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A Note on Representable Autometrized Algebras

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Abstract : In this paper, two remarkable results are obtained in a Representable Autometrized Algebra $A = (A, +, \leq, *)$ satisfying the condition $[R]$:

$$a * (a \wedge b) + a \wedge b = a \quad \text{for all } a, b \in A.$$

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1 Introduction

Representable Autometrized Algebras were introduced and studied by B. V. Subba Rao ([1–3]). In this paper we obtained two results regarding Representable Autometrized Algebra $A = (A, +, \leq, *)$ satisfying the condition $[R] : a * (a \wedge b) + a \wedge b = a$ for all $a, b \in A$.

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In the first theorem, if there exists $1 \in A$ such that $a + (1 * a) = 1 + 1$, for all $a \in A$, then we proved that it is unique.
In the second theorem, if $1 \in A$ such that $a + (1 * a) = 1 + 1$, for all $a \in A$, then we proved the following.

(1) If A has least element say α, then $\alpha = 0$ and

(2) If A has greatest element say β, then $\beta = 1$.

2 Lattice Ordered Autometrized Algebra and Representable Autometrized Algebra

Here we recall the definition of (i) Lattice Ordered Autometrized algebra (Swamy [4, 5]), (ii) Representable Autometrized Algebra, introduced and studied by Subba Rao ([1–3]).

Definition 2.1. A system $A = (A, +, \leq, *)$ is called a "Lattice Ordered Autometrized Algebra" if and only if

(i) $(A, +, \leq)$ is a commutative lattice ordered semigroup with $'0'$, and

(ii) $*$ is a metric operation on A.

i.e., $*$ is a mapping from $A \times A$ into A satisfying the formal properties of distance, namely,

(M_1) $a * b \geq 0$ for all a, b in A, equality, if and only if $a = b$,

(M_2) $a * b = b * a$ for all a, b in A, and

(M_3) $a * b \leq a * c + c * b$ for all a, b, c in A.

Definition 2.2. A lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is called a "Representable Autometrized Algebra", if and only if, A satisfies the following conditions:

(L_1) $A = (A, +, \leq, *)$ is a semiregular autometrized algebra, i.e., $a \in A$ and $a \geq 0$ implies $a * 0 = a$, and

(L_2) For every a in A all the mapping $x \mapsto a + x$, $x \mapsto a \vee x$, $x \mapsto a \wedge x$ and $x \mapsto a*x$ are contractions (i.e., if θ denotes any one of the operations $+, \vee, \wedge$ and $*$, then, for each a in A, $(a\theta x) * (a\theta y) \leq x * y$ for all x, y in A).

Theorem 2.3. *Let $A = (A, +, \leq, *)$ be a Representable Autometrized Algebra satisfying the condition $[R]$ $a * (a \wedge b) + a \wedge b = a$, for all $a, b \in A$. If there exists $1 \in A \ni a + (1 * a) = 1 + 1$ $\forall$ $a \in A$, then, it is unique.*

Proof. Let $A = (A, +, \leq, *)$ be a Representable Autometrized Algebra satisfying (R) mentioned in the theorem. Let $1 \in A$ and $1^1 \in A$ such that

$$a + (1 * a) = 1 + 1 \text{ for all } a \in A \tag{2.1}$$

and

$$a + \left(1^1 * a\right) = 1^1 + 1^1 \text{ for all } a \in A. \tag{2.2}$$

Since $1 \in A$, by (2.1) we have

$$1 + (1 * 1) = 1 + 1,$$

i.e.,

$$1 + 0 = 1 + 1.$$

So,

$$1 = 1 + 1. \tag{2.3}$$

In a similar manner since $1^1 \in A$,
from(2.2), we get

$$1^1 = 1^1 + 1^1. \tag{2.4}$$

Putting $a = 1^1$ in (2.1), we get

$$1^1 + \left(1 * 1^1\right) = 1 + 1 = 1 \quad (\text{from}(2.3)). \tag{2.5}$$

Also putting $a = 1$ in (2.2), we get

$$1 + \left(1^1 * 1\right) = 1^1 + 1^1 = 1^1 \quad (\text{from}(2.4)). \tag{2.6}$$

Therefore we get

$$\begin{aligned} 1 + 1^1 \quad &= 1^1 + 1 \\ &= 1^1 + [1^1 + \left(1 * 1^1\right)] \\ &= \left(1^1 + 1^1\right) + \left(1 * 1^1\right) \\ &= 1^1 + \left(1 * 1^1\right) \\ &= 1. \end{aligned}$$

Again from (2.6), we get

$$\begin{aligned} 1 + 1^1 \quad &= 1 + [1 + \left(1^1 * 1\right)] \\ &= 1 + 1 + \left(1^1 * 1\right) \\ &= 1 + \left(1 * 1^1\right) \\ &= 1^1, \quad (\text{from (2.6) since} 1 * 1^1 = 1^1 * 1) \end{aligned}$$

$\Rightarrow 1 = 1^1$.
Thus, there exists a unique $1 \in A$ such that $a + (1 * a) = 1 + 1$ for all $a \in A$, if it exists. □

Theorem 2.4. *Let $A = (A, +, \leq, *)$ be a Representable Autometrized Algebra satisfying the condition $[R]$ $a * (a \wedge b) + a \wedge b = a$, for all $a, b \in A$. If there exists $1 \in A \ni a + (1 * a) = 1 + 1 \;\forall\; a \in A$, then, we have*

(i) If A has least element α then $\alpha = 0$,

(ii) If A has greatest element β then $\beta = 1$.

Proof. Let $A = (A, +, \leq, *)$ be Representable Autometrized Algebra satisfying the condition (R) mentioned in the theorem.

$$\text{Let } 1 \in A \text{ such that } a + (1 * a) = 1 + 1 \;\; \forall\, a \in A. \tag{2.7}$$

Therefore, by Theorem 2.3 above, 1 is unique.

(i) Assume that A has a least element α.
Therefore $\alpha \leq x \; \forall x \in A$.
In particular $\alpha \leq 0$.
Therefore

$$\begin{aligned} 0 &= (0 * \alpha) + \alpha \quad \text{by } (R) \\ &= \alpha + (0 * \alpha)\,. \end{aligned} \tag{2.8}$$

We have $2\alpha \leq \alpha$ (since $\alpha \leq 0$)
but $\alpha \leq 2\alpha$ (since α is the least element of A).
Therefore $2\alpha = \alpha$.
By induction, it follows that $n\alpha = \alpha$ for every positive integer n.
Therefore

$$\begin{aligned} \alpha * 0 &= (\alpha + 0) * 0 \\ &= [\alpha + (\alpha + (0 * \alpha))] * [\alpha + (0 * \alpha)] \;\; \text{from (2.8)} \\ &= [2\alpha + (0 * \alpha)] * [\alpha + (0 * \alpha)] \\ &\leq 2\alpha * \alpha \;\; (\text{since } x \mapsto a + x \text{ is a contraction, } \forall a \in A) \\ &= 0 \;\; (\text{since } 2\alpha = \alpha), \end{aligned}$$

i.e., $\alpha * 0 \leq 0$,
but $\alpha * 0 \geq 0$,
therefore $\alpha * 0 = 0$.
Hence $\alpha = 0$.
Thus, if α is the least element of A, then it must be 0, the additive identity of A.

(ii) Assume that A has a greatest element, β, say.
So, $x \leq \beta \; \forall\, x \in A$.
In particular, we have $1 \leq \beta$ and $0 \leq \beta$.
Therefore, by (R), we have $(\beta * 1) + 1 = \beta$,
and by (2.7), we have $\beta + (1 * \beta) = 1 + 1$.
By (2.7) we also have
$1 + (1 * 1) = 1 + 1$
$\Rightarrow 1 + 0 = 1 + 1$
$\Rightarrow 1 = 1 + 1$.
Therefore $\beta + (1 * \beta) = 1$.
Since $0 \leq \beta$ we have $\beta \leq 2\beta$,
but $\beta \geq 2\beta$ (since β is the greatest element in A).
Therefore $2\beta = \beta$.

By induction, it follows that $n\beta = \beta$ for every positive integer $'n'$.
Now using the properties of contraction mappings
we have

$$\begin{aligned} 0 &= \beta * \beta \geq (\beta * 1) * (\beta * 1) \\ &\geq [(\beta * 1) + \beta] * [(\beta * 1) + \beta] \\ &\geq 1 * [(\beta * 1) + \beta] \\ &\geq (1 + 1) * [(\beta * 1) + 1 + \beta] \\ &= 1 * \quad (\beta + \beta) \\ &= 1 * 2\beta. \end{aligned}$$

So, $2\beta * 1 \leq 0$.
But $2\beta * 1 \geq 0$, therefore $2\beta * 1 = 0$
$\Rightarrow 2\beta = 1$
$\Rightarrow \beta = 1$ (since $2\beta = \beta$).
Thus, if there exists a greatest element β in A, then $\beta = 1$. □

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