpha O(X)_H^\gamma \subseteq \alpha|H$.

Proof. 1. Let $x \in B \cap H$. It follows from assumption that there exists an α-open subset U of (X,τ) such that $x \in U$ and $U^\gamma \subseteq B$ and hence $U^\gamma \cap H \subseteq B \cap H$. Thus, $B \cap H$ is α_γ-open relative to H. Let $V \in \alpha O(X)_\gamma|H$. There exists a subset $B \in \alpha O(X)_\gamma$ such that $V = B \cap H$ and so $V \in \alpha O(X)_H^\gamma$. Thus we have the implication $\alpha O(X)_\gamma|H \subseteq \alpha O(X)_H^\gamma$.

2. Let V be a nonempty α_γ-open set relative to H. For each point $x \in V$, there exists a subset $U(x) \in \alpha O(X)$ such that $x \in U(x)$ and $U(x)^\gamma \cap H \subseteq V$. Taking the union over all points $x \in V$, we obtain that
$V = \bigcup_{x\in V}\{U(x) \cap H : U(x) \in \alpha O(X), U(x)^\gamma \cap H \subseteq V\}$. Therefore, $V = U \cap H$ were $U = \bigcup_{x\in V}\{U(x) : U(x) \in \alpha O(X), U(x)^\gamma \cap H \subseteq V\}$. Hence, V is open in $(H,\alpha|H)$ and by Definition 2.9, we have $\alpha O(X)_H^\gamma \subseteq \alpha|H$.

3. Follows from (1) and (2). □

Theorem 2.15. *Suppose that $\gamma \neq id$ and H is α-open in (X,τ). Then we have the following properties:*

1. *Every γ_H^α-open set in $(H,\alpha|H)$ is α_γ-open relative to H, and hence, $(\alpha|H)^{\gamma_H^\alpha} \subseteq \alpha O(X)_H^\gamma$.*
2. *Moreover, if γ is an α-monotone operation, then every α_γ-open set relative to H is γ_H^α-open in $(H,\alpha|H)$ and hence $\alpha O(X)_H^\gamma \subseteq (\alpha|H)^{\gamma_H^\alpha}$ so that $\alpha O(X)_H^\gamma = (\alpha|H)^{\gamma_H^\alpha}$.*

3. *If A is any subset of H which is α_γ-open in (X,τ), then it is γ_H^α-open in $(H,\alpha|H)$ that is, $A \in (\alpha|H)^{\gamma_H^\alpha}$.*

Proof. 1. Let A be a γ_H^α-open set in $(H,\alpha|H)$ that is, $A \in (\alpha|H)^{\gamma_H^\alpha}$, where $\gamma_H^\alpha : \alpha|H \to P(H)$ is an operation on $\alpha|H$. Let $x \in A$, then there exists a subset $W \in \alpha|H$ such that $x \in W$ and $\gamma_H^\alpha(W) = W^\gamma \cap H \subseteq A$. Since H is α-open, we have $W \in \alpha O(X)$. Thus, for the point $x \in A$, $W \in \alpha O(X)$ such that $W^\gamma \cap H \subseteq A$. This shows that A is α_γ-open relative to H, that is, $A \in \alpha O(X)_H^\gamma$.

2. Let A be an α_γ-open set relative to H that is, $A \in \alpha O(X)_H^\gamma$. Let $x \in A$, then there exists a subset $U(x) \in \alpha O(X)$ such that $x \in U(x)$ and $U(x)^\gamma \cap H \subseteq A$. Since $H \in \alpha O(X)$ and γ is α-monotone, we have $\gamma_H^\alpha(U(x) \cap H) = (U(x) \cap H)^\gamma \cap H \subseteq U(x)^\gamma \cap H \subseteq A$. This shows that for the point $x \in A$, we have $\gamma_H^\alpha(U(x) \cap H) \subseteq A$ and $U(x) \cap H \in \alpha|H$, and so A is γ_H^α-open in $(H,\alpha|H)$ that is, $A \in (\alpha|H)^{\gamma_H^\alpha}$. Hence, combining to (1), we obtain that $\alpha O(X)_H^\gamma = (\alpha|H)^{\gamma_H^\alpha}$.

3. Let $x \in A$, then there exists a subset U of X such that $x \in U$, $U \in \alpha O(X)$ and $U^\gamma \subseteq A$. We have $x \in U \cap H = U$ and $U \in \alpha|H$ and so $\gamma_H^\alpha(U) = U^\gamma \cap H \subseteq A \cap H = A$. Thus, we show $A \in (\alpha|H)^{\gamma_H^\alpha}$. □

Theorem 2.16. *Let $H \subseteq X$ and $\gamma : \alpha O(X) \to P(X)$ be an α-regular operation on $\alpha O(X)$ such that $\gamma \neq id$. Then the following properties are true:*

1. *If A is γ_H^α-open in $(H,\alpha|H)$ and H is α_γ-open in (X,τ), then A is α_γ-open in (X,τ) that is, $A \in \alpha O(X)_\gamma$.*
2. *If H is α_γ-open in (X,τ), then $(\alpha|H)^{\gamma_H^\alpha} = \alpha O(X)_\gamma|H$.*

Proof. 1. Let $x \in A$. There exists a subset $U \in \alpha O(X)$ such that $x \in U$, $\gamma_H^\alpha(U\cap H) = (U\cap H)^\gamma \cap H \subseteq A$ because $A \in (\alpha|H)^{\gamma_H^\alpha}$, $\gamma \neq id$, $U\cap H \in \alpha|H$ and $U \cap H \in \alpha O(X)$. Since $H \in \alpha O(X)_\gamma$ and $x \in A \subseteq H$, for the point $x \in H$, there exists a subset $V \in \alpha O(X)$ such that $x \in V$ and $V^\gamma \subseteq H$. By the α-regularity of γ, for two α-open subsets $U \cap H$ and V containing x, there exists a subset $W \in \alpha O(X)$ such that $x \in W$ and $W^\gamma \subseteq (U\cap H)^\gamma \cap V^\gamma$ and so $W^\gamma \subseteq (U\cap H)^\gamma \cap H \subseteq A$. Therefore, for each point $x \in A$, we have a subset W such that $W \in \alpha O(X)$, $x \in W$ and $W^\gamma \subseteq A$ and so A is α_γ-open in (X,τ) (that is, $A \in \alpha O(X)_\gamma$).

2. Let $A \in \alpha O(X)_\gamma|H$, then there exists a subset B of X such that $B \in \alpha O(X)_\gamma$ and $A = B\cap H$. Since γ is α-regular, $\alpha O(X)_\gamma$ forms a topology of X ([2, Remark 2.19]). Thus, we have $B\cap H \in \alpha O(X)_\gamma$ and so $A \in \alpha O(X)_\gamma$ because $B \in \alpha O(X)_\gamma$ and $H \in \alpha O(X)_\gamma$. Since, $\alpha O(X)_\gamma \subseteq \alpha O(X)$ in general and so $H \in \alpha O(X)$), so by Theorem 2.15 (3), it is obtained that $A \in (\alpha|H)^{\gamma_H^\alpha}$. Thus, we prove $\alpha O(X)_\gamma|H \subseteq (\alpha|H)^{\gamma_H^\alpha}$.

   On the other hand if $A \in (\alpha|H)^{\gamma_H^\alpha}$, then it follows from (1) above that $A \in \alpha O(X)_\gamma$ and $A = A\cap H \in \alpha O(X)_\gamma|H$. Thus, we have the implication $(\alpha|H)^{\gamma_H^\alpha} \subseteq \alpha O(X)_\gamma|H$. Hence we obtain that $(\alpha|H)^{\gamma_H^\alpha} = \alpha O(X)_\gamma|H$. □

Since α-monotone operation is α-regular and any α_γ-open set is α-open for any operation γ, the following corollary is obtained from Theorem 2.15 and Theorem 2.16.

Corollary 2.17. *If $\gamma : \alpha O(X) \to P(X)$ is an α-monotone operation on $\alpha O(X)$ such that $\gamma \neq id$ and H is α_γ-open in (X, τ), then $(\alpha|H)^{\gamma_H^\alpha} = \alpha O(X)_\gamma|H = \alpha O(X)_H^\gamma$.*

Corollary 2.18. *If H is any subset of X and $\gamma = id$, then $\alpha O(X)_\gamma|H = \alpha O(X)_H^\gamma = (\alpha|H)^{\gamma_H^\alpha} = \alpha|H$.*

Proof. Follows from Remark 2.10 and Remark 2.13. □

Let H be a subspace of a topological space (X, τ), and B be any subset of H. We introduce some closure operation with respect to the subspace H and the α-open sets of X.

Recalling that, for a subset E of X and $x \in X$, we say that $x \in \alpha Cl(E)$ if and only if $U \cap E \neq \phi$ holds for every α-open set U of (X, τ) such that $x \in U$ and for a point $y \in H$ and a subset B of $(H, \alpha|H)$, $y \in Cl_H(B)$ if and only if $V \cap B \neq \phi$ holds for every open set V of $(H, \alpha|H)$ such that $y \in V$.

Let $\gamma : \alpha O(X) \to P(X)$ be a given operation. Let $(H, \alpha|H)$ be a subspace of (X, τ). Suppose that H is α-open in (X, τ) if $\gamma \neq id$. For the restriction $\gamma_H^\alpha : \alpha|H \to P(H)$ of γ, we define the following concepts of operation-closure in the subspace $(H, \alpha|H)$.

Definition 2.19. Let B be any subset of H, then

- $Cl_H(B) = H \cap \alpha Cl(B)$, where $\alpha Cl(B) = \bigcap\{F : B \subseteq F, F$ is α-closed in $(X, \tau)\}$.
- $(\alpha|H)$-$Cl(B) = \bigcap\{F : B \subseteq F, F$ is closed in $(H, \alpha|H)\}$.
- $Cl_{\gamma_H^\alpha}(B) = \{x \in H : \gamma_H^\alpha(U) \cap B \neq \phi$ holds for every open set U of $(H, \alpha|H)$ with $x \in U\}$.
- $(\alpha|H)^{\gamma_H^\alpha}$-$Cl(B) = \cap\{F : B \subseteq F, F$ is γ_H^α-closed in $(H, \alpha|H)\}$.

Remark 2.20. *Let $(H, \alpha|H)$ be a subspace of a topological space (X, τ) and $\gamma : \alpha O(X) \to P(X)$ be an operation on $\alpha O(X)$. Suppose that H is α-open in (X, τ) if $\gamma \neq id$ and A a subset of H. Then*

1. *For a point $x \in X$, $x \in (\alpha|H)^{\gamma_H^\alpha}$-$Cl(A)$ if and only if for every γ_H^α-open set V of $(H, \alpha|H)$ containing x such that $A \cap V \neq \phi$.*
2. *$A \subseteq Cl_{\gamma_H^\alpha}(A) \subseteq (\alpha|H)^{\gamma_H^\alpha}$-$Cl(A)$.*

It is known that the identity operation $id : \alpha O(X) \to P(X)$ is α-open on $\alpha O(X)$ and hence if H is a subset of X, then $id_H^\alpha : \alpha|H \to P(H)$ is the identity operation on $\alpha|H$, so it is α-open on $\alpha|H$.

In general, if $\gamma \neq id$, we give the following result.

Lemma 2.21. *Let $\gamma : \alpha O(X) \rightarrow P(X)$ be an α-regular operation on $\alpha O(X)$ such that $\gamma \neq id$ and H be an α_γ-open set of (X, τ). If γ is α-open on $\alpha O(X)$, then $\gamma_H^\alpha : \alpha|H \rightarrow P(H)$ is α-open on $\alpha|H$.*

Proof. Let $x \in H$ and V be an open set of $(H, \alpha|H)$ with $x \in V$. We show that there exists a γ_H^α-open set S in $(H, \alpha|H)$ such that $x \in S$ and $S \subseteq \gamma_H^\alpha(V)$. Since $H \in \alpha O(X)$ so $V \in \alpha O(X)$, by the α-openness of γ, there exists an α_γ-open set, say G in (X, τ) such that $x \in G$ and $G \subseteq V^\gamma$. We put $S = G \cap H$, then $x \in S$, $S \subseteq V^\gamma \cap H$ and by definition, $\gamma_H^\alpha(V) = V^\gamma \cap H$. Hence, $S \in \alpha O(X)_\gamma|H$. Since γ is α-regular and $H \in \alpha O(X)_\gamma$, by Theorem 2.16, we have $\alpha O(X)_\gamma|H \subseteq (\alpha|H)^{\gamma_H^\alpha}$. Thus, we have $S \in (\alpha|H)^{\gamma_H^\alpha}$. Therefore, for any point $x \in H$ and an open set V containing x in $(H, \alpha|H)$, the subset S is a γ_H^α-open set of $(H, \alpha|H)$ such that $x \in S$ and $S \subseteq \gamma_H^\alpha(V)$. Hence, $\gamma_H^\alpha : \alpha|H \rightarrow P(H)$ is an α-open operation on $\alpha|H$. □

If $\gamma = id$, then we have $id_H^\alpha : \alpha|H \rightarrow P(H)$ is the identity operation on $\alpha|H$, and so $Cl_{id_H^\alpha}(B) = Cl_H(B) = (\alpha|H)\text{-}Cl(B) = \alpha Cl(B) \cap H = \alpha Cl_{id}(B) \cap H$.

If the operation γ is not the identity operation on $\alpha O(X)$, we have the following result:

Theorem 2.22. *Let $\gamma \neq id$ be a given operation on $\alpha O(X)$ and $B \subseteq H \subseteq X$. If H is α-open in (X, τ), then $Cl_{\gamma_H^\alpha}(B) \supseteq \alpha Cl_\gamma(B) \cap H$.*
Moreover if γ is α-monotone, then $Cl_{\gamma_H^\alpha}(B) = \alpha Cl_\gamma(B) \cap H$.

Proof. Let $x \in \alpha Cl_\gamma(B) \cap H$. In order to prove $x \in Cl_{\gamma_H^\alpha}(B)$, let U be an open set of $(H, \alpha|H)$ with $x \in U$. Since H is α-open in (X, τ) and $x \in \alpha Cl_\gamma(B)$, we have $U \in \alpha O(X)$ and so $U^\gamma \cap B \neq \phi$. By definition, it is obtained that $\gamma_H^\alpha(U) \cap B = (U^\gamma \cap H) \cap B = U^\gamma \cap (H \cap B) = U^\gamma \cap B \neq \phi$ and so $x \in Cl_{\gamma_H^\alpha}(B)$.

On the other hand, let $x \notin \alpha Cl_\gamma(B) \cap H$. We have to show $x \notin Cl_{\gamma_H^\alpha}(B)$. For the point x, if $x \notin H$, then we have $x \notin Cl_{\gamma_H^\alpha}(B)$ and hence the proof. Suppose that $x \in H$, so we have $x \notin \alpha Cl_\gamma(B)$. Then, there exists a subset $U \in \alpha O(X)$ such that $x \in U$ and $U^\gamma \cap B = \phi$. Since γ is α-monotone, we have $(U \cap H)^\gamma \subseteq U^\gamma$ and so $\gamma_H^\alpha(U \cap H) \cap B = ((U \cap H)^\gamma \cap H) \cap B \subseteq U^\gamma \cap B = \phi$ (indeed, $U \cap H \in \alpha O(X)$). Thus, the subset $U \cap H$ is an open set of $(H, \alpha|H)$ such that $x \in U \cap H$ and $\gamma_H^\alpha(U \cap H) \cap B = \phi$ and so $x \notin Cl_{\gamma_H^\alpha}(B)$. Therefore, $Cl_{\gamma_H^\alpha}(B) = \alpha Cl_\gamma(B) \cap H$. □

Theorem 2.23. *Suppose that H is α_γ-open in (X, τ) and B is any subset of H. If $\gamma : \alpha O(X) \rightarrow P(X)$ is α-regular and α-open on $\alpha O(X)$, then we have the following properties:*

1. *$Cl_{\gamma_H^\alpha}(B) \subseteq \alpha Cl_\gamma(B) \cap H$ and so $Cl_{\gamma_H^\alpha}(B) = \alpha Cl_\gamma(B) \cap H$.*
2. *$Cl_{\gamma_H^\alpha}(B) = (\alpha|H)^{\gamma_H^\alpha}\text{-}Cl(B)$.*

Proof. 1. Let $x \notin \alpha Cl_\gamma(B) \cap H$. If $x \notin H$, then it is clear that $x \notin Cl_{\gamma_H^\alpha}(B)$ because $Cl_{\gamma_H^\alpha}(B) \subseteq H$. If $x \in H$, then, we have $x \notin \alpha Cl_\gamma(B)$. Hence, there exists a subset $U \in \alpha O(X)$ such that $x \in U$ and $U^\gamma \cap B = \phi$. Since γ is α-open on $\alpha O(X)$, so there exists an α_γ-open set S such that $x \in S$ and $S \subseteq U^\gamma$ and so $S \cap B \subseteq U^\gamma \cap B = \phi$ which implies that, $S \cap B = \phi$. Thus we have $S \cap H \in \alpha O(X)_\gamma|H$ and $x \in S \cap H$. By Theorem 2.16 (2), it is proved that $\alpha O(X)_\gamma|H \subseteq (\alpha|H)^{\gamma_H^\alpha}$ and so we have $S \cap H \in (\alpha|H)^{\gamma_H^\alpha}$. Therefore, the subset $S \cap H$ is a γ_H^α-open set of $(H, \alpha|H)$ such that $x \in S \cap H$ and $(S \cap H) \cap B = S \cap B = \phi$. From Remark 2.20 (1), we obtain that $x \notin (\alpha|H)^{\gamma_H^\alpha}\text{-}Cl(B)$ and from Remark 2.20 (2), we have $Cl_{\gamma_H^\alpha}(E) \subseteq (\alpha|H)^{\gamma_H^\alpha}\text{-}Cl(E)$ holds for any subset E of $(H, \alpha|H)$. Hence, we have $x \notin Cl_{\gamma_H^\alpha}(B)$. Therefore, we obtained that $Cl_{\gamma_H^\alpha}(B) \subseteq \alpha Cl_\gamma(B) \cap H$.

Moreover, since any α_γ-open set of (X, τ) is α-open in (X, τ), and applying Theorem 2.22, we conclude that $Cl_{\gamma_H^\alpha}(B) = \alpha Cl_\gamma(B) \cap H$.

2. By Lemma 2.21, $\gamma_H^\alpha : \alpha|H \to P(H)$ is α-open on $\alpha|H$. Using [[2], Theorem 2.26 (2)] for the topological space $(H, \alpha|H)$, the subset B of H and the operation $\gamma_H^\alpha : \alpha|H \to P(H)$, we obtain that $Cl_{\gamma_H^\alpha}(B) = (\alpha|H)^{\gamma_H^\alpha}\text{-}Cl(B)$. □

References

[1] O. Njastad, On some classes of nearly open sets, Pacific J. Math. 15 (1965) 961-970.

[2] H.Z. Ibrahim, On a class of α_γ-open sets in a topological space, Acta Scientiarum. Technology 35 (3) (2013) 539-545.

[3] A.B. Khalaf, H.Z. Ibrahim, Some properties of operations on $\alpha O(X)$, International Journal of Mathematics and Soft Computing 6 (1) (2016) 107-120.

(Received 9 June 2016)
(Accepted 29 January 2018)