Thai Journal of Mathematics
Volume 17 (2019) Number 1 : 205–216
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On Ćirić Type φ-Geraghty Contractions

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Abstract : In this paper we introduce the notions of φ-Geraghty contractions and Ćirić type φ-Geraghty contractions. We also investigate under which conditions such mappings posses a unique fixed point in the framework of complete metric spaces. We consider examples to show the validity of our main results.

Keywords : fixed point; metric space; Geraghty contraction; Ćirić contraction.
2010 Mathematics Subject Classification : 46T99; 47H10; 54H25.

1 Introduction and Preliminaries

Let (X, d) be a complete metric space. A map T is a contraction if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for each } x, y \in X. \tag{1.1}$$

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Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0. \tag{1.2}$$

An operator $T : X \to X$ is called a Geraghty contraction [1] if there exists a function $\beta \in \mathcal{F}$ which satisfies the condition

$$d(Tx,Ty) \le \beta(d(x,y))d(x,y) \text{ for each } x,y \in X. \tag{1.3}$$

In 1973, Geraghty [1] successfully obtained a unique fixed point for such contractions.

Theorem 1.1. [1] *Suppose that T is a self-mapping on the complete metric space (X,d). If T is a Geraghty contraction, then it posses a unique fixed point $x^* \in X$. Moreover, for any initial point $x_0 \in X$, the iterative sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.*

It is an undoubted generalization of the most celebrated result in the metric fixed point theory, the Banach contraction principle. Indeed, it is sufficient to take $\beta(t) = k$ for all $t \in [0,\infty)$. This idea has been appreciated and improved in several ways by many authors, see e.g. [2–12] and the related references therein.

Very recently, Suzuki [13] proved the following fixed point theorem that was inspired from the well-known results of Meir-Keeler [14].

Theorem 1.2. [13] *Let (X,d) be a complete metric space and a mapping $T : X \to X$. Define a function L from $X \times X$ into $[0,\infty)$ by*

$$L(x,y) = \max\left\{ d(x,y), \frac{d(x,Ty)+d(Tx,y)}{2}, d(x,Tx), d(y,Ty) \right\}. \tag{1.4}$$

Assume that there exists a function φ from $[0,\infty)$ into itself satisfying the following:

(φ1) *$\varphi(t) < t$ for any $t \in (0,\infty)$.*

(φ2) *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\epsilon < t < \varepsilon + \delta \text{ implies } \varphi(t) \le \varepsilon.$$

(φ3) *$d(Tx,Ty) \le \varphi \circ L(x,y)$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Remark 1.3. *By (φ1), is easy to see that (φ2) is equivalent to the following*

($\varphi2'$) *For any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$t < \varepsilon + \delta \text{ implies } \varphi(t) \le \varepsilon.$$

Indeed, if $0 < t \le \varepsilon$ from (φ1) we have $\varphi(t) < t \le \varepsilon$.

In this paper we revisit the notion of Geraghty contraction and propose a concept of φ-Geraghty contraction by inspired by the results of Suzuki [13]. Moreover, we observe a unique fixed point for such contractions. We also consider an example to indicate the validity of our results.

2 Main Results

We shall start this section by introducing the notion of φ-Geraghty contraction which is contraction by using the auxiliary functions defined in the first section.

Definition 2.1. Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a function and $\beta \in \mathcal{F}$. A self-mapping T on a complete metric space (X, d) is called φ-Geraghty contraction if it satisfies the following conditions:

(φ1) $\varphi(t) < t$ for any $t \in (0, \infty)$.

(φ2) For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon < t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon.$$

(φ3) $d(Tx, Ty) \leq \beta(d(x, y))(\varphi \circ d(x, y))$.

In what follows, we shall state and prove our first main result.

Theorem 2.2. *Let (X, d) be a complete metric space. If a self-mapping $T : X \to X$ forms a φ-Geraghty contraction, then T has a unique fixed point u. Moreover $\{T^n x\}$ converges to u for all $x \in X$.*

Proof. Let $x_0 \in X$. We shall build an iterative sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. As a first step, we shall show that the adjacent terms of the sequence $\{x_n\} \subset X$ should be distinct for a meaningful proof. Suppose on the contrary, that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. In this case, the point x_{n_0} forms a fixed point of T that completes the proof. From now on, we suppose that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Consequently, we have $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, from ($\varphi$3) and ($\varphi$1) we conclude that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \beta(d(x_{n-1}, x_n)\varphi\left(d(x_{n-1}, x_n)\right) \\ &< \varphi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Hence, the non-negative sequence $\{d(x_{n-1}, x_n)\}$ is non-increasing in $\mathbb{R}_+$. Accordingly, it is convergent to some non-negative real number ℓ. We assert that $\ell = 0$. We shall prove our assertion by the method of "*reductio to absurdum*". So, we suppose on the contrary, that $\ell > 0$. Hence, we have

$$0 < \ell < d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Set $\varepsilon = \ell > 0$. From ($\varphi 2'$) there exists $\delta > 0$ such that

$$t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon.$$

On the other hand, from the definition of ε we can choose $n_0 \in \mathbb{N}$ such that

$$\varepsilon < d(x_{n_0}, x_{n_0+1}) < \varepsilon + \delta$$

and taking into account the property ($\varphi 2$) we have

$$\begin{aligned}\varepsilon &< d(x_{n_0+2}, x_{n_0+3}) < d(x_{n_0+1}, x_{n_0+2}) = d(Tx_{n_0}, Tx_{n_0+1}) \\ &\leq \beta(d(x_{n_0}, x_{n_0+1})\varphi\left(d(x_{n_0}, x_{n_0+1})\right)) \\ &< \varphi\left(d(x_{n_0}, x_{n_0+1})\right)) \leq \varepsilon,\end{aligned}$$

which is a contradiction. Hence,

$$\ell = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.1}$$

As a next step, we shall indicate that the sequence $\{x_n\}$ is Cauchy. Fix $\varepsilon_1 > 0$. Then, by the hypothesis, there exists a $\delta_1 > 0$ such that

$$t < \varepsilon_1 + \delta_1 \text{ implies } \varphi(t) \leq \varepsilon_1. \tag{2.2}$$

Without loss of generality, we assume $\delta_1 < \varepsilon_1$. Due to (2.1), there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \delta_1, \text{ for all } n \geq N. \tag{2.3}$$

We will show that for any fixed $k \geq N$,

$$d(x_k, x_{k+l}) < \varepsilon_1 + \delta_1 \tag{2.4}$$

for all $l \in \mathbb{N}$. The inequality trivially holds for $l = 1$ by (2.3). We assume that the condition (2.4) is satisfied for some $j \in \mathbb{N}$. We shall show that it holds for $l = j + 1$. From (2.2), we get

$$\begin{aligned}d(x_k, x_{k+j+1}) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+j+1}) \\ &= d(x_k, x_{k+1}) + d(Tx_k, Tx_{k+j}) \\ &\leq d(x_k, x_{k+1}) + \beta(d(x_k, x_{k+j}))\varphi\left(d(x_k, x_{k+j})\right) \\ &< \varepsilon_1 + \delta_1.\end{aligned}$$

Consequently, (2.4) holds for $l = j + 1$. Hence we derive that

$$d(x_k, x_{k+l}) < \varepsilon_1 + \delta_1 \text{ for all } k \geq N \text{ and } l \geq 1.$$

Since ε_1 is arbitrary, we conclude that

$$\lim_{n\to\infty} d(x_n, x_m) = 0.$$

Thus the sequence $\{x_n\}$ is Cauchy. Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Arguing by contradiction, we assume that $Tu \neq u$, so there exists $r > 0$ such that $d(u, Tu) = r$. Since $\{x_n\}$ converges at u, we can choose $n_0 \in \mathbb{N}$ such that $d(x_l, u) < \frac{r}{2}$ for all $l \geq n_0$. Then, from ($\varphi 3$), we get that

$$\begin{aligned}r = d(u, Tu) &\leq d(u, x_{l+1}) + d(x_{l+1}, Tu) = d(u, x_{l+1}) + d(Tx_l, Tu) \\ &\leq d(u, x_{l+1}) + \beta(d(u, x_l)\varphi\left(d(u, x_l)\right)) \\ &< d(u, x_{l+1}) + \varphi\left(d(u, x_l)\right)) \\ &< d(u, x_{l+1}) + d(u, x_l) < r,\end{aligned}$$

which is a contradiction. Therefore we have shown that u is a fixed point of T. Suppose that $u \neq v$ are two fixed points of T. We then have

$$d(u,v) = d(Tu,Tv) \leq \beta(d(u,v))\varphi(d(u,v)) < \varphi(d(u,v)) < d(u,v)$$

which implies that $u = v$. □

Example 2.3. Let $X = [0, \frac{3}{4}] \cup \{1\}$ equipped with a standard metric $d(x,y) = |x-y|$ for all $x, y \in X$. We define a self-mapping $T : X \to X$ as follows

$$Tx = \begin{cases} \frac{x}{8}, & \text{if } x \in [0, \frac{3}{4}] \\ \frac{1}{16}, & \text{if } x = 1. \end{cases}$$

Moreover, the auxiliary functions β and φ are defined as follows:

$$\beta(t) = \begin{cases} \frac{1}{4}, & \text{if } t \in [0,2), \\ \frac{1}{2}, & \text{if } t \geq 2, \end{cases} \quad \text{and} \quad \varphi(t) = \begin{cases} \frac{t}{2}, & \text{if } t \in [0,2), \\ \frac{3}{2}, & \text{if } t = 2, \\ \frac{1}{t} + 1, & \text{if } t \in (2,\infty). \end{cases} \tag{2.5}$$

If $x, y \in \left[0, \frac{3}{4}\right]$, then

$$d(Tx,Ty) = \frac{|x-y|}{8} \leq \frac{1}{8}|x-y| = \beta(d(x,y))\varphi(d(x,y)).$$

If If $x \in \left[0, \frac{3}{4}\right]$ and $y = 1$, then ($\varphi 3$) becomes

$$d(Tx,Ty) = \frac{|2x-1|}{16} \leq \frac{1}{8}|1-x| = \beta(d(x,y))\varphi(d(x,y)),$$

or, equivalent

$$|2x-1| \leq 2-2x.$$

which implies that $x \leq \frac{3}{4}$.

Therefore, for any $x, y \in X$ all the conditions of Theorem 2.2 are satisfied. Moreover, $u = 0$ is a fixed point of T.

In what follows we introduce the family of refined Geraghty functions as follows: Let $\mathcal{F}'$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition:

$$\limsup_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0. \tag{2.6}$$

On the account of the very well-known Ćirić theorem, we extend the notion of φ-Geraghty contraction in the next definition.

Definition 2.4. Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a function and $\beta \in \mathcal{F}'$. A self-mapping T on a complete metric space (X,d) is called Ćirić type φ-Geraghty contraction if it satisfies the following conditions:

($\varphi 0$) φ is upper semicontinuous.

($\varphi 1$) $\varphi(t) < t$ for any $t \in (0,\infty)$.

($\varphi 2$) For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon < t < \varepsilon + \delta \text{ implies } \varphi(t) \le \varepsilon.$$

($\varphi 3'$) $d(Tx,Ty) \le \beta(L(x,y))(\varphi \circ L(x,y))$,

where

$$L(x,y) = \max\left\{d(x,y), \frac{d(x,Ty)+d(Tx,y)}{2}, d(x,Tx), d(y,Ty)\right\}. \tag{2.7}$$

Theorem 2.5. *Let (X,d) be a complete metric space and $\beta \in \mathcal{F}'$. If a self-mapping $T : X \to X$ forms a Ćirić type φ-Geraghty contraction, then T has a fixed point u. Moreover, $\{T^n x\}$ converges to u for any initial value $x \in X$.*

Proof. We shall use the same steps in the proof of Theorem 2.2. We begin by constructing an iterative sequence $\{x_n\}$ for an arbitrary initial value $x \in X$, as follows:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.8}$$

Regarding the discussion on the adjacent terms of the iterative sequence in the proof of Theorem 2.2, we can suppose that

$$x_n \ne x_{n-1} \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.9}$$

Thus, we have $d(x_n, x_{n+1}) > 0$ and consequently $L(x_n, x_{n+1}) > 0$. By ($\varphi 3'$) together with ($\varphi 1$) and the definition of function β, we have

$$\begin{aligned} d(x_{n+1},x_{n+2}) &= d(Tx_n, Tx_{n+1}) \le \beta(L(x_n,x_{n+1}))\varphi\left(L(x_n,x_{n+1})\right) \\ &< \varphi(L(x_n,x_{n+1})) < L(x_n,x_{n+1}), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} L(x_n,x_{n+1}) &= \max\left\{d(x_n,Tx_n), \frac{d(x_n,Tx_{n+1})+d(Tx_n,Tx_n)}{2},\right. \\ &\qquad \left. d(Tx_n,Tx_{n+1})\right\} \\ &= \max\left\{d(x_n,Tx_n), \frac{d(x_n,Tx_{n+1})}{2}, d(Tx_n,Tx_{n+1})\right\} \\ &= \max\left\{d(x_n,x_{n+1}), \frac{d(x_n,x_{n+2})}{2}, d(x_{n+1},x_{n+2})\right\}. \end{aligned}$$

Taking the triangle inequality into account, we find that

$$\frac{d(x_n,x_{n+2})}{2} \le \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2} \le \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\}.$$

Consequently, we derive that

$$L(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$$

and (2.10) becomes

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \tag{2.11}$$

It is clear that the case where $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$ is impossible due to (2.11). Indeed, by (2.11), this case yields

$$d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2}),$$

a contradiction. Accordingly, we have

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1})$$

and by (2.11) we get

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < L(x_n, x_{n+1}) = d(x_n, x_{n+1}), \tag{2.12}$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence the non-negative real number sequence $\{d(x_n, x_{n+1})\}$ is non-increasing. Consequently, this sequence converges to some $\varepsilon \geq 0$.

We claim that $\varepsilon = 0$. Firstly we note $\varepsilon < d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Arguing by contradiction, we assume $\varepsilon > 0$. Then, by $(\varphi 2')$ from Remark 1.3, there exists $\delta > 0$ such that

$$t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon.$$

On the other hand, for sufficiently large $N \in \mathbb{N}$, we have

$$0 < \varepsilon < L(x_N, x_{N+1}) = d(x_N, x_{N+1}) < \varepsilon + \delta.$$

Using (2.12) and $(\varphi 2')$ we get

$$\begin{aligned} 0 \quad &< \varepsilon \leq d(x_{N+1}, x_{N+2}) < d(x_{N+2}, x_{N+3}) \leq \beta(L(x_N, x_{N+1}))\varphi\left(L(x_N, x_{N+1})\right) \\ &< \varphi\left(L(x_N, x_{N+1})\right) \leq \varepsilon, \end{aligned}$$

a contradiction. Thus, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.13}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon_1 > 0$ fixed. Then, there exists $\delta_1 > 0$ which satisfies the following:

$$t < \varepsilon_1 + 2\delta_1 \Longrightarrow \varphi(t) \leq \varepsilon_1. \tag{2.14}$$

From (2.13), we can choose $k \in \mathbb{N}$ large enough to satisfy $d(x_k, x_{k+1}) < \delta_1(\varepsilon) = \delta_1$. We will show by induction that

$$d(x_k, x_{k+l}) < \varepsilon_1 + \delta_1, \tag{2.15}$$

for all $k \in \mathbb{N}$. (Without loss of generality, we assume that $\delta_1 = \delta_1(\varepsilon) < \varepsilon$.) We have already proved for $k = 1$, so we suppose the condition (2.15) is satisfied for some $j \in \mathbb{N}$. For $l = j + 1$, we get

$$\begin{aligned} L(x_k, x_{k+j}) &= \max\left\{d(x_k, x_{k+j}), d(x_k, x_{k+1}), d(x_{k+j}, x_{k+j+1}), \frac{d(x_k,x_{k+j+1})+d(x_{k+j},x_{k+1})}{2}\right\} \\ &\le \max\left\{d(x_k, x_{k+j}), d(x_k, x_{k+1}), d(x_{k+j}, x_{k+j+1}), \frac{d(x_k,x_{k+j})+d(x_{k+j},x_{k+j+1})+d(x_{k+j},x_k)+d(x_k,x_{k+1})}{2}\right\} \\ &< \max\left\{\varepsilon_1, \delta_1, \delta_1, \frac{2\varepsilon_1+2\delta_1+\delta_1+\delta_1}{2}\right\} = \varepsilon_1 + 2\delta_1. \end{aligned} \tag{2.16}$$

Then, by $(\varphi 3')$ and (2.14) we obtain

$$\begin{aligned} d(x_k, x_{k+j+1}) &\le d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+j+1}) = d(x_k, x_{k+1}) + d(Tx_k, Tx_{k+j}) \\ &\le d(x_k, x_{k+1}) + \beta(L(x_k, x_{k+j}))\varphi(L(x_k, x_{k+j})) < \varepsilon_1 + \delta_1. \end{aligned} \tag{2.17}$$

Consequently, (2.15) holds for $l = j+1$. Hence, $d(x_k, x_{k+l})) < \varepsilon_1$ for all $k \in \mathbb{N}$ and $l \le 1$, which means $\lim_{n\to\infty} \sup_{m>n} d(x_n, x_m) = 0$. Hence the sequence $\{x_n\}$ is Cauchy. Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$ when $n \to \infty$.

As a next step, we shall show that $Tu = u$. Suppose on the contrary, that there exists $r > 0$ such that $r := d(u, Tu) > 0$. Note that, due to the fact that the sequence $\{x_n\}$ is convergent to u, we can choose $l \in \mathbb{N}$ such that $d(u, x_n) < \frac{r}{2}$, for all $n \ge l$. So, we have the following estimation for $n \ge l$:

$$\begin{aligned} L(x_n, u) &= \max\left\{d(x_n, u), \frac{d(x_n,Tu)+d(Tx_n,u)}{2}, d(x_n, Tx_n), d(u, Tu)\right\} \\ &\le \max\left\{d(x_n, u), \frac{d(x_n,u)+d(u,Tu)+d(x_{n+1},u)}{2}, d(x_n, x_{n+1}), d(u, Tu)\right\} \\ &< \left\{\frac{r}{2}, \frac{\frac{r}{2}+r+\frac{r}{2}}{2}, \frac{r}{2}, r\right\} \\ &= r. \end{aligned}$$

It yields that

$$\limsup_{n\to\infty} L(x_n, u) = r. \tag{2.18}$$

By the triangle inequality together with $(\varphi 3)$ we derive that

$$0 < r < d(u,Tu) \le d(u,x_{n+1}) + d(Tx_n,Tu) \le d(u,x_{n+1}) + \beta(L(x_n,u))\varphi\left(L(x_n,u)\right).$$

Letting $n \to \infty$ in the previous inequality, together with $(\varphi 0)$ and $(\varphi 1)$ we get

$$\begin{aligned} 0 < \quad r = d(u, Tu) &\le \limsup_{n\to\infty} \left[d(u, x_{n+1}) + \beta(L(x_n, u))\varphi\left(L(x_n, u)\right)\right] \\ &= \limsup_{n\to\infty} \beta(L(x_n, u)) \limsup_{n\to\infty} \varphi(L(x_n, u)) \\ &\le \limsup_{n\to\infty} \beta(L(x_n, u))\varphi(r) \\ &< \varphi(r) < r. \end{aligned}$$

Thus, $\limsup_{n\to\infty}(\beta(L(x_n,u)) = 1$. Since $\beta \in \mathcal{F}'$ we have $\limsup_{n\to\infty} L(x_n,u) = 0$. Accordingly we have $d(u,Tu) = r = 0$, that is, u is a fixed point of T.

As a last step, we indicate that the limit point u of the iterative sequence $\{x_n\}$ is unique. Suppose on the contrary, that v is another fixed point of T, with $u \neq v$. It is clear that $L(u,v) = d(u,v)$. Thus, we have

$$0 < d(u,v) = d(Tu,Tv) \leq \beta(L(u,v))\varphi\left(L(u,v)\right) = \beta(d(u,v))\varphi(d(u,v)) < d(u,v),$$

a contradiction. □

Example 2.6. Let $X = \{a_1,a_2,a_3,a_4\}$ and $d : X \times X \to [0,\infty)$ defined by:

$$d(a_1,a_2) = d(a_2,a_1) = 1,\ d(a_3,a_4) = d(a_4,a_3) = 10,$$

$$d(a_1,a_4) = d(a_4,a_1) = d(a_2,a_4) = d(a_4,a_2) = 6,$$

$$d(a_1,a_3) = d(a_3,a_1) = d(a_2,a_3) = d(a_3,a_2) = 8,$$

$$d(a_i,a_i) = 0, \text{ for any } i = 1,2,3,4.$$

It is easy to see that the pair (X,d) forms a metric space. Assume $T : X \to X$ and $\varphi : [0,\infty) \to [0,\infty)$ be defined by

$$Ta_1 = Ta_2 = a_1, Ta_3 = Ta_4 = a_2$$

and

$$\varphi(t) = \begin{cases} \frac{t}{5}, & \text{if } t \in [0,4) \\ \frac{1}{t} + 3, & \text{if } t \in [4,\infty). \end{cases}$$

Let $\beta : [0,\infty) \to [0,1)$ be defined by $\beta(t) = \frac{1}{1+\frac{t}{4}}$. On the other hand, because $d(Ta_1,Ta_2) = d(Ta_3,Ta_4) = 0$ and $(\varphi 3')$ is obviously satisfied, relevant for our study only is only the set $\{(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4)\}$. For this reason, we consider the following cases:

Case(i) If $x = a_1$, $y = a_3$ then

$$d(a_1,Ta_1) = d(a_1,a_1) = 0, d(a_3,Ta_3) = d(a_3,a_2) = 8, d(Ta_1,Ta_3) = d(a_1,a_2) = 1$$
$$d(a_1,Ta_3) = d(a_1,a_2) = 1, d(a_3,Ta_1) = d(a_3,a_1) = 8, d(a_1,a_3) = 8$$

and

$$L(a_1,a_3) = \max\left\{8, \frac{1+8}{2}, 0, 8\right\} = 8, \beta(L(a_1,a_3)) = \beta(8) = \frac{1}{3},$$

$$\varphi(L(a_1,a_3)) = \varphi(8) = \frac{25}{8}.$$

In this case,

$$d(Ta_1,Ta_3) = 1 \leq \frac{25}{24} = \frac{1}{3} \cdot \frac{25}{8} = \beta(L(a_1,a_3))\varphi(L(a_1,a_3)).$$

Case(ii) If $x = a_1$, $y = a_4$ then

$$d(a_1, Ta_1) = d(a_1, a_1) = 0, d(a_4, Ta_4) = d(a_4, a_2) = 6, d(Ta_1, Ta_4) = d(a_1, a_2) = 1$$
$$d(a_1, Ta_4) = d(a_1, a_2) = 1, d(a_4, Ta_1) = d(a_4, a_1) = 6$$

and

$$L(a_1, a_4) = \max\left\{1, \frac{1+6}{2}, 0, 6\right\} = 6,\ \beta(L(a_1, a_4)) = \beta(6) = \frac{2}{5},$$
$$\varphi(L(a_1, a_4)) = \varphi(6) = \frac{19}{6}.$$

In this case,

$$d(Ta_1, Ta_4) = 1 \leq \frac{38}{30} = \frac{2}{5} \cdot \frac{19}{6} = \beta(L(a_1, a_4))\varphi(L(a_1, a_4)).$$

Case(iii) If $x = a_2$, $y = a_3$ then

$$d(a_2, Ta_2) = d(a_2, a_1) = 1, d(a_3, Ta_3) = d(a_3, a_2) = 8, d(Ta_2, Ta_3) = d(a_1, a_2) = 1$$
$$d(a_2, Ta_3) = d(a_2, a_2) = 0, d(a_3, Ta_2) = d(a_3, a_1) = 8, d(a_2, a_3) = 8.$$

and

$$L(a_2, a_3) = \max\left\{8, \frac{0+8}{2}, 1, 8\right\} = 8, \beta(L(a_2, a_3)) = \beta(8) = \frac{1}{3},$$
$$\varphi(L(a_2, a_3)) = \varphi(8) = \frac{25}{8}.$$

In this case,

$$d(Ta_2, Ta_3) = 1 \leq \frac{25}{24} = \frac{1}{3} \cdot \frac{25}{8} = \beta(L(a_2, a_3))\varphi(L(a_2, a_3)).$$

Case(iv) If $x = a_2$, $y = a_4$ then

$$d(a_2, Ta_2) = d(a_2, a_1) = 1, d(a_4, Ta_4) = d(a_4, a_2) = 6, d(Ta_2, Ta_4) = d(a_1, a_2) = 1$$
$$d(a_2, Ta_4) = d(a_2, a_2) = 0, d(a_4, Ta_2) = d(a_4, a_1) = 6, d(a_2, a_4) = 6$$

and

$$L(a_2, a_4) = \max\left\{6, \frac{0+6}{2}, 1, 6\right\} = 6, \beta(L(a_2, a_4)) = \beta(6) = \frac{2}{5},$$
$$\varphi(L(a_2, a_4)) = \varphi(6) = \frac{19}{6}.$$

In this case,

$$d(Ta_2, Ta_4) = 1 \leq \frac{38}{30} = \frac{2}{5} \cdot \frac{19}{6} = \beta(L(a_2, a_4))\varphi(L(a_2, a_4)).$$

Thus, all the conditions of Theorem 2.5 are satisfied. Moreover, $u = a_1$ is a fixed point of T.

Competing Interests : The authors declare that they have no competing interests.

Authors Contributions : All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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(Received 15 May 2018)
(Accepted 13 September 2018)