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Predictability in Atmospheric Model

P. Sangapate and D. Sukawat

Abstract: Atmosphere is a dynamical system, which is a system that changes over time. An interesting feature of the dynamical system is predictability. Predictability is an ability to make an accurate forecast, which depends on degree of freedom and uncertainty in the initial condition. The shallow water model is applied for the atmosphere. It is particularly well suited and often used to test numerical techniques for weather prediction. In this research, predictability of the spectral shallow water model is investigated by using the Lyapunov exponent, maximum Lyapunov exponent, and finite size Lyapunov exponent, which are the measures of the rate of growth of error in a dynamical system.

Keywords: Shallow water model, Spectral method, Lyapunov exponent, Predictability time.

1 Introduction

Fluid flow is a process that can be found at any place and any time. Examples are the flows of water in rivers, lakes, and oceans. Another important example is atmospheric flow which causes winds. The complete set of equations that describes fluid flow, the Navier-Stokes equations, is very complex which makes it impractical to apply in the real world. Thus, it is necessary to simplify the equations to applicable forms. Shallow water equations, a simplified version of the Navier-Stokes equations, are widely used in many fields including atmospheric and oceanographic studies. This results in spherical shallow water equations, which is important for atmospheric and oceanic numerical model development and applications. For examples, it is a basic model for atmospheric prediction, climate change study and pollution dispersion in the atmosphere. Although simplified from the full set of equations, the shallow water equations still maintain an important property, nonlinear behavior. As a consequence, the shallow water equations can sometimes behave in a chaotic manner that is unpredictable. Predictability is ability to make an accurate forecast, which depends on its degree of freedom and uncertainty in the initial condition. In this research, predictability of the spectral shallow water model is investigated using Lyapunov exponent to measure rate of divergence of nearby trajectories.

The outline of the paper is organized as follows: Section 2 introduces the spectral shallow water model, which is tested by standard test cases in Section 3. Sections 4 investigate predictability of the spectral shallow water model by using the Lyapunov exponent, maximum Lyapunov exponent, and finite size Lyapunov exponent. Section 5 is the conclusions.

2 Model

The experiments in this research are performed using a spectral transform shallow water equations model, STSWM (CHAMMP, 2005) with 2 standard test cases of (Williamson, 1992).

2.1 Spectral Shallow Water Model

The shallow water model is the simplest form of the equation of motion that can describes the horizontal structure, hydrostatic homogenous, and incompressible flow. The spectral shallow water model is refered to as the momentum equation and equation of continuity in vector form

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k} \times \mathbf{V} - \nabla\Phi \tag{2.1}$$

and

$$\frac{d\Phi}{dt} = -\Phi\nabla \cdot \mathbf{V} \tag{2.2}$$

where $\mathbf{V} \equiv \mathbf{i}u + \mathbf{j}v$ is the horizontal vector velocity,
$\Phi \equiv gh$ is the free surface geopotential,
h is the free surface height,
g is the acceleration of gravity,
$f \equiv 2\Omega \sin\varphi$ is the Coriolis parameter,
φ is latitude,
Ω is the angular velocity of the earth.
The derivative is given by

$$\frac{d}{dt}() = \frac{\partial}{\partial t}() + (\mathbf{V} \cdot \nabla)() \tag{2.3}$$

and the ∇ operator is defined in spherical coordinates as

$$\nabla() \equiv \frac{\mathbf{i}}{a\cos\varphi}\frac{\partial}{\partial\varphi}() + (\mathbf{V}\cdot\nabla)() \tag{2.4}$$

where λ denotes longitude, a is the radius of the earth.
The spectral method is approximated in term of the Fourier series, which is one of the most powerful solution techniques and has no truncation error. The spectral form of the shallow water model are

$$-n(n+1)\frac{\partial\psi_{m,n}}{\partial t} = -\frac{1}{2}\int_{-1}^{0}[\frac{im}{1-\mu^2}A_mP_{m,n} - \frac{\partial}{\partial\mu}B_mP_{m,n}]d\mu + 2\Omega[n(n-1)D_m\chi_{m,n-1} + (n+1)(n+2)D_{m,n+1}\chi_{m,n+1} - V_{m,n} \tag{2.5}$$

$$-n(n+1)\frac{\partial\chi_{m,n}}{\partial t} = -\frac{1}{2}\int_{-1}^{0}[\frac{im}{1-\mu^2}B_mP_{m,n} + A_m\frac{\partial}{\partial\mu}P_{m,n}]d\sigma - 2\Omega[n(n-1)D_m\psi_{m,n-1} + (n+1)(n+2)D_{m,n+1}\psi_{m,n+1} + U_{m,n} + n(n+1)(E_{m,n} + \Phi_{m,n}) \tag{2.6}$$

$$\frac{\partial\Phi_{m,n}}{\partial t} = -\frac{1}{2}\int_{-1}^{0}[\frac{im}{1-\mu^2}C_mP_{m,n} - D_m\frac{\partial}{\partial\mu}P_{m,n}]d\sigma + n(n+1)\Phi\chi_{m,n} \tag{2.7}$$

where $\psi_{m,n}$,$\chi_{m,n}$,and $\Phi_{m,n}$ are streamfunction, divergent, and geopotential height and A_m,B_m,C_m,D_m,E_m,P_m are coefficients.

3 Measurement of Predictability

Chaotic phenomena are common in nonlinear dynamic systems. They are deterministic and very sensitive to initial conditions. A chaotic system is mainly identified by testing Lyapunov exponents. There are many measurements of predictability. In this research, the following measurements are applied.

3.1 Lyapunov Exponent

Lyapunov exponents (LE) is the rate of divergence or convergence of two nearby initial points of a dynamical system. A positive Lyapunov exponent measures the average exponential divergence of two nearby trajectories whereas a negative Lyapunov exponent measures the average exponential convergence of two nearby trajectories.
The Lyapunov exponent is given by

$$\lambda = \frac{1}{\Delta t}\ln\frac{d_t}{d_0} \tag{3.1}$$

where d_0 is initial distance, d_t is growth of the difference between two nearby trajectories, and Δt is time period. The exponent can be used to identify whether the motion is periodic or chaotic by consider the Lyapunov exponents λ : if $\lambda < 0$, then the motion is periodic and if $\lambda > 0$, then the motion is chaotic (Jing-qing, 2004).

3.2 Maximum Lyapunov Exponent

The maximum Lyapunov exponent (MLE) is the average exponential separation of two infinitesimally close trajectories for a long time in the phase space and is given by

$$\lambda = \lim_{t \to \infty} \frac{1}{t - t_0} \ln \frac{d(t)}{d_0} \tag{3.2}$$

where $d(t)$ is distance between the trajectories at time t. The maximum Lyapunov exponent is based on the infinitesimal distance (Murison, 1995).

3.3 Finite Size Lyapunov Exponent

Generalize the maximum Lyapunov exponent to the average exponential separation of two trajectories at finite error δ. The finite size Lyapunov exponent (FSLE) is the average exponential separation of two trajectories at finite errors in the phase space and is given by

$$\lambda(\delta_n) = \frac{1}{\langle \tau(\delta_n, r) \rangle_e} \ln r \tag{3.3}$$

where $\langle \tau(\delta_n, r) \rangle_e$ is the doubling time, taken for a perturbation to grow from an initial size δ_n to a $r\delta_n$, average over many realizations, and r is a factor which in this case is chosen to be $r = 2$ or $r = \sqrt{2}$. In the limit of infinitesimal separation between trajectories, the finite size Lyapunov exponent tends to the maximum Lyapunov exponent (Aurell, 1997).

3.4 Maximum Predictable Time Scale

Predictability refers to the ability to make predictions of future events. Since a positive Lyapunov exponent of time series indicates divergent trajectories, so that the time series cannot be predicted in the long-term. The maximum predictable time scale can be estimated using the relation with the maximum Lyapunov exponent

$$T_p = \frac{1}{\lambda_{max}} \tag{3.4}$$

Therefore T_p is defined as an index for measuring the predictability of chaotic systems. The range of the maximum predictable time scale T_p is insensitive to the variations of prediction step length, while outside the range, the systematic error can be greatly magnified (Jing-qing, 2004).

4 Experiment Cases

In this research, the spectral transform shallow water model, STSWM, (CHAMMP, 2005) is run on a computer with Linux operating system. The test cases in the standard test set from Williamson et al. are implemented in the STSWM.
Williamson's First Case: Advection of Cosine Bell over the Pole. First case test is the advective component of the shallow water equations,

$$\frac{\partial}{\partial t}(h) + \frac{u}{a\cos\theta}\frac{\partial}{\partial\lambda}(h) + \frac{v}{a}\frac{\partial}{\partial\theta}(h) + \frac{h}{a\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\theta)}{\partial\theta}\right] = 0 \quad (4.1)$$

The solid body rotation is given by

$$u = u_0(\cos\theta\cos\alpha + \sin\theta\cos\lambda\sin\alpha) \quad (4.2)$$

$$v = -u_0\sin\lambda\sin\alpha \quad (4.3)$$

where $u_0 \approx 40\ m/s$, is the advecting wind velocity, $a = 6.37122\times10^6 m$ (the mean radius of the earth) and α is the angle between the axis of solid body rotation and the polar axis of the spherical coordinate system, θ is latitude and λ is longitude. The initial cosine bell test pattern to be advected is given by

$$h(\lambda,\theta) = \begin{cases} (\frac{h_0}{2})(1+\cos\frac{\pi r}{R}) & ;\ r < R \\ 0 & ;\ r \geq R \end{cases} \quad (4.4)$$

where $h_0 = 1000\ m$, $R = \frac{a}{3}$, r is the great circle distance between (λ,θ) and the center, $(\lambda_c,\theta_c) = \left(\frac{3\pi}{2}, 0\right)$.

Williamson's Sixth Case: Rossby-Haurwitz waves. The initial streamfunction and initial height field are given by

$$\psi = -a^2\omega\sin\theta + a^2K\cos^R\theta\sin\theta\cos R\lambda \quad (4.5)$$

$$gh = gh_0 + a^2A(\theta) + a^2B(\theta)\cos R\lambda + a^2C(\theta)\cos 2R\lambda \quad (4.6)$$

where ω, K, R, A, B,and C are constants.

Experiments in this study are summarized in Table 1.

Table 1. Parameters for the experiments

Experiment	truncation	α	Δt
case1-1	T42	0.0	1200
case1-2	T42	0.05	1200
case1-3	T42	1.52	1200
case1-4	T42	1.57	1200
case1-5	T42	1.52	600
case1-6	T63	1.52	900
case1-7	T106	1.52	600
case1-8	T170	1.52	450
case6-1	T42	1.52	1200
case6-2	T63	1.52	1200

5 Results

From the experiments, the initial height fields of the spectral shallow water model are shown in Figure 1.

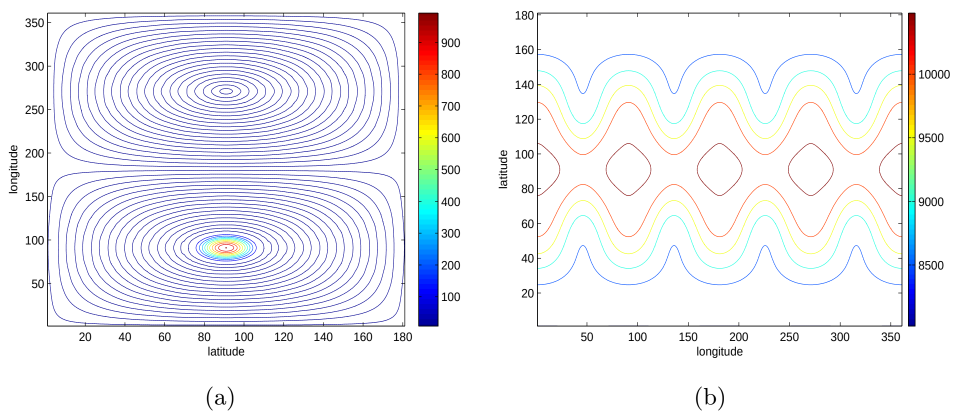
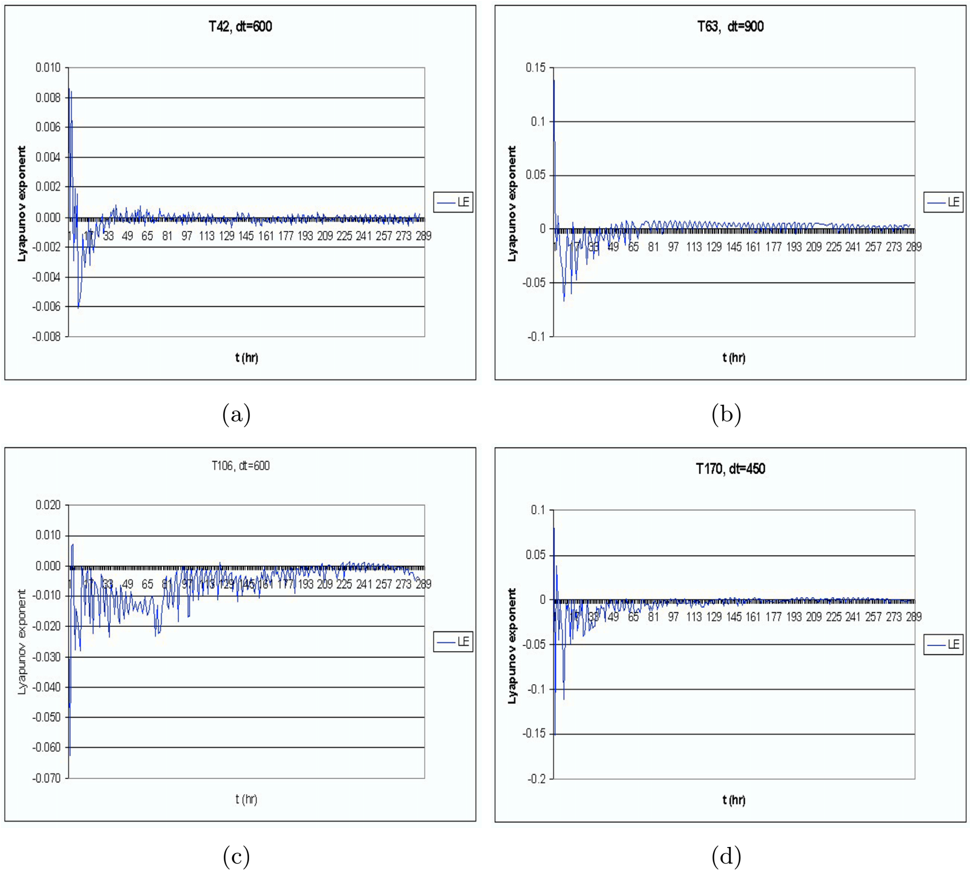


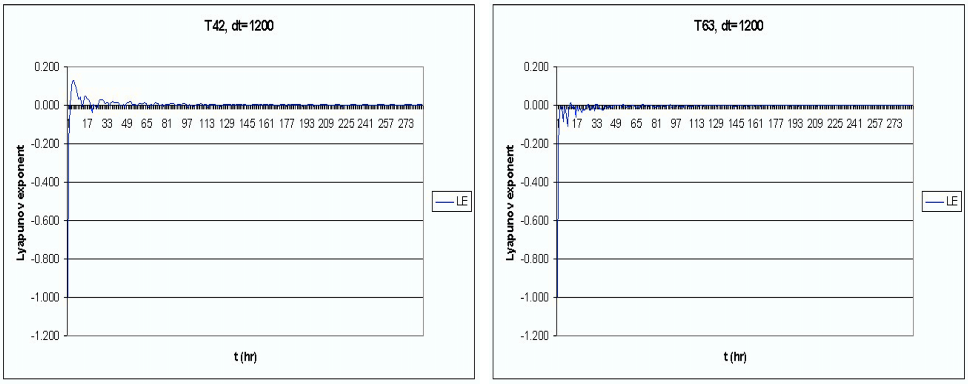
Figure 1: (a) Initial cosine bell and (b) Initial Rossby-Haurwitz waves.



(a) (b) (c) (d)

Figure 2: Time series of Lyapunov exponent for advection of cosine bell (a) T=42, α=1.52, $\Delta t = 600$,(b) T=63, α=1.52, $\Delta t = 900$,(c) T=106, α=1.52, $\Delta t = 600$, and (d) T=170, α=1.52, $\Delta t = 450$.

Figures 2-3 show time series of Lyapunov exponent as a function of geopotential height and longitude velocity component in different truncations T42, T63, T106, and T170.



(a) (b)

Figure 3: Time series of Lyapunov exponent for Rossby Haurwitz waves (a) T=42, α=1.52, $\Delta t = 1200$,(b) T=63, α=1.52, $\Delta t = 1200$.

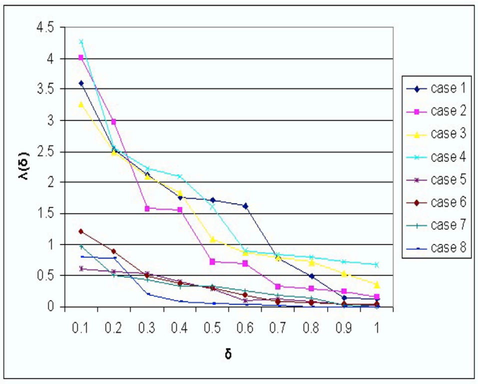
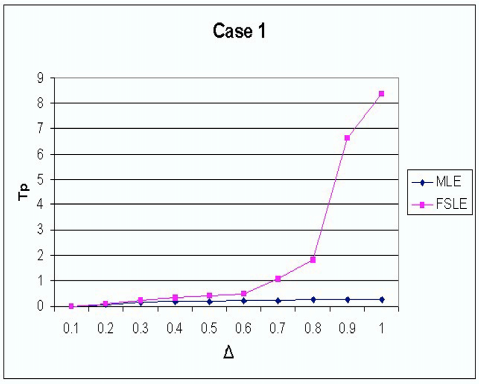
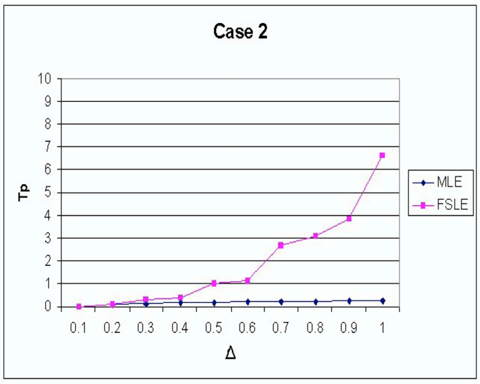


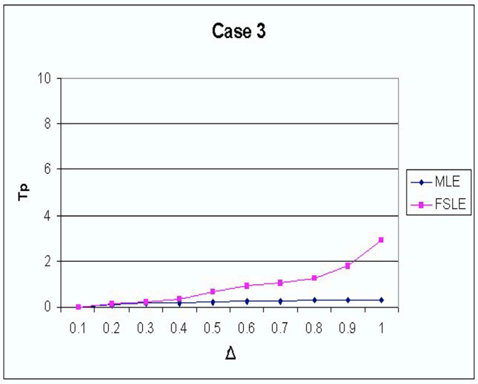
Figure 4: Finite size Lyapunov exponent as a function of error for Case 1-1 - Case 1-8.


Case 1
Tp
Δ
0 1 2 3 4 5 6 7 8 9
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
MLE
FSLE

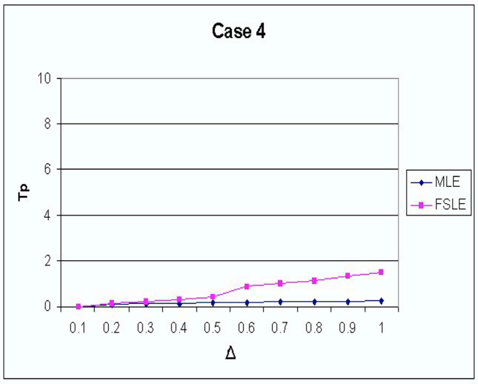
(a)


Case 2
Tp
Δ
0 1 2 3 4 5 6 7 8 9 10
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
MLE
FSLE

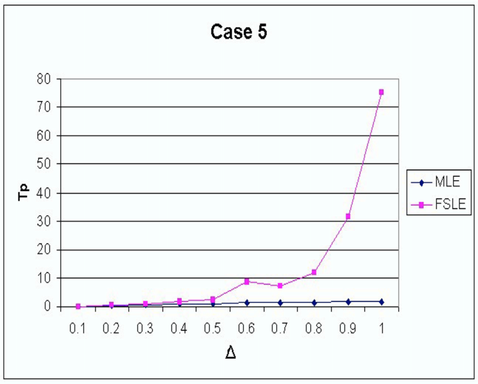
(b)


Case 3
Tp
Δ
0 2 4 6 8 10
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
MLE
FSLE

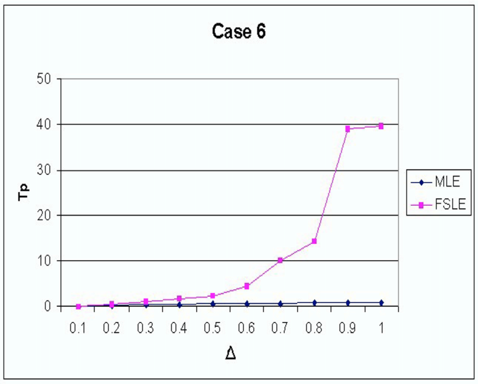
(c)


Case 4
Tp
Δ
0 2 4 6 8 10
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
MLE
FSLE

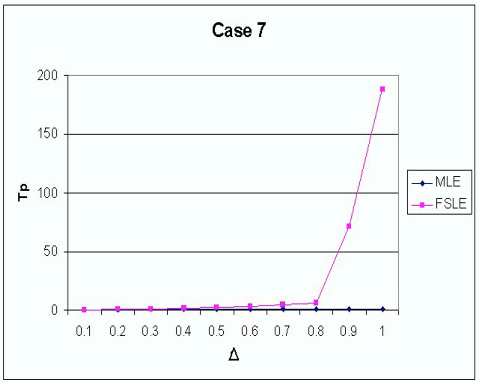
(d)


Case 5
Tp
Δ
0 10 20 30 40 50 60 70 80
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
MLE
FSLE

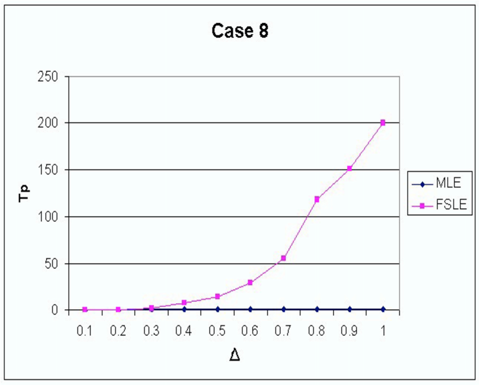
(e)


Case 6
Tp
Δ
0 10 20 30 40 50
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
MLE
FSLE

(f)



(g)



(h)

Figure 5: Predictability time based on MLE and FSLE for Case 1-1 - Case 1-8.

Figures 2(a)-2(d), the earier motions are sensitive to initial condition and then the motions are stable in the later time. For $T106$ and $T170$ in Figures 2(c)-2(d), the motion is more stable than $T42$ and $T163$ in Figures 2(a)-2(b). A positive Lyapunov exponent of time series indicates dissipative trajectories periodicity, so that the time series cannot be predicted in the long time (Jing-qing, 2004). The maximum predictable time scale T_p for cases $T42$, $T63$, $T106$, and $T170$ are 14.8628, 1.1338, 5.5556, and 2.5253 hrs, respectively.

Figures 3(a)-3(b), the earier motions are insensitive to initial condition and then the motions are unstable in the later time. For $T63$ in Figure 3(b), the motion is more stable than $T42$ in Figure 3(a). For a positive Lyapunov exponent, the motion cannot be predicted in the long time. The maximum predictable time scale T_p for cases $T42$ and $T63$ are 0.1235 and 9.8722 hrs, respectively.

Figure 4 shows the plot of finite size Lyapunov exponent $\lambda(\delta)$ as a function of an error δ for Cases 1-1 - 1-8. The $\lambda(\delta)$ of $\alpha = 1.57$ is more larger than $\alpha = 0.0$, $\alpha = 0.05$, and $\alpha = 1.52$, respectively. The $\lambda(\delta)$ of $T170$ is more smaller than $T42$, $T63$, and $T106$, respectively. For small error, $\lambda(\delta)$ tends to λ_{max}. For larger errors, $\lambda(\delta)$ decreases with δ and the analysis predicts an enhancement of predictability time.

Figure 5 shows the predictability time T_p based on FSLE and MLE as a function of the tolerance Δ for an initial error of 0.1. For figures 5(a)-5(h), the FSLE gives longer predictability time than MLE, which is an enhancement of predictability for a range of prediction error tolerances Δ is observed. For $\alpha = 1.57$ in Figure 5(d), the predictability time is smaller than $\alpha = 0.0$, $\alpha = 0.05$, and $\alpha = 1.52$ in Figures 5(a)-5(c). For $T170$ in Figure 5(h), the predictability time is larger than most of $T42$, $T63$, and $T106$ in Figures 5(e)-5(g).

6 Conclusions

In this research, numerical experiments of the shallow water equation model STSWM are performed and predictability time of 2 standard test cases (Williamson, 1992) are investigated. Results from the experiment show that, the predictability time of spectral shallow water model is relates to Lyapunov exponent and also depend on model resolution and angular velocity. For high resolution and low angular velocity, the maximum predictability time scale of the spectral shallow water model is more longer than that of the low resolution and high angular velocity.

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P. Sangapate and D. Sukawat
Department of Mathematics, Faculty of Science,
King Mongkut's University of Technology Thonburi, 10140 Thailand
e-mail: prisayarat1369@hotmail.com
e-mail: d_sukawat@hotmail.com