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Finite-Time Stabilization of Linear Systems with Time-varying Delays using New Integral Inequalities

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Abstract : In this research we propose stability conditions for guaranteeing finite-time stabilization of linear systems with time-varying delay. Based on Lyapunov theory, improved finite-time stability and stabilization criteria of the linear systems are formulated in the form of linear matrix inequalities. To obtain the improved stability criteria, we first propose two new inequalities in the forms of free-matrix based inequality for bounding the integral of the form $\int_{t-d_2}^{t-d_1} e^{\alpha(t-s)}\dot{x}^T(s)R\dot{x}(s)ds$ which is occurred in the derivative of Lyapunov-Krasovskii functional. By desiring a proper state-feedback controller, the non-linear terms occuring in the formulation can be managed without defining new variables. At the end, two numerical examples are given to show that the new criteria are praticable and can be applied to the case of continuous but not differentiable delay function.

Keywords : finite-time; stability; stabilization; Lyapunov-Krasovskii.
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1 Introduction

In the fields of sciences and engineering such as atmospheric flow, forest insects are often described in the forms of differential equations [1–3]. To understand the behavior or physical properties of these problems, researchers often assume some assumptions to simplify the problems; for example, assuming that the atmospheric flows having small amplitude and linear so that flows can be modeled by Boussinesq approximation [1, 2, 4]. Along with mathematical models, finite-time synchronization is of interests rather than the one over infinite time. The dynamic manners, such as the existence of the equilibrium points, the stability and boundedness of results, have drawn the wide attention and attracted research opportunities of many scholars.

Some systems may allow state variables to oscillate and finally reach their equilibrium states. In many practical circumstants, however, system may need to attain its state values to be within some certain threshold for a short or finite period of time. This situation is commonly known as finite-time stability (FTS) as proposed by Dorato [5]. This FTS concept can be considered as one of the practical situations for real world problems such as in environmental, industrial and sciences. In many system, a desired controller is required to maintain the values of state variables within their required values. This concept is called finite-time stabilization (FTU). Thus, many researchers have been paid attention to the FTS and FTU of the dynamical system.

During the last few decades, researchers have been proposing criteria that guarantee FTS of various systems by finding the smallest upper bound of the norm square of state variables or finding maximum time that guarantees state variables to be within the given bounds for a certain period of time. Some examples of FTS of linear system with constant delay are studied in [6–12]; FTS of linear system with time-varying delays in [13–16]; and FTS on other systems in [5, 9, 17–21].

The past studies of FTS on linear system with time-varying delay are mostly limited to the delay differentiable functions which lead to conservative conditions. Moreover, some FTU criteria based on Lyapunov-Krasovski functional (LKF) may need to define new variables for taking care of nonlinear terms occurring in their formulations that lead to some conservativeness (see [10, 12, 13, 16, 22]).

Another source that can cause the stability of systems is delay effect. In various scientific topics such as electrical engineering, neural networks, and chemical systems, time delay which is well observed that will deteriorate the systems performance and even make the system unstable are frequently confronted. The artificial neural network which has powerful scientific application background and great research capability is a very active framework for algorithm. However, it unavoidably causes to the emergence of diverse time-delays in the procedure of using large-scale integrated circuits to form a neural network. The synchronous discharge of neurons, in actuality, is a universal phenomenon. For example, in various synchronizations: local brain regions in patients with Parkinsons disease, the visual cortex of conscious monkeys, the hippocampus and the cerebral cortex during the maze task, neurons in circadian clock, etc. Therefore, many reseachers have paid attention to stabilize the systems using various methods. Lyapunov theory is one of the common method to investigate the stability of dynamical systems dealing with delays.

In formulating the stability criterion of the systems with time delay via Lyapunov theory, ones must deal with derivative of the proposed LKF. Ones can improve the stability condition by applying a better bound to the derivative of the LKF. Examples of well-known inequalities used in the control theory are free-matrix-based inequality described in [23, 24] and Jensen's inequality [25]. During the past decade, Seuret and Gouaisbaut have developed an improved inequality commonly known as Wirting-based integral inequality for bounding $\int_a^b \dot{x}^T(s)R\dot{x}(s)\,ds$ (see [26–28]). Lately, Stojanovic [29] proposed an integral inequality for bounding an integral inequality with exponential function.

As mentioned above, FTS is one of the important topics that should have been further investigated. Thus, in this study, we will formulate less conservative stability criteria for guaranteeing FTS and FTU of the linear systems with interval time-varying delay that does not need to be differentiable. To do so, we first develop two new inequalities in the form of free-matrix based for bounding the integral of the form $-\int_{t-d_2}^{t-d_1} e^{\alpha(t-s)}\dot{x}^T(s)R\dot{x}(s)\,ds$. By choosing an appropriated Lyapunov-Krasovski functional and desiring a proper state-feedback controller $u(t)$ to control the system, our criteria are derived without defining new variables for non-linear terms that can lead to conservative condition.

This article is organized as follow: the considered system and important lemmas, propositions and definition are introduced in section 2, follow by the derivation of FTS and FTU criteria in section 3. Section 4 is devoted to show the effectiveness of the proposed criteria. The last section is concluded the work.

Notations: The following notations will be used throughout this article. $\mathbb{R}^n$ is the n-dimensional space with the scalar product x^Ty; $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real-valued matrices; A^T denotes the transpose of the matrix A; $\lambda(A)$ are eigenvalues of A; $\lambda_{\max}(A)$ $(\lambda_{\min}(A))$ maximum (minimum) real part of $\lambda(A)$; $x_t := \{x(t+s) : s \in [-\tau_M, 0]\}$; $||x_t|| := \sup_{s\in[-\tau_M,0]}\{||x(t+s)||, ||\dot{x}(t+s)||\}$; $A > 0$ (< 0) means A is positive (negative) definite; $A \geq 0$ (≤ 0) means A is positive (negative) semi-definite; lower entries of any symmetric matrix are represented by $*$.

2 Preliminaries

Consider the following linear time-varying delay system:

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t)) + Bu(t) \tag{2.1}$$

where $t > 0, x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ is the state vector of the system. $u(t) \in \mathbb{R}^m$ is the control input. $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known constant matrices. The time-varying delay $\tau(t)$ is a continuous function satisfying $0 \leq \tau_m \leq \tau(t) \leq \tau_M, \tau_m \neq \tau_M$. An initial condition $x(t) = \phi(t)$ is defined to be a differentiable vector-valued function on $[-\tau_M, 0]$ with its norm defined by $\|\phi(t)\| := \sup_{\tau_m \le t \le \tau_M} \left\{\|\phi(t)\|, \|\dot{\phi}(t)\|\right\}$. Here we use a state-feedback control law in the form of $u(t) = B^T Kx(t)$, where K is a designed parameter that will be determined later.

To formulate the finite-time stabilization, we first introduce useful lemmas and definitions as following.

Lemma 2.1. [30] *For any symmetric matrix $M \in \mathbb{R}^{n\times n}$ and nonsingular matrix $A \in \mathbb{R}^{n\times n}$. The matrix M is positive definite if and only if $A^T MA$ is positive definite. Similarly, M is negative definite matrix if and only if $A^T MA$ is negative definite.*

Proposition 2.2. *For any symmetric matrix $M \in \mathbb{R}^{n\times n}$ and any real matrix $A \in \mathbb{R}^{n\times n}$. If the matrix M is positive definite then $A^T MA$ is positive semi-definite.*

Proof. Let M is positive definite then $v^T Mv > 0$, $\forall v \neq 0$. If $Av \neq 0$ then $v^T A^T MAv = (Av)^T MAv > 0$. If $Av = 0$ then $v^T A^T MAv = (Av)^T MAv = 0$. As a results, $v^T A^T MAv \ge 0$, $\forall v \neq 0$. Thus, $A^T MA \ge 0$. □

Proposition 2.3. *For any real symmetric matrices A, $\Sigma \in \mathbb{R}^{n\times n}$ with $\Sigma > 0$, the following inequality holds:*

$$-A^T \Sigma^{-1} A \le -2A + \Sigma.$$

Proof. Since $\Sigma > 0$, this implies that $\Sigma^{-1} > 0$. From Proposition 2.2, $\Sigma^{-1} > 0$ implies $Y^T \Sigma^{-1} Y \ge 0$, for any real matrix Y. Let $Y = A - \Sigma$. We obtain $-A^T \Sigma^{-1} A \le -2A + \Sigma$. So, the proof is complete. □

Proposition 2.4. *For any symmetric positive definite matrix $R \in \mathbb{R}^{n\times n}$ and scalars $\alpha > 0, d_1, d_2 \ge 0$ with $d = d_2 - d_1 > 0$, the following inequality holds:*

$$-\int_{t-d_2}^{t-d_1} e^{\alpha(t-s)} \dot{x}^T(s) R\dot{x}(s)\, ds \le \begin{bmatrix} x^T(t-d_1) \\ x^T(t-d_2) \end{bmatrix}^T \begin{bmatrix} -r_1 R & r_2 R \\ r_2 R & -r_3 R \end{bmatrix} \begin{bmatrix} x(t-d_1) \\ x(t-d_2) \end{bmatrix},$$

where $\varepsilon = \dfrac{e^{-\alpha d_1} - e^{-\alpha d_2}}{\alpha}$, $r_1 = 2\left(\dfrac{\alpha}{2} + \dfrac{1}{d}\right) e^{\alpha d_1} - \varepsilon \left(\dfrac{\alpha}{2} + \dfrac{1}{d}\right)^2 e^{2\alpha d_1}$,

$$r_2 = \left(\frac{\alpha}{2} + \frac{1}{d}\right) e^{\alpha d_1} - \left(\frac{\alpha}{2} - \frac{1}{d}\right) e^{\alpha d_2} + \varepsilon \left(\frac{\alpha^2}{4} - \frac{1}{d^2}\right) e^{\alpha(d_1+d_2)},$$

$$r_3 = -2\left(\frac{\alpha}{2} - \frac{1}{d}\right) e^{\alpha d_2} - \varepsilon \left(\frac{\alpha}{2} - \frac{1}{d}\right)^2 e^{2\alpha d_2}.$$

Proof. Define the function z as

$$z(s) = e^{\alpha(t-s)}\dot{x}(s) - e^{\alpha d_1} x(t-d_1)\left(\frac{\alpha}{2} + \frac{1}{d}\right) - e^{\alpha d_2} x(t-d_2)\left(\frac{\alpha}{2} - \frac{1}{d}\right).$$

Since $R > 0$, we have $0 \leq \displaystyle\int_{t-d_2}^{t-d_1} e^{-\alpha(t-s)} z^T(s) R z(s)\, ds$. Substituting $z(s)$ into the integral, we obtain

$$\begin{aligned}
0 \leq & \int_{t-d_2}^{t-d_1} e^{\alpha(t-s)} \dot{x}^T(s) R \dot{x}(s)\, ds \\
& -2\left(\int_{t-d_2}^{t-d_1} \dot{x}(s)\, ds\right)^T R\left(e^{\alpha d_1} x(t-d_1)\left(\frac{\alpha}{2}+\frac{1}{d}\right)+e^{\alpha d_2} x(t-d_2)\left(\frac{\alpha}{2}-\frac{1}{d}\right)\right) \\
& +\left(\int_{t-d_2}^{t-d_1} e^{-\alpha(t-s)}\, ds\right)\left(e^{\alpha d_1} x(t-d_1)\left(\frac{\alpha}{2}+\frac{1}{d}\right)+e^{\alpha d_2} x(t-d_2)\left(\frac{\alpha}{2}-\frac{1}{d}\right)\right)^T \\
& \quad \times R\left(e^{\alpha d_1} x(t-d_1)\left(\frac{\alpha}{2}+\frac{1}{d}\right)+e^{\alpha d_2} x(t-d_2)\left(\frac{\alpha}{2}-\frac{1}{d}\right)\right).
\end{aligned}$$

Because $\int_{t-d_2}^{t-d_1} \dot{x}(s)\, ds = x(t-d_1) - x(t-d_2)$ and $\int_{t-d_2}^{t-d_1} e^{-\alpha(t-s)}\, ds = (e^{-\alpha d_1} - e^{-\alpha d_2})/\alpha$, substituting these terms into the above inequality then rearranging the inequality, we obtain

$$-\int_{t-d_2}^{t-d_1} e^{\alpha(t-s)} \dot{x}^T(s) R \dot{x}(s)\, ds \quad \leq \begin{bmatrix} x^T(t-d_1) \\ x^T(t-d_2) \end{bmatrix}^T \begin{bmatrix} -r_1 R & r_2 R \\ r_2 R & -r_3 R \end{bmatrix} \begin{bmatrix} x^T(t-d_1) \\ x^T(t-d_2) \end{bmatrix}.$$

The proof is completed. □

Proposition 2.5. *For any symmetric positive definite matrix $R \in \mathbb{R}^{n\times n}$ and scalars $\alpha > 0, \tau_m, \tau_M, d_1, d_2$ with $0 \leq \tau_m \leq d_1 \leq d_2 \leq \tau_M$ and $\tau_{12} = \tau_M - \tau_m > 0$, the following inequality holds:*

$$-\int_{t-d_2}^{t-d_1} e^{\alpha(t-s)} \dot{x}^T(s) R \dot{x}(s)\, ds \leq \begin{bmatrix} x^T(t-d_1) \\ x^T(t-d_2) \end{bmatrix}^T \begin{bmatrix} -u_1 R & u_2 R \\ u_2 R & -u_3 R \end{bmatrix} \begin{bmatrix} x(t-d_1) \\ x(t-d_2) \end{bmatrix},$$

where

$$\begin{aligned}
u_1 =& 2\left(\frac{\alpha}{2}+\frac{1}{\tau_{12}}\right) e^{\alpha\tau_m} - \varepsilon_2\left(\frac{\alpha}{2}+\frac{1}{\tau_{12}}\right)^2 e^{2\alpha\tau_m}, \\
u_2 =& \left(\frac{\alpha}{2}+\frac{1}{\tau_{12}}\right) e^{\alpha\tau_m} - \left(\frac{\alpha}{2}-\frac{1}{\tau_{12}}\right) e^{\alpha\tau_M} + \varepsilon_2\left(\frac{\alpha^2}{4}-\frac{1}{\tau_{12}^2}\right) e^{\alpha(\tau_m+\tau_M)}, \\
u_3 =& -2\left(\frac{\alpha}{2}-\frac{1}{\tau_{12}}\right) e^{\alpha\tau_M} - \varepsilon_2\left(\frac{\alpha}{2}-\frac{1}{\tau_{12}}\right)^2 e^{2\alpha\tau_M}, \\
\varepsilon_2 =& \frac{e^{-\alpha\tau_m} - e^{-\alpha\tau_M}}{\alpha}.
\end{aligned}$$

Proof. Because $0 \leq \tau_m \leq d_1 \leq d_2 \leq \tau_M$, we obtain

$$\int_{t-d_2}^{t-d_1} e^{-\alpha(t-s)} ds \quad \leq \int_{t-\tau_M}^{t-\tau_m} e^{-\alpha(t-s)} ds \quad = \frac{e^{-\alpha\tau_m} - e^{-\alpha\tau_M}}{\alpha}. \tag{2.2}$$

By defining function z as

$$z(s) = e^{\alpha(t-s)}\dot{x}(s) - e^{\alpha\tau_m}x(t-d_1)\left(\frac{\alpha}{2}+\frac{1}{\tau_{12}}\right) - e^{\alpha\tau_M}x(t-d_2)\left(\frac{\alpha}{2}-\frac{1}{\tau_{12}}\right).$$

The proof is similar to the one in Proposition 2.4 but applying inequality (2.2) for bounding the integral $\int_{t-d_2}^{t-d_1} e^{-\alpha(t-s)}ds$ instead of $\int_{t-d_2}^{t-d_1} e^{-\alpha(t-s)}\,ds = (e^{-\alpha d_1} - e^{-\alpha d_2})/\alpha$. Thus, the proof is complete. □

Remark 2.6. Inequalities in Propositions 2.4 and 2.5 are similar except that the value of d_1, d_2, d are replaced by $\tau_m, \tau_M, \tau_{12}$; respectively. In this research, moreover, the inequality in Proposition 2.5 is suitable for applying to the integral with interval time delay such as $0 \le \tau_m \le \tau(t) \le \tau_M$.

Remark 2.7. One can notice that the well-known Jensen's inequality is a special case of our new inequalities when $\alpha = 0$.

Lemma 2.8. [12] *Given constant matrices, X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$, then $X + Z^TY^{-1}Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -Y & Z \\ Z^T & X \end{bmatrix} < 0.$$

Definition 2.9. [12] The system is said to be finite-time stability (FTS) with respect to (c_1, c_2, T), where $c_1, c_2 \ge 0$ if

$$\|\phi(t)\|^2 \le c_1, \ \forall t \in [-\tau_M, 0] \Longrightarrow \|x(t)\|^2 < c_2, \ \forall t \in [0, T]. \tag{2.3}$$

Remark 2.10. It is important to point out that finite-time stability is totally different from Lyapunov asymptotic stability, and they are independent of each other.

In the next section, we first formulate the finite-time stability condition for the linear systems without controller ($u(t) = 0$) follow by deriving the finite-time stabilization of the linear system (2.1). The use of the inequalities stated in Propositions 2.4 and 2.5 play important roles in obtaining less conservative FTS and finite-time stabilization conditions in this research as will be seen in the following theorem.

3 Main Results

To formulate the finite-time stability and stabilization criteria, we first define the following constants: $0 \leq \tau_m \leq \tau(t) \leq \tau_M, \tau_m \neq \tau_M, \tau_{12} = \tau_M - \tau_m$, and

$$\varepsilon_1 = \frac{1 - e^{-\alpha\tau_m}}{\alpha}, \quad \varepsilon_2 = \frac{e^{-\alpha\tau_m} - e^{-\alpha\tau_M}}{\alpha}, \tag{3.1}$$

$$r_1 = 2\left(\frac{\alpha}{2} + \frac{1}{\tau_m}\right) - \varepsilon_1\left(\frac{\alpha}{2} + \frac{1}{\tau_m}\right)^2, \tag{3.2}$$

$$r_2 = \left(\frac{\alpha}{2} + \frac{1}{\tau_m}\right) - \left(\frac{\alpha}{2} - \frac{1}{\tau_m}\right)e^{\alpha\tau_m} + \varepsilon_1\left(\frac{\alpha^2}{4} - \frac{1}{\tau_m^2}\right)e^{\alpha\tau_m}, \tag{3.3}$$

$$r_3 = -2\left(\frac{\alpha}{2} - \frac{1}{\tau_m}\right)e^{\alpha\tau_m} - \varepsilon_1\left(\frac{\alpha}{2} - \frac{1}{\tau_m}\right)^2 e^{2\alpha\tau_m}, \tag{3.4}$$

$$u_1 = 2\left(\frac{\alpha}{2} + \frac{1}{\tau_{12}}\right)e^{\alpha\tau_m} - \varepsilon_2\left(\frac{\alpha}{2} + \frac{1}{\tau_{12}}\right)^2 e^{2\alpha\tau_m}, \tag{3.5}$$

$$u_2 = \left(\frac{\alpha}{2} + \frac{1}{\tau_{12}}\right)e^{\alpha\tau_m} - \left(\frac{\alpha}{2} - \frac{1}{\tau_{12}}\right)e^{\alpha\tau_M} + \varepsilon_2\left(\frac{\alpha^2}{4} - \frac{1}{\tau_{12}^2}\right)e^{\alpha(\tau_m + \tau_M)}, \tag{3.6}$$

$$u_3 = -2\left(\frac{\alpha}{2} - \frac{1}{\tau_{12}}\right)e^{\alpha\tau_M} - \varepsilon_2\left(\frac{\alpha}{2} - \frac{1}{\tau_{12}}\right)^2 e^{2\alpha\tau_M}. \tag{3.7}$$

We begin the section by formulating the FTS condition for the linear system with time-varying delay (2.1) with $u(t) = 0$ as stated in the following theorem.

Theorem 3.1. *The linear system with time-varying delay as in Eq.* (2.1) *without controller* ($u(t) = 0$) *is finite-time stable with respect to* (c_1, c_2, T) *if there exist positive scalars* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha$, *and symmetric positive definite matrices* P, Q_1, Q_2, R, *such that the following inequalities hold:*

$$\Delta :\equiv \begin{bmatrix} -e^{-\alpha T}\frac{c_2}{c_1} & 1 & \sqrt{e^{\alpha\tau_m} - 1} & \sqrt{e^{\alpha\tau_M} - 1} & \sqrt{e^{\alpha\tau_M} - \alpha\tau_M - 1} \\ * & -\lambda_1 & 0 & 0 & 0 \\ * & * & -\lambda_2\alpha & 0 & 0 \\ * & * & * & -\lambda_3\alpha & 0 \\ * & * & * & * & -\lambda_4\alpha^2 \end{bmatrix} < 0, \tag{3.8}$$

$$\lambda_1 I < P < I, \quad \lambda_2 I < Q_1, \quad \lambda_3 I < Q_2, \quad \lambda_4 I < R, \tag{3.9}$$

$$\Omega^* :\equiv \begin{bmatrix} & & & & \tau_M P A_0^T & r_2 P & P & P \\ & [\Omega] & & & 0 & r_2 Q_1 & 0 & 0 \\ & & & & \tau_M R A_1^T & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ * & * & * & * & -\tau_M R & 0 & 0 & 0 \\ * & * & * & * & * & -r_2 R & 0 & 0 \\ * & * & * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & * & * & -Q_2 \end{bmatrix} < 0, \tag{3.10}$$

where

$$[\Omega] = \begin{bmatrix} \Omega_{11} & 0 & A_1 R & 0 \\ * & \Omega_{22} & u_2 Q_1 & 0 \\ * & * & -(u_3+u_1)R & u_2 Q_2 \\ * & * & * & \Omega_{44} \end{bmatrix} \tag{3.11}$$

with $\Omega_{11} = A_0 P + P A_0^T - (\alpha + 2r_1 + 2r_2)P + (r_1 + r_2)R$, $\Omega_{22} = -e^{\alpha\tau_m} Q_1 - (r_2 + r_3 + u_1)(2Q_1 - R)$ *and* $\Omega_{44} = -(e^{\alpha\tau_M} + 2u_3)Q_2 + u_3 R$.

Proof. Define $\xi^T = \begin{bmatrix} x^T(t) & x^T(t-\tau_m) & x^T(t-\tau(t)) & x^T(t-\tau_M) \end{bmatrix}$. Choose Lyapunov-Krasovskii functional (LKF) $V(x(t)) = V_1 + V_2 + V_3 + V_4$ where

$$V_1 = x^T(t) P^{-1} x(t), \qquad V_2 = \int_{t-\tau_m}^{t} e^{\alpha(t-s)} x^T(s) Q_1^{-1} x(s)\, ds,$$
$$V_3 = \int_{t-\tau_M}^{t} e^{\alpha(t-s)} x^T(s) Q_2^{-1} x(s)\, ds, \quad V_4 = \int_{-\tau_M}^{0} \int_{t+\theta}^{t} e^{\alpha(t-\theta)} \dot{x}^T(\theta) R^{-1} \dot{x}(\theta)\, d\theta\, ds.$$

Differentiating the proposed LKF along the system (2.1), we obtain

$$\begin{aligned} \dot{V}_1 &= 2x^T(t) P^{-1} \dot{x}(t) - \alpha x^T(t) P^{-1} x(t) + \alpha V_1, \\ \dot{V}_2 &= x^T(t) Q_1^{-1} x(t) - e^{\alpha\tau_m} x^T(t-\tau_m) Q_1^{-1} x(t-\tau_m) + \alpha V_2, \\ \dot{V}_3 &= x^T(t) Q_2^{-1} x(t) - e^{\alpha\tau_M} x^T(t-\tau_M) Q_2^{-1} x(t-\tau_M) + \alpha V_3, \\ \dot{V}_4 &= \int_{-\tau_M}^{0} \Big[\dot{x}^T(t) R^{-1} \dot{x}(t) - e^{-\alpha s} \dot{x}^T(t+s) R^{-1} \dot{x}(t+s) \\ &\quad + \alpha \int_{t+s}^{t} e^{\alpha(t-\theta)} \dot{x}^T(\theta) R^{-1} \dot{x}(\theta)\, d\theta \Big]\, ds \\ &= \tau_M \dot{x}^T(t) R^{-1} \dot{x}(t) - \int_{t-\tau_M}^{t} e^{\alpha(t-s)} \dot{x}^T(s) R^{-1} \dot{x}(s)\, ds + \alpha V_4. \end{aligned}$$

Defining $\Psi_{11} = P^{-1} A_0 + A_0^T P^{-1} + Q_1^{-1} + Q_2^{-1} - \alpha P^{-1}$. Thus, we have

$$\begin{aligned} \dot{V} - \alpha V = \xi^T &\begin{bmatrix} \Psi_{11} & 0 & P^{-1} A_1 & 0 \\ 0 & -e^{\alpha\tau_m} Q_1^{-1} & 0 & 0 \\ A_1^T P^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\alpha\tau_M} Q_2^{-1} \end{bmatrix} \xi \\ &+ \tau_M \xi^T \begin{bmatrix} A_0^T \\ 0 \\ A_1^T \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} A_0 & 0 & A_1 & 0 \end{bmatrix} \xi - \int_{t-\tau_M}^{t} e^{\alpha(t-s)} \dot{x}^T(s) R^{-1} \dot{x}(s)\, ds, \end{aligned} \tag{3.12}$$

From Propositions 2.4 and 2.5, we have

$$-\int_{t-\tau_m}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R^{-1}\dot{x}(s)\,ds \le \xi^T \begin{bmatrix} -r_1R^{-1} & r_2R^{-1} & 0 & 0 \\ r_2R^{-1} & -r_3R^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xi, \quad (3.13)$$

$$-\int_{t-\tau(t)}^{t-\tau_m} e^{\alpha(t-s)}\dot{x}^T(s)R^{-1}\dot{x}(s)\,ds \le \xi^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -u_1R^{-1} & u_2R^{-1} & 0 \\ 0 & u_2R^{-1} & -u_3R^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xi, \quad (3.14)$$

$$-\int_{t-\tau_M}^{t-\tau(t)} e^{\alpha(t-s)}\dot{x}^T(s)R^{-1}\dot{x}(s)\,ds \le \xi^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -u_1R^{-1} & u_2R^{-1} \\ 0 & 0 & u_2R^{-1} & -u_3R^{-1} \end{bmatrix} \xi. \quad (3.15)$$

Applying inequalities (3.13) - (3.15), we have

$$\begin{aligned} &-\int_{t-\tau_M}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R^{-1}\dot{x}(s)\,ds \\ &= -\left(\int_{t-\tau_m}^{t} + \int_{t-\tau(t)}^{t-\tau_m} + \int_{t-\tau_M}^{t-\tau(t)}\right)\left\{e^{\alpha(t-s)}\dot{x}^T(s)R^{-1}\dot{x}(s)\,ds\right\} \\ &\le \xi^T \begin{bmatrix} -r_2R^{-1} & r_2R^{-1} & 0 & 0 \\ r_2R^{-1} & (-r_3-u_1)R^{-1} & u_2R^{-1} & 0 \\ 0 & u_2R^{-1} & (-u_3-u_1)R^{-1} & u_2R^{-1} \\ 0 & 0 & u_2R^{-1} & -u_3R^{-1} \end{bmatrix} \xi. \end{aligned}$$

Thus, we can rewrite the relation (3.12) as

$$\begin{aligned} \dot{V} - \alpha V \le\ & \xi^T \begin{bmatrix} \Psi_{11} & 0 & P^{-1}A_1 & 0 \\ 0 & -e^{\alpha\tau_m}Q_1^{-1} & 0 & 0 \\ A_1^TP^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\alpha\tau_M}Q_2^{-1} \end{bmatrix} \xi \\ &+ \tau_M \xi^T \begin{bmatrix} A_0^T & 0 & A_1^T & 0 \end{bmatrix}^T R^{-1} \begin{bmatrix} A_0 & 0 & A_1 & 0 \end{bmatrix} \xi \\ &+ \xi^T \begin{bmatrix} -r_2R^{-1} & r_2R^{-1} & 0 & 0 \\ r_2R^{-1} & (-r_3-u_1)R^{-1} & u_2R^{-1} & 0 \\ 0 & u_2R^{-1} & (-u_3-u_1)R^{-1} & u_2R^{-1} \\ 0 & 0 & u_2R^{-1} & -u_3R^{-1} \end{bmatrix} \xi, \\ &\le \xi^T \Psi \xi, \end{aligned}$$

where

$$\Psi = \begin{bmatrix} \Psi_{11} & 0 & P^{-1}A_1 & 0 \\ 0 & -e^{\alpha\tau_m}Q_1^{-1} & 0 & 0 \\ A_1^TP^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\alpha\tau_M}Q_2^{-1} \end{bmatrix} + \tau_M \begin{bmatrix} A_0^T \\ 0 \\ A_1^T \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} A_0 & 0 & A_1 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} -r_2R^{-1} & r_2R^{-1} & 0 & 0 \\ r_2R^{-1} & (-r_3-u_1)R^{-1} & u_2R^{-1} & 0 \\ 0 & u_2R^{-1} & (-u_3-u_1)R^{-1} & u_2R^{-1} \\ 0 & 0 & u_2R^{-1} & -u_3R^{-1} \end{bmatrix}.$$

Pre- and post-multiplying the matrix Ψ by $H = \text{diag}(P, Q_1, R, Q_2)$, we obtain

$$H^T\Psi H = \Pi + \begin{bmatrix} \hat{\Psi}_{11} & 0 & A_1R & 0 \\ 0 & -e^{\alpha\tau_m}Q_1 & 0 & 0 \\ RA_1^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\alpha\tau_M}Q_2 \end{bmatrix}$$
$$+ \tau_M \begin{bmatrix} PA_0^T \\ 0 \\ RA_1^T \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} A_0P & 0 & A_1R & 0 \end{bmatrix}$$

where $\hat{\Psi}_{11} = A_0P + PA_0^T + P(Q_1^{-1} + Q_2^{-1})P - \alpha P$ and

$$\Pi = \begin{bmatrix} -r_1PR^{-1}P & r_2PR^{-1}Q_1 & 0 & 0 \\ r_2Q_1R^{-1}P & (-r_3-u_1)Q_1R^{-1}Q_1 & u_2Q_1 & 0 \\ 0 & u_2Q_1 & (-u_3-u_1)R & u_2Q_2 \\ 0 & 0 & u_2Q_2 & -u_3Q_2R^{-1}Q_2 \end{bmatrix}.$$

Since $r_2PR^{-1}P - r_2PR^{-1}P = 0$ and $r_2Q_1R^{-1}Q_1 - r_2Q_1R^{-1}Q_1 = 0$, we have

$$\Pi = \begin{bmatrix} -(r_1+r_2)PR^{-1}P & 0 & 0 & 0 \\ 0 & -(r_3+r_2+u_1)Q_1R^{-1}Q_1 & u_2Q_1 & 0 \\ 0 & u_2Q_1 & (-u_3-u_1)R & u_2Q_2 \\ 0 & 0 & u_2Q_2 & -u_3Q_2R^{-1}Q_2 \end{bmatrix}$$
$$+ \begin{bmatrix} r_2P & r_2Q_1 & 0 & 0 \end{bmatrix}^T (r_2R)^{-1} \begin{bmatrix} r_2P & r_2Q_1 & 0 & 0 \end{bmatrix}.$$

Using Proposition 2.3, we have

$$-(r_3+r_2+u_1)Q_1R^{-1}Q_1 \le -2(r_3+r_2+u_1)Q_1 + (r_3+r_2+u_1)R,$$
$$-u_3Q_2R^{-1}Q_2 \le -2u_3Q_2 + u_3R,$$
$$-(r_1+r_2)PR^{-1}P \le -2(r_1+r_2)P + (r_1+r_2)R.$$

Thus, we have

$$\Pi \le \begin{bmatrix} -2(r_1+r_2)P+(r_1+r_2)R & 0 & 0 & 0 \\ 0 & -(r_3+r_2+u_1)(2Q_1-R) & u_2Q_1 & 0 \\ 0 & u_2Q_1 & (-u_3-u_1)R & u_2Q_2 \\ 0 & 0 & u_2Q_2 & -2u_3Q_2+u_3R \end{bmatrix} + \begin{bmatrix} r_2P & r_2Q_1 & 0 & 0 \end{bmatrix}^T (r_2R)^{-1} \begin{bmatrix} r_2P & r_2Q_1 & 0 & 0 \end{bmatrix}.$$

Let $\Omega_{11} = A_0P + PA_0^T - (\alpha+2r_1+2r_2)P + (r_1+r_2)R$ and $\Omega_{22} = -e^{\alpha\tau_m}Q_1 - (r_2+r_3+u_1)(2Q_1-R)$. Thus, we obtain

$$\begin{aligned} H^T\Psi H \le & \begin{bmatrix} \Omega_{11} & 0 & A_1R & 0 \\ 0 & \Omega_{22} & u_2Q_1 & 0 \\ RA_1^T & u_2Q_1 & -(u_3+u_1)R & u_2Q_2 \\ 0 & 0 & u_2Q_2 & -(e^{\alpha\tau_M}+2u_3)Q_2+u_3R \end{bmatrix} \\ & + \tau_M \begin{bmatrix} PA_0^T \\ 0 \\ RA_1^T \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} A_0P & 0 & A_1R & 0 \end{bmatrix} + \begin{bmatrix} r_2P \\ r_2Q_1 \\ 0 \\ 0 \end{bmatrix} (r_2R)^{-1} \begin{bmatrix} r_2P & r_2Q_1 & 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} P(Q_1^{-1}+Q_2^{-1})P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & = [\Omega] + \Upsilon^T\chi\Upsilon, \end{aligned}$$

where $[\Omega]$ is defined in Eq. (3.11), $\chi = diag(\frac{1}{\tau_M}R^{-1}, \frac{1}{r_2}R^{-1}, Q_1^{-1}, Q_2^{-1})$, and

$$\Upsilon = \begin{bmatrix} \tau_M A_0 P & 0 & \tau_M A_1 R & 0 \\ r_2P & r_2Q_1 & 0 & 0 \\ P & 0 & 0 & 0 \\ P & 0 & 0 & 0 \end{bmatrix}.$$

From LMI (3.10), we have $\Omega^* < 0$. By Schur's complement (Lemma 2.8), $\Omega^* < 0$ is equivalent to $H^T\Psi H < 0$. Using Proposition 2.1, it implies that $\Psi < 0$. Therefore, $\dot{V} - \alpha V < 0$. Multiplying this inequality by $e^{-\alpha t}$ then integrating from 0 to t, with $t \in [0, T]$, we have

$$V(x(t)) < e^{\alpha t}V(x(0)).$$

From relations (3.9), these inequalities imply that $I < P^{-1}$, $P^{-1} < \dfrac{1}{\lambda_1}I$, $Q_1^{-1} < \dfrac{1}{\lambda_2}I$, $Q_2^{-1} < \dfrac{1}{\lambda_3}I$, and $R^{-1} < \dfrac{1}{\lambda_4}I$. Thus, we have

$$V(x(t)) \ge x^T(t)P^{-1}x(t) > x^T(t)x(t) = \|x(t)\|^2,$$

and

$$\begin{aligned}V(x(0)) &= x^T(0)P^{-1}x(0) + \int_{-\tau_m}^{0} e^{-\alpha s}x^T(s)Q_1^{-1}x(s)\,ds \\ &\quad + \int_{-\tau_M}^{0} e^{-\alpha s}x^T(s)Q_2^{-1}x(s)\,ds + \int_{-\tau_M}^{0}\int_{\theta}^{0} e^{-\alpha s}\dot{x}^T(s)R^{-1}\dot{x}(s)\,ds\,d\theta \\ &\le \frac{1}{\lambda_1}\|\phi\|^2 + \frac{e^{\alpha\tau_m}-1}{\alpha\lambda_2}\|\phi\|^2 + \frac{e^{\alpha\tau_M}-1}{\alpha\lambda_3}\|\phi\|^2 + \frac{e^{\alpha\tau_M}-\alpha\tau_M-1}{\alpha^2\lambda_4}\|\phi\|^2.\end{aligned}$$

From LMI (3.8), we have $\Delta < 0$, by applying Schur's complement, this LMI is equivalent to

$$\left(\frac{1}{\lambda_1} + \frac{e^{\alpha\tau_M}-1}{\alpha\lambda_2} + \frac{e^{\alpha\tau_m}-1}{\alpha\lambda_3} + \frac{e^{\alpha\tau_M}-\alpha\tau_M-1}{\alpha^2\lambda_4}\right) - e^{-\alpha T}\frac{c_2}{c_1} < 0.$$

Because the initial condtion $\|\phi\|^2 < c_1$, thus, we obtain

$$\|x(t)\|^2 < e^{\alpha T}c_1\left(\frac{1}{\lambda_1} + \frac{e^{\alpha\tau_M}-1}{\alpha\lambda_2} + \frac{e^{\alpha\tau_m}-1}{\alpha\lambda_3} + \frac{e^{\alpha\tau_M}-\alpha\tau_M-1}{\alpha^2\lambda_4}\right) < c_2.$$

Therefore, the linear system (2.1) without controller is finite-time stable. □

Based on the finite-time stability condition in Theorem 3.1, we next formulate finite-time stabilization condition of the linear system (2.1) with time-varying delays. Here we desire a feedback controller in the form of $u(t) = B^TKx(t)$ where $K = YP^{-1}$. The finite-time stabilization of the system is formulated as follow.

Theorem 3.2. *The linear system* (2.1) *under the controller* $u(t) = B^TKx(t)$ *with* $K = YP^{-1}$ *is finite-time stabilization with respect to* (c_1, c_2, T) *if there exist positive scalars* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha$ *and symmetric positive definite matrices* P, Q_1, Q_2, R *and matrix* Y *such that LMIs* (3.8) - (3.9) *hold and*

$$\Sigma^* :\equiv \begin{bmatrix} & & & & \tau_M(PA_0^T + Y^TBB^T) & r_2P & P & P \\ & [\Sigma] & & & 0 & r_2Q_1 & 0 & 0 \\ & & & & \tau_M RA_1^T & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ * & * & * & * & -\tau_M R & 0 & 0 & 0 \\ * & * & * & * & * & -r_2R & 0 & 0 \\ * & * & * & * & * & * & -Q_1 & 0 \\ * & * & * & * & * & * & * & -Q_2 \end{bmatrix} < 0 \quad (3.16)$$

where

$$\Sigma = \begin{bmatrix} \bar{\Sigma}_{11} & 0 & A_1R & 0 \\ * & \Sigma_{22} & u_2Q_1 & 0 \\ * & * & -(u_3+u_1)R & u_2Q_2 \\ * & * & * & -(e^{\alpha\tau_M}+2u_3)Q_2 + u_3R \end{bmatrix} \quad (3.17)$$

with $\Sigma_{11} = A_0P + BB^TY + PA_0^T + Y^TBB^T - (\alpha + 2r_1 + 2r_2)P + (r_1 + r_2)R$ *and* $\Sigma_{22} = -e^{\alpha\tau_m}Q_1 - (r_2 + r_3 + u_1)(2Q_1 - R)$.

Proof. Replacing A_0 in LMIs (3.10) by $A_0 + BB^T K$ then using the fact that $KP = Y$, then the finite-time stabilization with controller $u(t) = B^T Kx(t)$ is obtained. □

Remark 3.3. In the derivations of our finite-time stability and stabilization criteria, we overcome the process of defnining new variables for non-linear terms occuring in the derivation by desiring proper controller and Lyapunov-Krasovskii functional. Unlike the derivations in other works (see [10, 12, 13, 16, 22]) their proposed finite-time stability conditions required to find new variables for the occuring non-linear terms. Thus, solving the required LMIs will need extra care for assuring the consistency of the feasible solutions regarding the non-linear terms.

4 Numerical Examples

In this section, we present two numerical examples to show the effectiveness of our proposed stability conditions by investigating the FTS (example 4.1) and FTU (example 4.2) of the linear system (2.1) of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) + Bu(t).$$

Example 4.1. Consider the linear system with time-varying delay (2.1) with:

$$A_0 = \begin{bmatrix} -0.2 & 2 \\ -1 & -0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}, \tau_m \leq \tau(t) \leq \tau_M. \tag{4.1}$$

Note that the linear system (2.1) with given constant matrices in (4.1) and $u(t) = 0$ is asymtotically stable with initial condition $\phi^T(t) = \begin{bmatrix} 0.1t + 0.2, -0.1t - 0.2 \end{bmatrix}$, $t \in [-\tau_M, 0]$. However, in this example, we aim to compare the smallest values of c_2 guaranteeing FTS with respect to (c_1, c_2, T) of our proposed FTS condition in theorem 3.1 with some existing conditions. With the initial given above, we choose $\|\phi(t)\|^2 = c_1 = 0.18$.

Case i: Comparing the smallest value of c_2 guarantees FTS with criteria proposed in [16] and [29]. For $\tau_m = 1 \leq \tau(t) \leq 1.5 = \tau_M, c_1 = 0.18$, we solve LMIs (3.8) - (3.10) using standard LMI solver and obtain the smallest c_2 for different final time $T = 1, 2, 3, 4, 5$ as seen in Table 1. Results show that our conditon gives smaller value of c_2 than those from [16] but larger than [29].

Case ii: Investigating the effect of range of interval time delay $(\tau_M - \tau_m)$ to the the value of c_2 guaranteeing FTS with respect to $(0.18, c_2, 2)$. First, we investigate the effect of the interval time delay of range $\tau_M - \tau_m = 0.5$ for different values of τ_m. Applying Theorem 3.1, results show that the smallest values of c_2 increase as τ_m increases (see Table 2). Next, we further investigate the range of interval time delay affecting the smallest value of c_2 guaranteeing FTS. We observe that increasing the range of time delay $(\tau_M - \tau_m)$ requires larger values of the smallest value of c_2 guaranteeing FTS (results not shown).

Table 1: Comparing smallest values of c_2 for $\tau(t) \in [1, 1.5]$ with various T.

T	1	2	3	4	5
[29]	0.33	0.33	0.33	0.33	0.33
[16]	759.5	1.21×10^6	$1.79{\times}10^9$	$2.58{\times}10^{12}$	$3.72{\times}10^{15}$
Theorem 3.1	20.68	151.99	1089	7733	54894

Table 2: Smallest values of c_2 for interval time delay with $\tau_M - \tau_m = 0.5$.

τ_m	0.1	0.2	0.3	0.4	0.6	0.8	1.0
τ_M	0.6	0.7	0.8	0.9	1.1	1.3	1.5
c_2	4.96	7.89	11.87	17.34	35.61	72.67	151.99

Remark 4.1. The FTS condition proposed in [29] requires delay to be differentiable function but our Theorem 3.1 and condition in [16] do not. Moreover, the condition in [29] is formulated using one more term of single integral and one more term of double integral in their LKF than ours; while the condition in [16] used two more terms of double integral in their LKF than ours.

Next, we investigate the FTU of the linear system with interval time-varying delay (2.1) with $\tau(t) = 0.1 + 0.01|\cos(t)|$,

$$A_0 = \begin{bmatrix} -1.7 & 1.7 & 0 \\ 1.3 & -1 & 0.7 \\ 0.7 & 1 & -0.6 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1.5 & -1.7 & 0.1 \\ -1.3 & 1 & -0.5 \\ -0.7 & 1 & 0.6 \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ 10 \\ 20 \end{bmatrix}. \qquad (4.2)$$

Remark 4.2. The linear system (2.1) with given constant matrices defined in Eq. (4.2) and $u(t) = 0$ is not asymtotically stable (see Figure 1). This figure reveals that the state variables $x_i(t) \to \infty, i = 1, 2, 3$ as $t \to \infty$ with initial condition $\phi^T(t) = \begin{bmatrix}0.4, & 0.2, & 0.4\end{bmatrix}$.

Example 4.2. Consider the FTU of the linear system (2.1) with respect to $(c_1, c_2, T) = (0.36, 5, 10)$ of linear system (2.1) with nonzero feedback controller $u(t) = B^T K x(t)$ to control the norm of the solution to be within $c_2 = 5$ during $0 \le t \le 10$.

From Figure 1, it is easy to observe that, for $t \to 10$, the largest value of the state variables reach above 2000 that yields the norm of the state variables reach very high value of about 10^6. With the initial condition above satifying $\|\phi(t)\|^2 = 0.36 = c_1$, we want to control the norm of the state variables to be within the value of $c_2 = 5$ for $t \in [0, 10]$. By solving LMIs (3.8), (3.9) and (3.16) as required in Theorem 3.2, result shows that for $\alpha = 0.2$, we obtain the set of feasible solutions guaranteeing finite-time stabilization as follows: $\lambda_1 =$

$0.6851, \lambda_2 = 0.5628, \lambda_3 = 0.5802, \lambda_4 = 0.4874$ and

$$P = \begin{bmatrix} 0.9447 & 5.0022 \times 10^{-4} & -0.0035 \\ 5.0022 \times 10^{-4} & 0.9302 & -0.0032 \\ -0.0035 & -0.0032 & 0.9328 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.9932 & -0.0534 & -0.0916 \\ -0.0534 & 1.4371 & 0.5322 \\ -0.0916 & 0.5322 & 2.2782 \end{bmatrix}, \; Q_2 = \begin{bmatrix} 0.9976 & -0.0467 & -0.0878 \\ -0.0467 & 1.4763 & 0.5641 \\ -0.0878 & 0.5641 & 2.3534 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.9909 & -0.0544 & -0.0869 \\ -0.0544 & 1.3427 & 0.4300 \\ -0.0869 & 0.4300 & 2.0441 \end{bmatrix}, \; Y = -\begin{bmatrix} 330.3973 & 704.3972 & 882.5567 \\ 5.3727 & 20.8157 & 25.6198 \\ 13.8299 & 24.8350 & 31.3583 \end{bmatrix}.$$

Here, the state-feedback controller guaranteeing finite-time stabilization of the linear system (2.1) is designed by $u(t) = \left[0.0736, -0.4971, -0.8678\right] x(t)$.

With this controller, we have the lower value of $c_2 = 4.8581$ and the solution of closed loop system with the initial condition $\phi^T = [0.4, 0.2, 0.4]$ is shown in Figure 2. We notice that the state variables oscillate for short period of time then converge to zero; while its norm is stay below $c_2 = 5$ as expected.

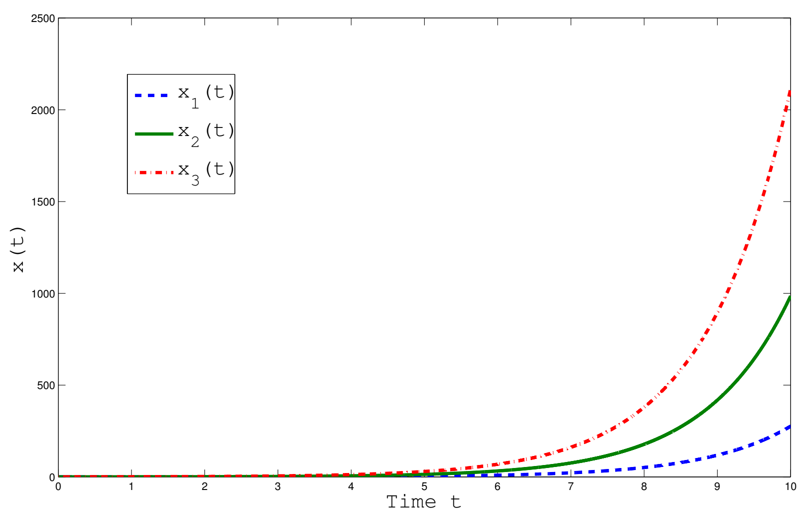

Figure 1: State trajectories of the linear system (2.1) with initial $\phi^T(t) = [0.4,\ 0.2,\ 0.4]$ for $T = 10$.

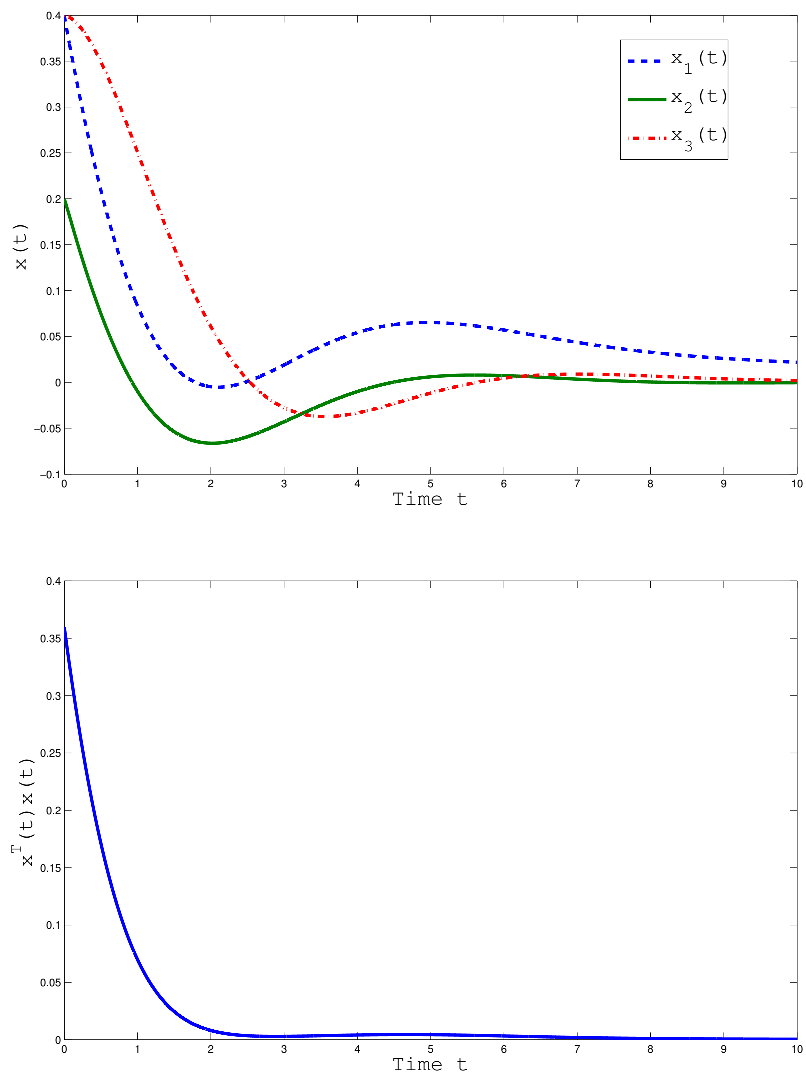

Figure 2: State (top) and norm (bottom) of the state trajectories of the closed loop system as in Eq. (2.1) with controller.

Conclusion

In this research, we propose two new integral inequalities for bounding an integral found in the derivative of LKF. Then the finite-time stability and stabilization criteria are formulated in the form of LMIs. Two numerical examples are given to show that our proposed FTS and FTU criteria are practicle and can be applied to non-differentiable continuous delay function.

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