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Solving Fractional Ordinary Differential Equations Using FNTM

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Abstract : Our goal for the present paper is to propose new method called the Fractional Natural Decomposition Method (FNTM). We use the FNTM to construct analytic approximate and exact solutions to some applications of Linear Fractional Ordinary Differential Equation (LFODEs), namely; the fractional nonhomogeneous Bagley–Torvik equation, the composite fractional oscillation equation and three other fractional ordinary differential equations. The fractional derivatives are described in the Caputo sense. One can conclude the new method is easy and efficient.

Keywords : natural transform; Sumudu transform; Laplace transform; fractional derivatives; ordinary differential equations.
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1 Introduction

There are many physical applications in physical sciences and engineering can be represented by models using fractional differential equations, such as; viscoelasticity, diffusion, control, relaxation process and so on [1–6]. These equations are represented by linear and nonlinear ODEs and solving such fractional differential equations is very important. In recent years, fractional calculus starts to attract much more attention of Physicists and Mathematicians. Most of fractional differential equations do not have exact analytical solutions; hence considerable need has been focused on approximate and numerical solutions of these equations.

In this article, we develop a new computational algorithm for solving linear fractional ordinary differential equations called the fractional Natural Transform Method (FNTM). The proposed method always lead to exact or approximate solution in the form of a rapidly convergence series with elegant computational terms. This shows the efficiency, flexibility and applicability of the new method. Hence the FNTM is a powerful mathematical tool for solving differential equations and it is a refinement of the existing methods.

Recently, several numerical methods were proposed to solve the fractional differential equations, such as; the Fractional Sumudu Transform Method (FSTM) [7, 8], Fractional Matrix method [9], Fractional Homotopy Perturbation Method (FHPM) [10, 11], Fractional Adomian decomposition method (FADM) [12], the Fractional Reduced Differential Transform Method (FRDTM) [13–16], the Natural Decomposition Method [17]. The Adomian decomposition method (ADM) [18, 19], proposed by George Adomian, has been applied to a wide class of linear and nonlinear PDEs. For the nonlinear models, the ADM shows reliable results in supplying exact solutions and analytical approximate solutions that converges rapidly to the exact solutions. One can conclude the FNTM is easy to use and efficient.

In this paper, we give exact solutions to the following fractional ordinary differential equations:

First, consider the initial value problem in the case of nonhomogeneous Bagley–Torvik equation of the form:

$$D^2 y(t) + D^{3/2} y(t) + y(t) = 1 + t, \tag{1.1}$$

subject to the initial conditions

$$y(0) = y'(0) = 1. \tag{1.2}$$

Second, consider the nonhomogeneous initial value problem of the form:

$$D^\alpha y(t) + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - x, \tag{1.3}$$

subject to the initial condition

$$y(0) = 0, \quad 0 < \alpha \leq 1. \tag{1.4}$$

Third, consider the linear fractional initial value problem of the form:

$$D^\alpha y(t) + y(t) = 0, \tag{1.5}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \; \alpha > 1. \tag{1.6}$$

Fourth, consider the linear fractional initial value problem of the form:

$$D^\alpha y(t) = y(t) + 1, \;\; 0 < \alpha \leq 1, \tag{1.7}$$

subject to the initial condition

$$y(0) = 0. \tag{1.8}$$

Fifth, consider the composite fractional oscillation equation of the form:

$$y''(t) - a\, D^\alpha y(t) - b\, y(t) = 8, \tag{1.9}$$

subject to the initial condition

$$y(0) = y'(0) = 0, \tag{1.10}$$

where a and b are constants, $D = \frac{d}{dt}$ and $1 < \alpha \leq 2$.

The remaining structure of this paper is organized as follows: In Section 2, we present preliminaries and definitions of fractional calculus. In Section 3, the Natural Transform Method (NTM) is presented. In Section 4, the Fractional Natural Transform Method (FNTM) is introduced. Section 5 is devoted to apply the method to five test problems. Section 6 is for discussion and conclusion of this paper.

2 Preliminaries of Fractional Calculus

In this section, we give some of the main definitions and notations that related to fractional calculus. These basic definitions are due to Liouville [20–22]:

Definition 2.1. A real function $f(x),\ x > 0$ is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $q(> \mu)$, such that $f(x) = x^q g(x)$, where $g(x) \in C\,[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu,\ m \in \mathbb{N}$.

Definition 2.2. For a function f, the Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, is defined as:

$$\left\{ \begin{array}{l} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \;\; \alpha > 0, \;\; x > 0 \\ \\ J^0 f(x) = f(x) \end{array} \right\}. \tag{2.1}$$

Caputo and Mainardi [21] presented a modified fractional differentiation operator D^α in their work on the theory of viscoelasticity to overcome the disadvantages of the Riemann–Liouville derivative when someone tries to model real world problems.

Definition 2.3. The fractional derivative of f in the Caputo sense can be defined as:

$$\left\{ \begin{array}{l} {}^cD^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \\ \\ m-1 < \alpha \leq m, \;\; m \in \mathbb{N}, \;\; x > 0, \;\; f \in C_{-1}^m \end{array} \right\}. \tag{2.2}$$

Definition 2.4. [23] A one-parameter function of the Mittag–Leffler type is defined by the series expansion:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C}. \tag{2.3}$$

Lemma 2.5. *If* $m-1 < \alpha \le m,\ m \in \mathbb{N}$ *and* $f \in C_\mu^m,\ \mu \ge -1$, *then*

$$\left\{ \begin{array}{l} D^\alpha J^\alpha f(x) = f(x),\ x > 0 \\ \\ J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!},\ m-1 < \alpha < m \end{array} \right\}. \tag{2.4}$$

It should be mentioned here, the Caputo fractional derivative is used because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Remark 2.6. *Note that,* Γ *represents the Gamma function, which is defined by:*

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{C}. \tag{2.5}$$

Notice that the Gamma function is the continuous extension to the fractional function. Throughout this paper, we use the recursive relation $\Gamma(z+1) = z\Gamma(z)$, $z > 0$ to calculate the value of the Gamma function of all real numbers by knowing only the value of the Gamma function between 1 and 2.

3 History of the Natural Transform Method

In this section, we present some background about the nature of the Natural Transform Method (NTM). Given a function $f(t)$, where $t \in \mathbb{R}$ then the general integral transform is defined by [24, 25]:

$$\Im\left[f(t)\right](s) = \int_{-\infty}^{\infty} K(s,t)\, f(t)\, dt, \tag{3.1}$$

where $K(s,t)$ represent the kernel of the transform, s is the real (complex) number which is independent of t. Note that when $K(s,t)$ is $e^{-st},\ t\,J_n(st)$ and $t^{s-1}(st)$, then Eq. (3.1) gives, respectively, Laplace transform, Hankel transform and Mellin transform.
Now, for $f(t),\ t \in (-\infty, \infty)$ consider the integral transforms defined by

$$\Im\left[f(t)\right](u) = \int_{-\infty}^{\infty} K(t)\, f(ut)\, dt, \tag{3.2}$$

and

$$\Im\left[f(t)\right](s,u) = \int_{-\infty}^{\infty} K(s,t)\, f(ut)\, dt. \tag{3.3}$$

It is worth mentioning when $K(t) = e^{-t}$, Eq. (3.2) gives the integral Sumudu transform, where the parameter s replaced by u. Moreover, for any value of n the generalized Laplace and Sumudu transform are respectively defined by [7, 24]:

$$\ell\left[f(t)\right] = F(s) = s^n \int_0^\infty e^{-s^{n+1}t} f(s^n t)\, dt, \tag{3.4}$$

and

$$\mathbb{S}\left[f(t)\right] = G(u) = u^n \int_0^\infty e^{-u^n t} f(tu^{n+1})\, dt. \tag{3.5}$$

Note that when $n = 0$, Eq. (3.4) and Eq. (3.5) are the Laplace and Sumudu transform respectively. The natural transform of the function $f(t)$ for $t \in \mathbb{R}$ is defined by [24, 25]:

$$\mathbb{N}\left[f(t)\right] = R(s,u) = \int_{-\infty}^\infty e^{-st} f(ut)\, dt; \quad s,\, u \in (-\infty, \infty)\,, \tag{3.6}$$

where $\mathbb{N}\left[f(t)\right]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables. Note that Eq. (3.6) can be written in the form [7, 24]:

$$\mathbb{N}\left[f(t)\right] = R^-(s,u) + R^+(s,u).$$

It should be mentioned here, if the function $f(t)H(t)$ is defined on the positive real axis, with $t \in \mathbb{R}$, then we define the Natural transform (N–Transform) on the set

$$A = \left\{ f(t) : \exists\, M,\, \tau_1,\, \tau_2 > 0, \text{ such that } |f(t)| < M e^{\frac{|t|}{\tau_j}}, \right.$$
$$\left. \text{where } t \in (-1)^j \times [0, \infty)\,,\ j \in \mathbb{Z}^+ \right\} \text{ as:}$$

$$\mathbb{N}\left[f(t)H(t)\right] = \mathbb{N}^+\left[f(t)\right] = R^+(s,u) = \int_0^\infty e^{-st} f(ut)\, dt; \quad s,\, u \in (0, \infty)\,, \tag{3.7}$$

where $H(.)$ is the Heaviside function. Note if $u = 1$, Eq. (3.7) can be reduced to the Laplace transform and if $s = 1$, Eq. (3.7) can be reduced to the Sumudu transform.

Now we give some of the N–Transforms and the conversion to Sumudu and Laplace.

Table 1. Special N–Transforms and the conversion to Sumudu and Laplace

$f(t)$	$\mathbb{N}[f(t)]$	$\mathbb{S}[f(t)]$	$\ell[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n=1,2,...$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Remark 3.1. *The reader can read more about the Natural transform in* [24, 25].

Now we give some important properties of the N–Transforms are given as follows [24–26]:

Table 2. Properties of N–Transforms

Functional Form	Natural Transform
$y(t)$	$Y(s,u)$
$y(at)$	$\frac{1}{a}Y(s,u)$
$y'(t)$	$\frac{s}{u}Y(s,u) - \frac{f(0)}{u}$
$y''(t)$	$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u^2}y(0) - \frac{y'(0)}{u}$
$\gamma f(t) + \beta g(t)$	$\gamma F^{+}(s,u) \pm \beta G^{+}(s,u)$

Important Properties:

We give some basic properties of the N-Transforms as follows [24, 25]:

1. $\mathbb{N}^{+}[t^{\alpha}] = \frac{\Gamma(\alpha+1)\, u^{\alpha}}{s^{\alpha+1}}, \;\; \alpha > -1.$

2. $\mathbb{N}^{+}\left[f^{(n)}(t)\right] = \frac{s^n}{u^n}\, R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0).$

3. $\mathbb{N}^{+}[f(t) * g(t)] = u\, R(s,u)\, G(s,u)$. This is called the convolution theorem of N–Transform.

4. If $\mathrm{R}(\mathrm{s},\mathrm{u})$ and $\mathrm{G}(\mathrm{u})$ are, respectively, the Natural and Sumudu transforms

of the function $f\,(\mathrm{t}) \in A$, then $\mathbb{N}^{+}\,[f(t)] = R(s,u) = \int_0^\infty e^{-t}\, f\left(\frac{ut}{s}\right)\, dt = \frac{1}{s} G\left(\frac{u}{s}\right)$.

This property is called the Natural–Sumudu Duality (NSD).

4 Theories of the FNTM

In this section, we give some important theorems and we illustrate the analysis of the Fractional Transform Method to some nonlinear ordinary differential equations. Also, it should be mentioned, these theorems were part of a master thesis (In Progress) supervised by the author of this paper.

Theorem 4.1. *If for any positive integer n, where $n-1 \le \alpha < n$ and $R(s,u)$ is the Natural transform of the function $f(t)$, then the Natural transform, $R_\alpha^c(s,u)$ of the Caputo fractional derivative of the function $f(t)$ of order α denoted by ${}^cD^\alpha f(t)$ is given by:*

$$R_\alpha^c(s,u) = \mathbb{N}^{+}\left[{}^cD^\alpha f(t)\right] = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} \left[D^k f(t)\right]_{t=0}.$$

Theorem 4.2. *For $\alpha,\ \beta > 0,\ a \in \mathbb{R}$ and $\frac{s^\alpha}{u^\alpha} > |a|$, then:*

$$\mathbb{N}^{-1}\left[\frac{u^{\beta-1} s^{\alpha-\beta}}{s^\alpha + a u^\alpha}\right] = t^{\beta-1} E_{\alpha,\beta}(-a t^\alpha).$$

Theorem 4.3. *For $\alpha \ge \beta > 0,\ a \in \mathbb{R}$ and $\left(\frac{u}{s}\right)^{\alpha-\beta} > |a|$, then:*

$$\mathbb{N}^{-1}\left[\frac{u^{(n+1)(\alpha+\beta)-1}}{\left(s^\alpha u^\beta + a u^\alpha s^\beta\right)^{n+1}}\right] = t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma\left(k(\alpha-\beta)+(n+1)\alpha\right)} t^{k(\alpha-\beta)}.$$

Theorem 4.4. *For $\alpha \ge \beta,\ \alpha > \gamma,\ a \in \mathbb{R}$, $\left(\frac{u}{s}\right)^{\alpha-\beta} > |a|$ and $\left|\frac{b u^{\alpha+\beta}}{s^\alpha u^\beta + a u^\alpha s^\beta}\right| < 1$, we have*

$$\begin{aligned} &\mathbb{N}^{-1}\left[\frac{u^{\alpha-(\gamma+1)+\beta} s^\gamma}{s^\alpha u^\beta + a u^\alpha s^\beta + b u^{\alpha+\beta}}\right] \\ &= t^{\alpha-(\gamma+1)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (-a)^k \binom{n+k}{k}}{\Gamma\left(k(\alpha-\beta)+(n+1)\alpha-\gamma\right)} t^{k(\alpha-\beta)+n\alpha}. \end{aligned}$$

5 Worked Examples

In this section, we present five worked examples to show the efficiency of the FNTM then we will compare our solutions with existing one.

Example 5.1. Consider the initial value problem in the case of nonhomogeneous Bagley–Torvik equation of the form:

$$D^2 y(t) + D^{3/2} y(t) + y(t) = 1 + t, \tag{5.1}$$

subject to the initial conditions

$$y(0) = y'(0) = 1. \tag{5.2}$$

Solution: Using the Natural transform of derivative and Natural transform of the Caputo fractional derivative of order $\frac{3}{2}$, we obtain:

$$\frac{s^2}{u^2} Y(s,u) - \frac{s\,y(0)}{u^2} + \frac{s^{3/2}}{u^{3/2}} Y(s,u) - \frac{s^{1/2}}{u^{3/2}} y(0) - \frac{s^{-1/2}}{u^{1/2}} y'(0) + Y(s,u) = \frac{1}{s} + \frac{u}{s^2}. \tag{5.3}$$

Substitute Eq. (5.2) into Eq. (5.3) to get:

$$Y(s,u)\left[\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right] = \frac{s}{u^2} + \frac{1}{u} + \frac{s^{1/2}}{u^{3/2}} + \frac{1}{s} + \frac{u}{s^2} + \frac{1}{(us)^{1/2}}. \tag{5.4}$$

Then Eq. (5.4) becomes:

$$Y(s,u)\left[\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right] = \left(\frac{1}{s} + \frac{u}{s^2}\right)\left(\frac{s}{u^2} + \frac{s^{1/2}}{u^{3/2}} + 1\right). \tag{5.5}$$

From Eq. (5.5) we obtain:

$$Y(s,u) = \left(\frac{1}{s} + \frac{u}{s^2}\right). \tag{5.6}$$

Now we take the inverse N-Transform of Eq. (5.6) to get:

$$y(t) = 1 + t.$$

This is the exact solution of Eq. (5.1).

Example 5.2. Consider the nonhomogeneous initial value problem of the form:

$$D^\alpha y(t) + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - x, \tag{5.7}$$

subject to the initial condition

$$y(0) = 0, \quad 0 < \alpha \leq 1. \tag{5.8}$$

Solution: Using Theorem 1 and the Natural transform of derivative, we obtain:

$$\begin{aligned}&\frac{s^\alpha}{u^\alpha}Y(s,u)-\frac{y(0)}{u^\alpha}+Y(s,u)\\&=\frac{2u^{2-\alpha}}{\Gamma(3-\alpha)}\frac{\Gamma(3-\alpha)}{s^{3-\alpha}}-\frac{1}{\Gamma(2-\alpha)}\frac{\Gamma(2-\alpha)u^{1-\alpha}}{s^{2-\alpha}}+\frac{2u^2}{s^3}-\frac{u}{s^2}.\end{aligned}$$

Then

$$Y(s,u)\left(\frac{s^\alpha}{u^\alpha}+1\right)=\frac{2u^{2-\alpha}}{s^{3-\alpha}}-\frac{u^{1-\alpha}}{s^{2-\alpha}}+\frac{2u^2}{s^3}-\frac{u}{s^2}. \tag{5.9}$$

Thus

$$Y(s,u)=\left(\frac{2u^2}{s^3}-\frac{u}{s^2}\right). \tag{5.10}$$

Now we take the inverse N-Transform of Eq. (5.11) to get

$$y(t)=t^2-t. \tag{5.11}$$

This is the exact solution of Eq. (5.7).

Example 5.3. Consider the linear fractional initial value problem of the form:

$$D^\alpha y(t)+y(t)=0, \tag{5.12}$$

subject to the initial conditions

$$y(0)=1,\quad y'(0)=0,\ \alpha>1. \tag{5.13}$$

Solution: The two cases of α, $\mathbb{N}^+\left[D^\alpha y(t)\right]$ are given as follows: For both $\alpha<1$ and $\alpha>1$ using Theorem 1, we obtain:

$$\frac{s^\alpha}{u^\alpha}Y(s,u)-\frac{s^{\alpha-1}y(0)}{u^\alpha}-\frac{s^{\alpha-2}y'(0)}{u^{\alpha-1}}+Y(s,u)=0. \tag{5.14}$$

Substituting Eq. (5.13) we have:

$$\frac{s^\alpha}{u^\alpha}Y(s,u)-\frac{s^{\alpha-1}}{u^\alpha}+Y(s,u)=0.$$

Then

$$Y(s,u)\left(\frac{s^\alpha}{u^\alpha}+1\right)=\frac{s^{\alpha-1}}{u^\alpha}. \tag{5.15}$$

Thus

$$Y(s,u)=\frac{\frac{s^{\alpha-1}}{u^\alpha}}{\left(\frac{s}{u}\right)^\alpha+1}. \tag{5.16}$$

Then by using Theorem 2, we obtain the exact solution as:

$$y(t)=E_\alpha\left(-t^\alpha\right).$$

This is the exact solution of Eq. (5.12).

Example 5.4. Consider the linear fractional initial value problem of the form:

$$D^{\alpha}y(t) = y(t) + 1, \ \ 0 < \alpha \leq 1, \tag{5.17}$$

subject to the initial condition

$$y(0) = 0. \tag{5.18}$$

Solution: Using Theorem 1, we obtain:

$$\frac{s^{\alpha}}{u^{\alpha}}Y(s,u) = Y(s,u) + \frac{1}{s}. \tag{5.19}$$

Then

$$Y(s,u)\left(\frac{s^{\alpha}}{u^{\alpha}} - 1\right) = \frac{1}{s}.$$

Thus

$$Y(s,u) = \frac{s^{-1}u^{\alpha}}{s^{\alpha} - u^{\alpha}} = \frac{s^{-1}}{\left(\frac{s}{u}\right)^{\alpha} - 1}. \tag{5.20}$$

Then by using Theorem 2, we obtain the exact solution as:

$$y(t) = t^{\alpha}E_{\alpha,\alpha+1}\left(t^{\alpha}\right).$$

This is the exact solution of Eq. (5.17).

Example 5.5. Consider the composite fractional oscillation equation of the form:

$$y''(t) - a\,D^{\alpha}y(t) - b\,y(t) = 8, \ \ 1 < \alpha \leq 2, \tag{5.21}$$

subject to the initial conditions

$$y(0) = y'(0) = 0. \tag{5.22}$$

Solution: Using Theorem 1 and the Natural transform of derivative, we obtain:

$$\frac{s^2}{u^2}Y(s,u) - \frac{s\,y(0)}{u^2} - \frac{y'(0)}{u} - a\left[\frac{s^{\alpha}}{u^{\alpha}}Y(s,u) - \frac{s^{\alpha-1}y(0)}{u^{\alpha}} - \frac{s^{\alpha-2}y'(0)}{u^{\alpha-1}}\right] - b\,Y(s,u) = \frac{8}{u}. \tag{5.23}$$

Substituting Eq. (5.22) into Eq. (5.23) to get:

$$\frac{s^2}{u^2}Y(s,u) - \frac{a\,s^{\alpha}}{u^{\alpha}}Y(s,u) - b\,Y(s,u) = \frac{8}{s}. \tag{5.24}$$

Then Eq. (5.24) becomes:

$$Y(s,u)\left(\frac{s^2}{u^2} - \frac{a\,s^{\alpha}}{u^{\alpha}} - b\right) = \frac{8}{s}. \tag{5.25}$$

Thus

$$Y(s,u) = \frac{8s^{-1}}{\left(\frac{s}{u}\right)^{2}u^{\alpha} - a\left(\frac{s}{u}\right)^{\alpha}u^{2} - b}. \tag{5.26}$$

Using Theorem 4 we obtain the exact solution given by:

$$y(t) = 8t^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^n a^k \begin{pmatrix} n+k \\ k \end{pmatrix} t^{(2-\alpha)k+2n}}{\Gamma\left((2-\alpha)k + 2(n+1) + 1\right)}.$$

This is the exact solution of Eq. (5.21).

6 Conclusion

In this paper, the Fractional Natural Transform Method (FNTM) has been successfully applied to obtain analytical solutions to the fractional ordinary differential equations, such as; the fractional nonhomogeneous Bagley–Torvik equation, the composite fractional oscillation equation and three other fractional ordinary differential equations. We successfully found exact solutions to all physical models. The FNTM introduces a significant improvement in the fields over existing techniques. Our goal in the future is to apply the FNTM to other linear fractional ODEs that arise in other areas of science.

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References

[1] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[3] D. Kumar, J. Singh, A. Kiliman, An efficient approach for fractional Harry Dym equation by using Sumudu transform, Abstract and Applied Analysis (2013) Article ID 608943.

[4] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.

[5] S. Kumar, M.P. Tripathi, O.P. Singh, A fractional model of Harry Dym equation and its approximate solution, Ain Shams Engineering Journal 4 (2013) 111-115.

[6] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[7] Q.K. Katatbeh, F.B.M. Belgacem, Applications of the Sumudu transform to fractional differential equations, Nonlinear Studies Journal 18 (1) (2011) 99-112.

[8] D. Kumar, J. Singh, A. Kiliman, An efficient approach for fractional Harry Dym equation by using Sumudu transform, Abstract and Applied Analysis (2013) Article ID 608943.

[9] M. Garg, P. Manohar, Numerical solution of fractional diffusion-wave equation with two space variables by matrix method, Fractional Calculus and Applied Analysis 13 (2) (2010) 191-207.

[10] Q. Wang, Homotopy perturbation method for fractional KdV-Burgers equation, Chaos Solitons & Fractals 35 (2008) 843-850.

[11] A. Yildirim, He's homtopy perturbation method for solving the space and time fractional telegraph equations, International Journal of Computer Mathematics 87 (13) (2010) 2998-3006.

[12] S.S. Ray, R.K. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method, Applied Mathematics and Computation 167 (2005) 561-571.

[13] M. Rawashdeh, Improved approximate solutions for nonlinear evolutions equations in Mathematical Physics using the RDTM, Journal of Applied Mathematics and Bioinformatics 3 (2) (2013) 1-14.

[14] M. Rawashdeh, Using the eeduced differential transform method to solve nonlinear PDEs arises in Biology and Physics, World Applied Sciences Journal 23 (8) (2013) 1037-1043.

[15] M. Rawashdeh, N. Obeidat, On finding exact and approximate solutions to some PDEs using the reduced differential transform method, Applied Mathematics and Information Sciences 8 (5) (2014) 1-6.

[16] M. Rawashdeh, Approximate solutions for coupled systems of nonlinear PDES using the reduced differential transform method, Mathematical and Computational Applications, An International Journal 19 (2) (2014) 161-171.

[17] M. Rawashdeh, S. Maitama, Solving coupled system of nonlinear PDEs using the natural decomposition method, International Journal of Pure and Applied Mathematics 92 (5) (2014) 757-776.

[18] G. Adomian, Solving frontier problems of physics: the decomposition method, Kluwer Acad. Publ (1994).

[19] G. Adomian, A new approach to nonlinear partial differential equations, J. Math. Anal. Appl. 102 (1984) 420-434.

[20] M. Caputo, Elasticitae dissipazione, Zanichelli: Bologna (1969).

[21] M. Caputo, F. Mainardi, Linear models of dissipation in anelastic solids, Rivista del Nuovo Cimento 1 (1971) 161-168.

[22] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, J. Roy. Austral. Soc. 13 (1967) 529-539.

[23] G.M. Leffler, Sur la nouvelle fonction E_α (x), C. R. Acad. Sci. Paris 137 (1903) 554-558.

[24] F.B.M. Belgacem, R. Silambarasan, Theoretical investigations of the natural transform, Progress In Electromagnetics Research Symposium Proceedings, Suzhou, China, Sept. 12 16 (2011).

[25] Z.H. Khan, W.A. Khan, N-transform properties and applications, NUST Jour. of Engg. Sciences 1 (1) (2008) 127-133.

[26] F.B.M. Belgacem, Sumudu applications to Maxwell's equations, PIERS Online 5 (4) (2009) 355-360.

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