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Fixed Point Theorems for $s - \alpha$ Contractions in Dislocated and b-Dislocated Metric Spaces

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Abstract : In this paper, we prove some unique fixed point results for quasicontraction and T-Hardy Rogers contraction in the setting of complete dislocated and b-dislocated metric spaces. Our theorems involve one and two self-mappings and extend and generalize some several known results of literature in a wider class as b-spaces.

Keywords : dislocated metric; b-dislocated metric; $s - \alpha$ quasicontraction; T-Hardy-Rogers contraction; common fixed point.
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1 Introduction

Fixed point theory is one of the cornerstones in the development of mathematics since it plays a basic role in applications of many branches of mathematics. The famous banach contraction principle is one of the power tools to study in this field. In 1989, Bakhtin [1] (and also Czerwik [2]) introduced the concept of b-metric space and presented the contraction mapping in b-metric spaces that is a generalization of Banach contraction principle in metric spaces. Recently there are

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a number of generalizations of metric space. Some of them are the notions of dislocated metric spaces and b-dislocated metric spaces where the distance of a point in the self may not be zero, introduced and studied by Hitzler and Seda [3], Nawab Hussain et.al [4]. Also in [4] are presented some topological aspects and properties of b-dislocated metrics. Subsequently, several authors have studied the problem of existence and uniqueness of a fixed point for single-valued and set-valued mappings and different types of contractions in these spaces. Quasicontractions and g-quasicontractions in metric spaces were first studied in [2, 5]. The purpose of this paper is to present some fixed point theorems for $s-\alpha$-quasicontractions and T-Hardy-Rogers contractions in the context of dislocated and b-dislocated metric spaces. The presented theorems extend and generalize some comparable results in the literature in a larger class of spaces.

2 Preliminaries

Definition 2.1. [6] Let X be a nonempty set and a mapping $d_l : X \times X \to [0, \infty)$ is called a dislocated metric (or simply d_l-metric) if the following conditions hold for any $x, y, z \in X$:

1. If $d_l(x, y) = 0$, then $x = y$;
2. $d_l(x, y) = d_l(y, x)$;
3. $d_l(x, y) \le d_l(x, z) + d_l(z, y)$.

The pair (X, d_l) is called a dislocated metric space (or d-metric space for short). Note that when $x = y$, $d_l(x, y)$ may not be 0.

Example 2.2. If $X = R$, then $d(x, y) = |x| + |y|$ defines a dislocated metric on X.

Definition 2.3. [6] A sequence (x_n) in d_l-metric space (X, d_l) is called: (1) a Cauchy sequence if, for given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \ge n_0$, we have $d_l(x_m, x_n) < \varepsilon$ or $\lim_{n,m\to\infty} d_l(x_n, x_m) = 0$, (2) convergent with respect to d_l if there exists $x \in X$ such that $d_l(x_n, x) \to 0$ as $n \to \infty$. In this case, x is called the limit of (x_n) and we write $x_n \to x$.

A d_l-metric space X is called complete if every Cauchy sequence in X converges to a point in X.

Definition 2.4. [7] Let X be a nonempty set and a mapping $b_d : X \times X \to [0, \infty)$ is called a b-dislocated metric (or simply b_d-dislocated metric) if the following conditions hold for any $x, y, z \in X$ and $s \ge 1$:

1. If $b_d(x, y) = 0$, then $x = y$;
2. $b_d(x, y) = b_d(y, x)$;

3. $b_d(x,y) \leq s\left[b_d(x,z)+b_d(z,y)\right]$.

The pair (X,b_d) is called a b-dislocated metric space. And the class of b-dislocated metric space is larger than that of dislocated metric spaces, since a b-dislocated metric is a dislocated metric when $s=1$.

In [7] it was showed that each b_d-metric on X generates a topology τ_{b_d} whose base is the family of open b_d-balls $B_{b_d}(x,\varepsilon)=\{y\in X : b_d(x,y)<\varepsilon\}$.

Also in [7] there are presented some topological properties of b_d-metric spaces.

Definition 2.5. Let (X,b_d) be a b_d-metric space, and $\{x_n\}$ be a sequence of points in X. A point $x\in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim\limits_{n\to\infty} b_d(x_n,x)=0$ and we say that the sequence $\{x_n\}$ is b_d-convergent to x and denote it by $x_n\to x$ as $n\to\infty$.

The limit of a b_d-convergent sequence in a b_d-metric space is unique [7, Proposition 1.27].

Definition 2.6. A sequence $\{x_n\}$ in a b_d-metric space (X,b_d) is called a b_d-Cauchy sequence iff , given $\varepsilon>0$, there exists $n_0\in N$ such that for all $n,m>n_0$, we have $b_d(x_n,x_m)<\varepsilon$ or $\lim\limits_{n,m\to\infty} b_d(x_n,x_m)=0$. Every b_d-convergent sequence in a b_d-metric space is a b_d-Cauchy sequence.

Remark 2.7. The sequence $\{x_n\}$ in a b_d-metric space (X,b_d) is called a b_d-Cauchy sequence iff $\lim\limits_{n,m\to\infty} b_d(x_n,x_{n+p})=0$ for all $p\in N^*$

Definition 2.8. A b_d-metric space (X,b_d) is called complete if every b_d-Cauchy sequence in X is b_d-convergent.

In general a b_d-metric is not continuous, as in Example 1.31 in [7] showed.

Definition 2.9. [8] Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be sequentially convergent (respectively, subsequentially convergent) if, for a sequence $\{x_n\}$ in X for wich $\{Tx_n\}$ is convergent, $\{x_n\}$ is also convergent (respectively, $\{x_n\}$ has a convergent subsequence).

Lemma 2.10. *Let (X,b_d) be a b-dislocated metric space with parameter $s\geq 1$. Suppose that $\{x_n\}$and $\{y_n\}$are b_d-convergent to $x,y\in X$, respectively. Then we have*

$$\frac{1}{s^2}b_d(x,y)\leq \lim_{n\to\infty}\inf b_d(x_n,y_n)\leq \lim_{n\to\infty}\sup b_d(x_n,y_n)\leq s^2 b_d(x,y)$$

In particular, if $b_d(x,y)=0$, then we have $\lim\limits_{n\to\infty} b_d(x_n,y_n)=0=b_d(x,y)$. Moreover, for each $z\in X$, we have

$$\frac{1}{s}b_d(x,z)\leq \lim_{n\to\infty}\inf b_d(x_n,z)\leq \lim_{n\to\infty}\sup b_d(x_n,z)\leq s b_d(x,z)$$

In particular, if $b_d(x,z)=0$, then we have $\lim\limits_{n\to\infty} b_d(x_n,z)=0=b_d(x,z)$.

Some examples in the literature shows that in general a b-dislocated metric is not continuous.

Example 2.11. Let $X = R^+ \cup \{0\}$ and any constant $\alpha > 0$. Define the function $d_l : X \times X \to [0, \infty)$ by $d_l(x, y) = \alpha(x + y)$. Then, the pair (X, d_l) is a dislocated metric space.

Example 2.12. If $X = R^+ \cup \{0\}$, then $b_d(x, y) = (x + y)^2$ defines a b-dislocated metric on X with parameter $s = 2$.

3 Main Results

Based in the definition of quasi-contraction from Ciric we introduced the following definition in the setting of b-dislocated metric space.

Definition 3.1. Let (X, b_d) be complete b-dislocated metric space with parameter $s \geq 1$. If $T : X \to X$ is a self mapping that satisfies:

$$s^2 b_d(Tx, Ty) \leq \alpha \max\{b_d(x, y), b_d(x, Tx), b_d(y, Ty), b_d(x, Ty), b_d(y, Tx)\} \tag{3.1}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{2}\right)$. Then T is called a $s - \alpha$ quasi-contraction.

In this section, we obtain the existence of some fixed point theorems for $s - \alpha$ quasi-contraction mappings in a class of space which is larger than metric and b-metric spaces.

Theorem 3.2. *Let (X, b_d) be complete b-dislocated metric space with parameter $s \geq 1$. If $T : X \to X$ is a self mapping that is a $s - \alpha$ quasi-contraction, then T has a unique fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Define the iterative sequence $\{x_n\}$ as follows: $x_1 = T(x_0), x_2 = T(x_1), ..., x_{n+1} = T(x_n)$, ...

If assume that $x_{n+1} = x_n$ for some $n \in N$, then we have $x_n = x_{n+1} = T(x_n)$, so x_n is a fixed point of T and the proof is completed. From now on we will assume that for each $n \in N$, $x_{n+1} \neq x_n$. By condition (3.1) we have:

$$\begin{aligned}
& s^2 b_d(x_n, x_{n+1}) = s^2 b_d(Tx_{n-1}, Tx_n) \\
& \leq \alpha \max\big\{b_d(x_{n-1}, x_n), b_d(x_{n-1}, Tx_{n-1}), \\
& \qquad b_d(x_n, Tx_n), b_d(x_{n-1}, Tx_n), b_d(x_n, Tx_{n-1})\big\} \\
& = \alpha \max\big\{b_d(x_{n-1}, x_n), b_d(x_{n-1}, x_n), \\
& \qquad b_d(x_n, x_{n+1}), b_d(x_{n-1}, x_{n+1}), b_d(x_n, x_n)\big\} \\
& \leq \alpha \max\left\{\begin{array}{l} b_d(x_{n-1}, x_n), b_d(x_{n-1}, x_n), b_d(x_n, x_{n+1}), \\ s\left[b_d(x_{n-1}, x_n) + b_d(x_n, x_{n+1})\right], 2sb_d(x_{n-1}, x_n) \end{array}\right\}.
\end{aligned} \tag{3.2}$$

If $b_d(x_{n-1}, x_n) \leq b_d(x_n, x_{n+1})$ for some $n \in N$, then from the above inequality (3.2) we have

$b_d(x_n, x_{n+1}) \leq \frac{2\alpha}{s} b_d(x_n, x_{n+1})$ a contradiction since $\frac{2\alpha}{s} < 1$.

Hence for all $n \in N$, $b_d(x_n, x_{n+1}) \le b_d(x_{n-1}, x_n)$ and also by the above inequality (3.2) we get

$$b_d(x_n, x_{n+1}) \le \frac{2\alpha}{s} b_d(x_{n-1}, x_n). \tag{3.3}$$

Similarly by the contractive condition of theorem we have:

$$b_d(x_{n-1}, x_n) \le \frac{2\alpha}{s} b_d(x_{n-2}, x_{n-1}). \tag{3.4}$$

Generally from (3.3) and (3.4) we have for all $n \ge 2$

$$b_d(x_n, x_{n+1}) \le c b_d(x_{n-1}, x_n) \le ... \le c^n b_d(x_0, x_1) \tag{3.5}$$

where $c = \frac{2\alpha}{s}$ and $0 \le c < 1$. Taking limit as $n \to \infty$ in (3.5) we have

$$b_d(x_n, x_{n+1}) \to 0. \tag{3.6}$$

Now, we prove that $\{x_n\}$ is a b_d-Cauchy sequence, and to do this let be $m, n > 0$ with $m > n$, and using definition 2.4 (3) we have

$$\begin{aligned} & b_d(x_n, x_m) \le s[b_d(x_n, x_{n+1}) + b_d(x_{n+1}, x_m)] \\ & \le s b_d(x_n, x_{n+1}) + s^2 b_d(x_{n+1}, x_{n+2}) + s^3 b_d(x_{n+2}, x_{n+3}) + ... \\ & \le s c^n b_d(x_0, x_1) + s^2 c^{n+1} b_d(x_0, x_1) + s^3 c^{n+2} b_d(x_0, x_1) + ... \\ = \quad & s c^n b_d(x_0, x_1) \left[1 + sc + (sc)^2 + (sc)^3 + ...\right] \\ & \le \tfrac{sc^n}{1-sc} b_d(x_0, x_1). \end{aligned}$$

On taking limit for $n, m \to \infty$ we have $b_d(x_n, x_m) \to 0$ as $cs < 1$. Therefore $\{x_n\}$ is a b_d-Cauchy sequence in complete b-dislocated metric space (X, b_d). So there is some $u \in X$ such that $\{x_n\}$ dislocated converges to u.

If T is a continuous mapping we get:

$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} (x_{n+1}) = u$. Thus u is a fixed point of T.

If the self-map T is not continuous then, we consider,

$$\begin{aligned} & s^2 b_d(x_{n+1}, Tu) = s^2 b_d(Tx_n, Tu) \\ & \le \alpha \max\{b_d(x_n, u), b_d(x_n, Tx_n), b_d(u, Tu), b_d(x_n, Tu), b_d(u, Tx_n)\} \\ & = \alpha \max\{b_d(x_n, u), b_d(x_n, x_{n+1}), b_d(u, Tu), b_d(x_n, Tu), b_d(u, x_{n+1})\}. \end{aligned} \tag{3.7}$$

Using Lemma 2.10, result 3.6 and taking the upper limit in (3.7) follows that

$$s^2 \frac{1}{s} b_d(u, Tu) \le \alpha s b_d(u, Tu).$$

From this inequality have $b_d(u, Tu) \le \alpha b_d(u, Tu)$ and this implies $Tu = u$ since $\alpha < \frac{1}{2}$. Hence u is a fixed point of T.

Uniqueness: Let us suppose that u and v are two fixed points of T where $Tu = u$ and $Tv = v$. Using condition (3.1), we have:

$$\begin{aligned} s^2 b_d(u,v) &= s^2 b_d(Tu,Tv) \\ &\leq \alpha \max\{b_d(u,v), b_d(u,Tu), b_d(v,Tv), b_d(u,Tv), b_d(v,Tu)\} \\ &= \alpha \max\{b_d(u,v), b_d(u,u), b_d(v,v), b_d(u,v), b_d(v,u)\} \\ &\leq 2\alpha s b_d(u,v). \end{aligned}$$

So $b_d(u,v) \leq c b_d(u,v)$ where $c = \frac{2\alpha}{s}$, since $0 \leq c < 1$ we get $b_d(u,v) = 0$. Therefore, $b_d(u,v) = b_d(v,u) = 0$ implies $u = v$. Hence the fixed point is unique. □

Example 3.3. Let $X = [0,1]$ and $b_d(x,y) = (x+y)^2$ for all $x, y \in X$. It is clear that b_d is a b-dislocated metric on X with parameter $s = 2$ and (X, b_d) is complete. Also b_d is not a dislocated metric or a b-metric or a metric on X. Define the self-mapping $T : X \to X$ by $Tx = \frac{x}{5}$. For $x, y \in [0,1]$, we have

$$\begin{aligned} s^2 b_d(Tx,Ty) &= 2^2\left(\tfrac{x}{5}+\tfrac{y}{5}\right)^2 \\ &= 4\tfrac{(x+y)^2}{25} \\ &= \tfrac{4}{25} b_d(x,y) \\ &\leq \alpha \max\{b_d(x,y), b_d(x,Tx), b_d(y,Ty), b_d(x,Ty), b_d(y,Tx)\} \end{aligned}$$

for $\frac{4}{25} \leq \alpha < \frac{1}{2}$. Clearly $x = 0$ is a unique fixed point of T.

If we take parameter $s = 1$ in Theorem 3.2, we obtain the following corollary in the setting of dislocated metric spaces.

Corollary 3.4. *Let (X, d_l) be a complete dislocated metric space. If $T : X \to X$ is a self mapping that satisfies:*

$$d_l(Tx,Ty) \leq \alpha \max\{d_l(x,y), d_l(x,Tx), d_l(y,Ty), d_l(x,Ty), d_l(y,Tx)\}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point in X.

The following example shows that Theorem 3.4 is a proper generalization.

Example 3.5. Let $X = [0,1]$ and $d_l : X^2 \to R^+$ by $d_l(x,y) = (x+y)$ for all $x, y \in X$. It is clear that d_l is a dislocated metric on X and (X, d_l) is complete. Also d_l is not a metric on X. Define the self-mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{8}, & 0 \leq x < 1 \\ \frac{1}{16}, & x = 1. \end{cases}$$

We have the following cases.

Case 1. For $x = y = 0$ have $d_l(Tx,Ty) = d_l(0,0) = 0 \leq d_l(0,0)$.

Case2. If $1 > x = y > 0$, then

$$d_l(Tx,Ty) = d_l\left(\frac{x}{8}, \frac{x}{8}\right) = \frac{2x}{8} = \frac{1}{8}2x = \frac{1}{8} d_l(x,y) < d_l(x,y).$$

Case 3. If $x=1, y=\frac{1}{2}$, then

$$d_l(Tx,Ty)=d_l\left(T(1),T\left(\frac{1}{2}\right)\right)=d_l\left(\frac{1}{16},\frac{1}{16}\right)=\frac{1}{8}<\frac{3}{2}=d_l(x,y).$$

Case 4. if $0<x<y=1$, then

$$d_l(Tx,T1)=d_l\left(\frac{x}{8},\frac{1}{16}\right)=\frac{x}{8}+\frac{1}{16}<\frac{x}{8}+\frac{1}{8}=\frac{1}{8}(x+1)=\frac{1}{8}d_l(x,1)<d_l(x,1).$$

Case 5. If $1>x>y>0$, then

$$d_l(Tx,Ty)=d_l\left(\frac{x}{8},\frac{y}{8}\right)=\frac{x}{8}+\frac{y}{8}=\frac{1}{8}(x+y)<d_l(x,y).$$

Thus all conditions of theorem are satisfied and T has $x=0$ a unique fixed point in X.

Therefore, we note that for $x=1$ and $y=\frac{99}{100}$ in the usual metric space (X,d) where $d(x,y)=|x-y|$ in the special case of Banach contraction, we have

$$d\left(T(1),T\left(\frac{99}{100}\right)\right)=d\left(\frac{1}{16},\frac{99}{800}\right)=\frac{49}{800}\leq\alpha\frac{1}{100}=d\left(1,\frac{99}{100}\right).$$

This inequality implies that $\alpha\geq\frac{49}{8}$ and this means that the contractive condition is not true in the usual metric on X. Also, this example shows that the contractive condition of theorem failed in the setting of b-metric space (X,d) where $d(x,y)=|x-y|^2$.

In the following we are giving a result in which T is not continuous in X, but T^p is continuous for some positive integer p.

Theorem 3.6. *Let (X,b_d) be a complete b-dislocated metric space with parameter $s\geq 1$ and $T:X\to X$ a self-mapping satisfying the condition* (3.1)

$$s^2 b_d(Tx,Ty)\leq\alpha\max\{b_d(x,y),b_d(x,Tx),b_d(y,Ty),b_d(x,Ty),b_d(y,Tx)\}$$

for all $x,y\in X$ and $\alpha\in\left[0,\frac{1}{2}\right)$. If for some positive integer p, T^p is continuous, then T has a unique fixed point in X.

Proof. Similarly as in above theorem we can construct a sequence $\{x_n\}$ and conclude that the sequence $\{x_n\}$ converges to some point $u\in X$. Thus its subsequence $\{x_{n_k}\}$ $(n_k=k_p)$ converges to u. Also, we have

$$T^p(u)=T^p\left(\lim_{k\to\infty}x_{n_k}\right)=\lim_{k\to\infty}\left(T^p(x_{n_k})\right)=\lim_{k\to\infty}x_{n_{k+1}}=u.$$

Therefore, u is a fixed point of T^p. Further we have to show that u is a fixed point of T.

Let m be the smallest positive integer such that $T^m u = u$. If suppose that $m > 1$ we consider:

$$\begin{aligned}
&s^2 b_d(u, Tu) = s^2 b_d(T^m u, Tu)\\
&= s^2 b_d\left(TT^{m-1}u, Tu\right)\\
&\le \alpha \max\left\{b_d\left(T^{m-1}u, u\right), b_d\left(T^{m-1}u, T^m u\right), b_d(u, Tu), b_d\left(T^{m-1}u, Tu\right), b_d(u, T^m u)\right\}\\
&= \alpha \max\left\{b_d\left(T^{m-1}u, u\right), b_d\left(T^{m-1}u, u\right), b_d(u, Tu), b_d\left(T^{m-1}u, Tu\right), b_d(u, u)\right\}\\
&\le \alpha \max\left\{\begin{array}{l} b_d\left(T^{m-1}u, u\right), b_d\left(T^{m-1}u, u\right), b_d(u, Tu),\\ s\left[b_d\left(T^{m-1}u, u\right) + b_d(u, Tu)\right], 2sb_d\left(T^{m-1}u, u\right)\end{array}\right\}\\
&\le 2\alpha s b_d\left(T^{m-1}u, u\right) \Rightarrow b_d(u, Tu) < b_d\left(T^{m-1}u, u\right)
\end{aligned}$$

Again from the condition of theorem, have

$$\begin{aligned}
&s^2 b_d\left(T^{m-1}u, u\right) = s^2 b_d\left(T^{m-1}u, T^m u\right)\\
&= s^2 b_d\left(TT^{m-2}u, T^m u\right)\\
&\le \alpha \max\left\{b_d\left(T^{m-2}u, u\right), b_d\left(T^{m-2}u, TT^{m-2}u\right),\right.\\
&\qquad \left. b_d(u, T^m u), b_d\left(T^{m-2}u, T^m u\right), b_d\left(u, TT^{m-2}u\right)\right\}\\
&= \alpha \max\left\{b_d\left(T^{m-2}u, u\right), b_d\left(T^{m-2}u, T^{m-1}u\right),\right.\\
&\qquad \left. b_d(u, u), b_d\left(T^{m-2}u, u\right), b_d\left(u, T^{m-1}u\right)\right\}\\
&\le \alpha \max\left\{\begin{array}{l} b_d\left(T^{m-2}u, u\right), s\left[b_d\left(T^{m-2}u, u\right) + b_d\left(u, T^{m-1}u\right)\right],\\ 2\alpha s b_d\left(u, T^{m-2}u\right), b_d\left(T^{m-2}u, u\right), b_d\left(u, T^{m-1}u\right)\end{array}\right\}\\
&\le 2\alpha s b_d\left(T^{m-2}u, u\right) \Rightarrow\\
&b_d\left(T^{m-1}u, u\right) < b_d\left(T^{m-2}u, u\right).
\end{aligned}$$

In general using this process inductively, we get

$$b_d(u, Tu) < b_d\left(T^{m-1}u, u\right) < b_d\left(T^{m-2}u, T^{m-1}u\right) < \ldots.. < b_d(u, Tu).$$

As a result we have, $b_d(u, Tu) < b_d(u, Tu)$ that is a contradiction. Hence $Tu = u$ and u is a fixed point of T.

Clearly the uniqueness of fixed point follows as in above theorem. □

Theorem 3.7. *Let (X, b_d) be a complete b-dislocated metric space with parameter $s \ge 1$ and $T : X \to X$ a self-mapping such that for some positive integer m, T satisfies the following condition* (3.1)*:*

$$\begin{aligned}
&s^2 b_d(T^m x, T^m y)\\
&\le \alpha \max\left\{b_d(x, y), b_d(x, T^m x), b_d(y, T^m y), b_d(x, T^m y), b_d(y, T^m x)\right\}
\end{aligned}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{2}\right)$. If $T^m$is continuous, then Thas a unique fixed point in X.

Proof. If we set $F = T^m$, then from Theorem 3.2 F has a unique fixed point. We call it u. Then $T^m u = u$ and this implies,

$$T^{m+1}u = T^m(Tu) = T(T^m u) = Tu.$$

From this Tu is a fixed point of T^m. Since T^m has a unique fixed point, then $Tu = u$.

Uniqueness. From condition of theorem, we get the uniqueness of fixed point u. □

If we take the parameter $s=1$ in Theorem 3.6 and 3.7, we reduce the following corollaries in the setting of dislocated metric spaces.

Corollary 3.8. *Let (X,d_l) be a complete dislocated metric space and $T:X\to X$ a self- mapping satisfying the condition:*

$$d_l(Tx,Ty)\le\alpha\max\{d_l(x,y),d_l(x,Tx),d_l(y,Ty),d_l(x,Ty),d_l(y,Tx)\}$$

for all $x,y\in X$ and $\alpha\in\left[0,\frac{1}{2}\right)$. If for some positive integer p, T^p is continuous, then T has a unique fixed point in X.

Corollary 3.9. *Let (X,d_l) be a complete dislocated metric space and $T:X\to X$ a self- mapping such that for some positive integer m, T satisfies the following condition:*

$$d_l(T^mx,T^my)\le\alpha\max\{d_l(x,y),d_l(x,T^mx),d_l(y,T^my),d_l(x,T^my),d_l(y,T^mx)\}$$

for all $x,y\in X$ and $\alpha\in\left[0,\frac{1}{2}\right)$. If T^m is continuous, then T has a unique fixed point in X.

Further we prove existence of unique fixed point for a mapping that is said to be a T-Hardy-Rogers contraction in the setup of a b-dislocated metric space.

Theorem 3.10. *Let (X,b_d) be a complete b-dislocated metric space with parameter $s\ge1$ and $T,f:X\to X$ are such that T is one-to-one, continuous and the contractive condition,*

$$\begin{aligned}&s^2b_d(Tfx,Tfy)\\&\le Ab_d(Tx,Ty)+Bb_d(Tx,Tfx)+Cb_d(Ty,Tfy)+Db_d(Tx,Tfy)+Eb_d(Ty,Tfx)\end{aligned}$$

holds for all $x,y\in X$, where the constants A,B,C,D,E are non negative and $0\le A+B+C+2D+2E<1$. Then we have the following:

1. *For every $x_0\in X$ the sequence $\{Tf^nx_0\}$ is convergent;*
2. *If T is subsequentially convergent, then f has a unique fixed point in X;*
3. *If T is sequentially convergent, then for each $x_0\in X$ the sequence $\{f^nx_0\}$ converges to the fixed point of f.*

Proof. Let $x_0\in X$ be an arbitrary point. We define the sequence $\{x_n\}$ by $x_{n+1}=fx_n=f^{n+1}x_0, n=0,1,2,\ldots$. If there exists $n\in N$ such that $x_n=x_{n+1}$ then we get $x_n=x_{n+1}=fx_n$ thus f has a fixed point and the proof is completed. Thus we can suppose $x_n\ne x_{n+1}$ for all $n\in N$.

From contractive condition of theorem, we have:

$$\begin{aligned}&s^2b_d(Tx_{n+1},Tx_n)=s^2b_d(Tfx_n,Tfx_{n-1})\\&\le Ab_d(Tx_n,Tx_{n-1})+Bb_d(Tx_n,Tfx_n)+Cb_d(Tx_{n-1},Tfx_{n-1})+\\&+Db_d(Tx_n,Tfx_{n-1})+Eb_d(Tx_{n-1},Tfx_n)\\&=Ab_d(Tx_n,Tx_{n-1})+Bb_d(Tx_n,Tx_{n+1})+Cb_d(Tx_{n-1},Tx_n)+\\&+Db_d(Tx_n,Tx_n)+Eb_d(Tx_{n-1},Tx_{n+1})\\&\le Ab_d(Tx_n,Tx_{n-1})+Bb_d(Tx_n,Tx_{n+1})+Cb_d(Tx_{n-1},Tx_n)+\\&+2sDb_d(Tx_n,Tx_{n-1})+sE\left[b_d(Tx_{n-1},Tx_n)+b_d(Tx_n,Tx_{n+1})\right].\end{aligned}$$

Hence

$$\begin{aligned}&b_d\left(Tx_{n+1},Tx_n\right)\\&\leq \tfrac{1}{s^2}\left[\begin{array}{c} sAb_d\left(Tx_n,Tx_{n-1}\right)+sBb_d\left(Tx_n,Tx_{n+1}\right)+sCb_d\left(Tx_{n-1},Tx_n\right)+\\ +2sDb_d\left(Tx_n,Tx_{n-1}\right)+sE\left[b_d\left(Tx_{n-1},Tx_n\right)+b_d\left(Tx_n,Tx_{n+1}\right)\right]\end{array}\right]\\&=\tfrac{1}{s}\left[\begin{array}{c} Ab_d\left(Tx_n,Tx_{n-1}\right)+Bb_d\left(Tx_n,Tx_{n+1}\right)+Cb_d\left(Tx_{n-1},Tx_n\right)+\\ +2Db_d\left(Tx_n,Tx_{n-1}\right)+E\left[b_d\left(Tx_{n-1},Tx_n\right)+b_d\left(Tx_n,Tx_{n+1}\right)\right]\end{array}\right].\end{aligned}\tag{3.8}$$

If $b_d\left(Tx_n,Tx_{n-1}\right)\leq b_d\left(Tx_{n+1},Tx_n\right)$ for some $n\in N$, then from the above inequality (3.8) we have

$$\begin{aligned}&b_d\left(Tx_{n+1},Tx_n\right)\\&\leq \tfrac{1}{s}\left[\begin{array}{c} Ab_d\left(Tx_n,Tx_{n+1}\right)+Bb_d\left(Tx_n,Tx_{n+1}\right)+Cb_d\left(Tx_{n+1},Tx_n\right)+\\ +2Db_d\left(Tx_n,Tx_{n+1}\right)+E\left[b_d\left(Tx_{n+1},Tx_n\right)+b_d\left(Tx_n,Tx_{n+1}\right)\right]\end{array}\right]\\&=\tfrac{1}{s}\left[A+B+C+2D+2E\right]b_d\left(Tx_{n+1},Tx_n\right)\\&< b_d\left(Tx_{n+1},Tx_n\right).\end{aligned}$$

That is a contradiction, since $0\leq A+B+C+2D+2E<1$. Thus, we have $b_d\left(Tx_{n+1},Tx_n\right)\leq b_d\left(Tx_n,Tx_{n-1}\right)$ for all $n\in N$. As a result we get,

$$b_d\left(Tx_{n+1},Tx_n\right)\leq \frac{1}{s}\left[A+B+C+2D+2E\right]b_d\left(Tx_n,Tx_{n-1}\right). \tag{3.9}$$

Also in a same way we have

$$b_d\left(Tx_n,Tx_{n-1}\right)\leq \frac{1}{s}\left[A+B+C+2D+2E\right]b_d\left(Tx_{n-1},Tx_{n-2}\right). \tag{3.10}$$

Now from (3.9) and (3.10) we have

$$b_d\left(Tx_{n+1},Tx_n\right)\leq kb_d\left(Tx_n,Tx_{n-1}\right)\leq ...\leq k^nb_d\left(Tx_1,Tx_0\right) \tag{3.11}$$

where $k=\frac{A+B+C+2D+2E}{s}$, so $0<k<1$ as $s\geq 1$. Taking in limit in inequality (3.11) we get

$$b_d\left(Tx_{n+1},Tx_n\right)\to 0. \tag{3.12}$$

Let we prove that $\{Tx_n\}$ is a b_d-Cauchy sequence.

By the triangle inequality, for $m\geq n$ we have:

$$\begin{aligned}&b_d\left(Tx_n,Tx_m\right)\leq s\left[b_d\left(Tx_n,Tx_{n+1}\right)+b_d\left(Tx_{n+1},Tx_m\right)\right]\\&\leq sb_d\left(Tx_n,Tx_{n+1}\right)+s^2b_d\left(Tx_{n+1},Tx_{n+2}\right)+s^3b_d\left(Tx_{n+2},Tx_{n+3}\right)+...\\&\leq sk^nb_d\left(Tx_0,Tx_1\right)+s^2k^{n+1}b_d\left(Tx_0,Tx_1\right)+s^3k^{n+2}b_d\left(Tx_0,Tx_1\right)+...\\&=sk^nb_d\left(Tx_0,Tx_1\right)\left[1+sk+(sk)^2+(sk)^3+....\right]\\&\leq \tfrac{sk^n}{1-sk}b_d\left(Tx_0,Tx_1\right).\end{aligned}$$

As $0\leq sk<1$ letting $n,m\to\infty$, we have $\lim\limits_{n,m\to\infty}b_d\left(Tx_n,Tx_m\right)=0$. So $\{Tx_n\}$ is a b_d-Cauchy sequence in X.

Since (X, b_d) is a complete b-dislocated metric space then $\{Tx_n\} = \{Tf^n x_0\}$ is a b_d-Cauchy convergent sequence, so there exists a point $z \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = z. \tag{3.13}$$

Assuming that T is subsequentially convergent then $\{f^n x_0\}$ has a b_d-convergent subsequence. Hence there exists $u \in X$ and a subsequence $\{n_i\}$ such that $\lim_{i\to\infty} f^{n_i} x_0 = u$. Since the mapping T is continuous, we obtain

$$\lim_{i\to\infty} Tf^{n_i} x_0 = Tu \tag{3.14}$$

and by (3.13), (3.14) we conclude that $Tu = z$.

In the contractive condition of theorem, we have

$s^2 b_d(Tfu, Tfx_n)$
$\le Ab_d(Tu, Tx_n) + Bb_d(Tu, Tfu) + Cb_d(Tx_n, Tfx_n) + Db_d(Tu, Tfx_n)$
$+Eb_d(Tfx_n, Tfu)$.

Taking upper limit as $n \to \infty$and using Lemma 2.10, and results in (3.12) and (3.13), we have

$$\begin{aligned}
&s^2 \tfrac{1}{s} b_d(Tfu, Tu)\\
&\le Bb_d(Tu, Tfu) + sCb_d(Tu, Tfu) + sDb_d(Tu, Tfu) + sEb_d(Tu, Tfu)\\
&\le s(B+E)\, b_d(Tu, Tfu)\\
&\le s(A+B+C+2D+2E)\, b_d(Tu, Tfu)
\end{aligned}$$

which implies that

$$b_d(Tfu, Tu) \le (A+B+C+2D+2E)\, b_d(Tfu, Tu).$$

Since $0 \le A+B+C+2D+2E < 1$ we obtain $b_d(Tfu, Tu) = 0$ that means $Tfu = Tu$.

As T is one-to-one, we get $fu = u$. Thus f has a fixed point.

Also if T is sequentially convergent, similarly we get that $\lim_{n\to\infty} f^n x_0 = u$ replacing $\{n\}$ with $\{n_i\}$.

Uniqueness. Firstly we will prove that if u is a fixed point of f then $b_d(Tu, Tu) = 0$. Using the contractive condition of Theorem 3.10 replacing $x = y = u$, we have

$$\begin{aligned}
&s^2 b_d(Tu, Tu)\\
&= s^2 b_d(Tfu, Tfu)\\
&\le Ab_d(Tu, Tu) + Bb_d(Tu, Tfu) + Cb_d(Tu, Tfu) + Db_d(Tu, Tfu)\\
&+ Eb_d(Tu, Tfu)\\
&= Ab_d(Tu, Tu) + Bb_d(Tu, Tu) + Cb_d(Tu, Tu) + Db_d(Tu, Tu) + Eb_d(Tu, Tu)\\
&= (A+B+C+D+E)\, b_d(Tu, Tu)\\
&\le (A+B+C+2D+2E)\, b_d(Tu, Tu)
\end{aligned}$$

From this inequality we get,

$b_d(Tu,Tu) \leq \frac{A+B+C+2D+2E}{s^2} b_d(Tu,Tu)$ and this implies $b_d(Tu,Tu) = 0$ since $0 \leq \frac{A+B+C+2D+2E}{s^2} < 1$.

If we assume that w is another fixed point of f, then we have,

$$\begin{aligned}
& s^2 b_d(Tu,Tw) \\
&= s^2 b_d(Tfu,Tfw) \\
&\leq Ab_d(Tu,Tw) + Bb_d(Tu,Tfu) + Cb_d(Tw,Tfw) + Db_d(Tu,Tfw) \\
&+ Eb_d(Tw,Tfu) \\
&= Ab_d(Tu,Tw) + Bb_d(Tu,Tu) + Cb_d(Tw,Tw) + Db_d(Tu,Tw) \\
&+ Eb_d(Tw,Tu) \\
&= Ab_d(Tu,Tw) + Db_d(Tu,Tw) + Eb_d(Tw,Tu) \\
&\leq (A+B+C+2D+2E)\, b_d(Tu,Tw).
\end{aligned}$$

The above inequality implies that

$b_d(Tu,Tw) \leq \frac{A+B+C+2D+2E}{s^2} b_d(Tu,Tw)$ and this implies $b_d(Tu,Tw) = 0$ and by property 2, we have $Tu = Tw$. Since T is continuous and one-to-one, we get $u = w$.

Thus the fixed point is unique. □

Example 3.11. Let $X = [0,\infty)$ be equipped with the b-dislocated metric $b_d(x,y) = (x+y)^2$ for all $x,y \in X$, where $s = 2$. It is clear that (X,b_d) is a complete b-dislocated metric space. Also let be the self-mappings $T,f : X \to X$ defined by $T(x) = \frac{x}{3}$, $f(x) = \frac{x}{6}$. We note, that f is a T-Hardy-Rogers contraction, also T is continuous and subsequntially convergent.

For each $x,y \in X$, we have

$$\begin{aligned}
s^2 b_d(Tfx,Tfy) &= 2^2 b_d\left(\tfrac{x}{18},\tfrac{y}{18}\right) \\
&= 4\tfrac{(x+y)^2}{324} \\
&\leq 4\tfrac{(x+y)^2}{144} \\
&= \tfrac{1}{4}\tfrac{(x+y)^2}{9} \\
&= \tfrac{1}{4}\left(\tfrac{x}{3}+\tfrac{y}{3}\right)^2 \\
&= \tfrac{1}{4} b_d(Tx,Ty) \\
&\leq Ab_d(Tx,Ty) + Bb_d(Tx,Tfx) + Cb_d(Ty,Tfy) \\
&\quad + Db_d(Tx,Tfy) + Eb_d(Ty,Tfx).
\end{aligned}$$

Thus T,f satisfy all the conditions of Theorem 3.10. Moreover 0 is the unique fixed point of f.

As a consequence of Theorem 3.10 for taking the parameter $s = 1$ or the identity mapping $Tx = x$ we can establish the following corollaries.

Corollary 3.12. *Let (X,d_l) be a complete dislocated metric space and $T,f : X \to X$ are such that T is one-to-one, continuous and the contractive condition*

$$\begin{aligned}
& d_l(Tfx,Tfy) \\
&\leq Ad_l(Tx,Ty) + Bd_l(Tx,Tfx) + Cd_l(Ty,Tfy) + Dd_l(Tx,Tfy) \\
&+ Ed_l(Ty,Tfx)
\end{aligned}$$

holds for all $x, y \in X$, where the constants A, B, C, D, E are non negative and $0 \leq A+B+C+2D+2E<1$. Then we have the following

1. *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent;*
2. *If T is subsequentially convergent, then f has a unique fixed point in X;*
3. *If T is sequentially convergent, then for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Corollary 3.13. *Let (X, b_d) be a complete b-dislocated metric space with parameter $s \geq 1$ and $f : X \to X$ is a self-mapping such that the contractive condition*

$$s^2 b_d(fx, fy) \leq Ab_d(x, y) + Bb_d(x, fx) + Cb_d(y, fy) + Db_d(x, fy) + Eb_d(y, fx)$$

holds for all $x, y \in X$, where the constants A, B, C, D, E are non negative and $0 \leq A+B+C+2D+2E<1$. Then f has a unique fixed point in X.

Corollary 3.14. *Let (X, b_d) be a complete dislocated metric space and $f : X \to X$ is a self-mapping such that the contractive condition*

$$b_d(fx, fy) \leq Ab_d(x, y) + Bb_d(x, fx) + Cb_d(y, fy) + Db_d(x, fy) + Eb_d(y, fx)$$

holds for all $x, y \in X$, where the constants A, B, C, D, E are non negative and $0 \leq A+B+C+2D+2E<1$. Then f has a unique fixed point in X.

Remark 3.15. From Theorem 3.10 and its corollaries by specifying condition on the given constants we derive as corollaries (special cases) fixed point results for T-Kannan contraction, T-Chatterjea contractions and T-Reich contraction in the framework of b-dislocated metric spaces.

References

[1] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped. Inst. 30 (1989) 26-37.

[2] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993) 5-11.

[3] P. Hitzler, A.K. Seda, Dislocated topologies, J. Electr. Engin. 51 (12) (2000) 3-7.

[4] N. Hussain, J.R. Roshan, V. Parvaneh, M. Abbas, Common fixed point results for weak contractive mappings in ordered b-dislocated metric spaces with applications, Journal of Inequalities and Applications (2013) https://doi.org/10.1186/1029-242X-2013-486.

[5] L.B. Ciric, A generalization of Banach's contraction principle, Prooceedings of the American Mathematical Society 45 (1974) 267-273.

[6] R. Shrivastava, Z.K. Ansari, M. Sharma, Some results on fixed points in dislocated and dislocated quasi-metric spaces, Journal of Advanced Studies in Topology 3 (1) (2012) 25-31.

[7] M.A. Kutbi, M. Arshad, J. Ahmad, A. Azam, Generalized common fixed point results with applications, Abstract and Applied Analysis (2014) http://dx.doi.org/10.1155/2014/363925.

[8] A. Beiranvand, S. Moradi, M. Omid, H. Pazandeh, Two fixed point theorems for special mapping, arXiv:0903.1504v1[math.FA].

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