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An Alternative Quadratic Functional Equation on 2-Divisible Commutative Groups

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Abstract : We prove that the alternative quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)\pm 2f(y)$$

is equivalent to the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for the class of functions from a 2-divisible commutative group to a linear space.

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1 Introduction

One of the most well-known functional equation is probably the additive Cauchy functional equation

$$f(x+y)=f(x)+f(y), \tag{C}$$

where its general solution is simply defined to be an *additive* function and may be regarded as the foundation of the *polynomial-type* functional equation. To study

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a basic polynomial-type functional equation, ones usually start with the classical quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{Q}$$

where its general solution can be written as a *diagonalization* of a *bi-additive* symmetric function. For example, a function $f:\mathbb{R}\to\mathbb{R}$ satisfies (Q) for all $x,y\in\mathbb{R}$ if and only if there exists a bi-additive symmetric function $B:\mathbb{R}^2\to\mathbb{R}$, i.e.

$$\begin{aligned} B(x_1+x_2,y)&=B(x_1,y)+B(x_2,y), && \text{(additive in the first variable)}\\ B(x,y_1+y_2)&=B(x,y_1)+B(x,y_2), && \text{(additive in the second variable)}\\ \text{and}\quad B(x,y)&=B(y,x) && \text{(symmetric)} \end{aligned}$$

for all $x,x_1,x_2,y,y_1,y_2\in\mathbb{R}$, such that

$$f(x)=B(x,x)\quad \text{for all } x\in\mathbb{R}.$$

Interested readers may want to consult [1] for more details.

Solving a functional equation may require quite a few replacements of the independent variables as well as some ingenious manipulation of those resulting equations. Yet an even more challenging class of functional equations are the so-called *alternative* functional equations, where at each point in the space of independent variables, there are two or more alternatives. For instance, an alternative Cauchy functional equation derived from (C) may read

$$f(x)+f(y)=\pm f(x+y),$$

which its general solution for functions defined on semigroups has been discussed by M. Kuczma [2]. A more general alternative Cauchy functional equation of the form

$$[f(x+y)-af(x)-bf(y)]\cdot[f(x+y)-cf(x)-df(y)]=0,$$

where f is a function defined on a commutative group, has been solved by R. Ger [3].

In 1995, F. Skof [4] proposed the following four alternative quadratic functional equations:

$$|f(x+y)|=|2f(x)+2f(y)-f(x-y)|, \tag{AQ$_1$}$$
$$|f(x-y)|=|2f(x)+2f(y)-f(x+y)|, \tag{AQ$_2$}$$
$$|2f(y)|=|f(x+y)+f(x-y)-2f(x)|, \tag{AQ$_3$}$$
$$\text{and}\quad |2f(x)|=|f(x+y)+f(x-y)-2f(y)| \tag{AQ$_4$}$$

and proved that for the class of functions $f:X\to\mathbb{R}$, where X is a real linear space, each of the above functional equation is equivalent to the quadratic functional equation (Q). Nevertheless, the alternative quadratic functional equations:

$$|f(x+y)+f(x-y)|=|2f(x)+2f(y)| \tag{AQ$_5$}$$

is considerably subtle, and it can only be proved that f is rationally homogeneous of degree 2, i.e. $f(rx) = r^2 f(x)$ for all rational numbers r and for all $x \in X$. But for a specific case when $X = \mathbb{R}$ and f is a continuous function, it can be successfully shown that the alternative quadratic functional equation (AQ_5) is equivalent to the quadratic functional equation (Q).

A more recent result concerning an alternative quadratic functional equation is due to G.L. Forti [5] who studied the solution of the following functional equation:

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \in \{0, 1\},$$

where f is a function from a group $(G, \cdot)$ to $\mathbb{R}$ and G possesses certain additional properties.

In this paper, we will study the alternative quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) \pm 2f(y), \tag{AQ}$$

which is similar to (AQ_3) to some aspects, but we will consider a more general class of functions from a 2-divisible commutative group $(G, +)$ to a real (or rational or complex) linear space Y. We will prove that the alternative quadratic functional equation (AQ) is equivalent to the quadratic functional equation (Q) for functions $f : G \to Y$. It should be noted that the 2-divisibility property of the domain of the function has been extensively used in the work of Skof [4], but here we will present a proof which relies on a minimal use of 2-divisibility.

2 Auxiliary Lemmas

Many substitutions will be made through the alternative quadratic functional equation (AQ), it therefore is quite convenient to adopt the following notations for the rest of this paper:

$$\begin{aligned}\mathcal{AQ}_f(x, y) := \Big(& f(x + y) + f(x - y) = 2f(x) + 2f(y) \\ &\text{or} \quad f(x + y) + f(x - y) = 2f(x) - 2f(y)\Big).\end{aligned}$$

The following lemma will give two of the most basic properties that can derived directly from the alternative quadratic functional equation (AQ).

Lemma 2.1. *If a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, then*

1. *$f(0) = 0$, and*
2. *$f(-x) = f(x)$ for all $x \in G$, i.e. f is an even function.*

Proof. Suppose that $f : G \to Y$ satisfies (AQ) for all $x, y \in G$.

1. Considering $\mathcal{AQ}_f(0, 0)$, we immediately get that $f(0) = 0$.

2. Assume that there exists $x_0 \in G$ such that

$$f(-x_0) \neq f(x_0) \tag{2.1}$$

Considering $\mathcal{AQ}_f(0, x_0)$ and substituting $f(0) = 0$, we can see that

$$f(-x_0) = f(x_0) \quad \text{or} \quad f(-x_0) = -3f(x_0). \tag{2.2}$$

Similarly, $\mathcal{AQ}_f(0, -x_0)$ with $f(0) = 0$ gives

$$f(-x_0) = f(x_0) \quad \text{or} \quad f(x_0) = -3f(-x_0). \tag{2.3}$$

Taking the assumption (2.1) into account, we can derive from (2.2) and (2.3) that

$$\begin{cases} f(-x_0) = -3f(x_0), \\ \quad f(x_0) = -3f(-x_0). \end{cases}$$

Solving the above system of equations gives

$$f(-x_0) = 0 \quad \text{and} \quad f(x_0) = 0,$$

which contradicts the assumption (2.1).
Therefore, $f(-x) = f(x)$ for all $x \in G$ as desired. □

The following lemma is the only lemma in this paper that uses the 2-divisibility property of the group G to prove that $f(2x) = 4f(x)$ for all $x \in G$, where f is any function satisfying (AQ).

Lemma 2.2. *If a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, then $f(2x) = 4f(x)$ for all $x \in G$.*

Proof. Suppose that $f : G \to Y$ satisfies (AQ) for all $x, y \in G$.
For any $x \in G$, $\mathcal{AQ}_f(x, x)$ with $f(0) = 0$ simplifies to

$$f(2x) = 4f(x) \quad \text{or} \quad f(2x) = 0 \quad \text{for all } x \in G. \tag{2.4}$$

Assume that there exists $x_0 \in G$ such that $f(2x_0) \neq 4f(x_0)$. Since G is a 2-divisible group, we let $z_0 \in G$ such that $2z_0 = x_0$. Therefore,

$$f(4z_0) \neq 4f(2z_0). \tag{2.5}$$

With $x = 2z_0$ in (2.4) and taking the assumption (2.5) into account, we have

$$f(4z_0) = 0. \tag{2.6}$$

From (2.5) and (2.6), we know that

$$f(2z_0) \neq 0 \tag{2.7}$$

With $x = z_0$ in (2.4) and taking (2.7) into account, we now have

$$f(2z_0) = 4f(z_0) \tag{2.8}$$

$\mathcal{AQ}_f(2z_0, z_0)$ with $f(2z_0)$ from (2.8) gives

$$f(3z_0) = 9f(z_0) \quad \text{or} \quad f(3z_0) = 5f(z_0). \tag{2.9}$$

$\mathcal{AQ}_f(3z_0, z_0)$ with $f(4z_0)$ from (2.6) and $f(2z_0)$ from (2.8) gives

$$f(3z_0) = f(z_0) \quad \text{or} \quad f(3z_0) = 3f(z_0). \tag{2.10}$$

Considering all possibilities in (2.9) and (2.10), we infer that $f(z_0) = 0$, which contradicts (2.7) and (2.8).

Therefore, $f(2x) = 4f(x)$ for all $x \in G$ as desired. □

We will now generalize Lemma 2.1 and Lemma 2.2 to the following important lemma which actually proves that f is integrally homogeneous of degree 2.

Lemma 2.3. *If a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, then*

$$f(nx) = n^2 f(x) \quad \textit{for all } x \in G \textit{ and for all } n \in \mathbb{Z}.$$

Proof. Suppose that $f : G \to Y$ satisfies (AQ) for all $x, y \in G$, and let $x \in G$. We will first prove by mathematical induction that, for all $n = -1, 0, 1, \ldots$,

$$f(nx) = n^2 f(x). \tag{2.11}$$

For $n = 1$, (2.11) is trivial. While for $n = -1, 0, 2$, (2.11) follows from Lemma 2.1 and Lemma 2.2.

Now suppose that $f(kx) = k^2 f(x)$ for all $k = -1, 0, 1, \ldots, n$ for an integer $n \geq 2$. We will prove that $f\big((n+1)x\big) = (n+1)^2 f(x)$ by a contradiction; i.e., assume that

$$f\big((n+1)x\big) \neq (n+1)^2 f(x) \tag{2.12}$$

$\mathcal{AQ}_f(nx, x)$ with $f\big((n-1)x\big)$ and $f(nx)$ from the induction hypothesis will simplify to

$$f\big((n+1)x\big) = (n+1)^2 f(x) \quad \text{or} \quad f\big((n+1)x\big) = \big(n^2 + 2n - 3\big) f(x).$$

Taking the assumption (2.12) into account, we are left with

$$f\big((n+1)x\big) = \big(n^2 + 2n - 3\big) f(x). \tag{2.13}$$

$\mathcal{AQ}_f\big((n-1)x, 2x\big)$ with $f\big((n-3)x\big), f\big((n-1)x\big)$ and $f(2x)$ from the induction hypothesis, will simplify to

$$f\big((n+1)x\big) = (n+1)^2 f(x) \quad \text{or} \quad f\big((n+1)x\big) = \big(n^2 + 2n - 15\big) f(x).$$

Taking the assumption (2.12) into account, we are left with

$$f\big((n+1)x\big) = (n^2 + 2n - 15)f(x). \tag{2.14}$$

Equating (2.13) and (2.14) will lead to the conclusion that

$$f(x) = 0 \quad \text{and} \quad f\big((n+1)x\big) = 0,$$

which contradicts (2.12).

Therefore, (2.11) holds for all $n = -1, 0, 1, \ldots$.

Lemma 2.1 tells us that f is an even function; therefore, (2.11) also holds for all negative integers n. This completes the proof of the lemma. □

3 Equivalence of (AQ) and (Q)

We are now ready to establish the equivalence of the alternative quadratic functional equation (AQ) and the quadratic functional equation (Q) in the following theorem. The fact that f is integrally homogeneous of degree 2 will be used extensively in the proof without an explicit reference to Lemma 2.3.

Theorem 3.1. *A function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$ if and only if f satisfies (Q) for all $x, y \in G$.*

Proof. ($\Leftarrow$) A function $f : G \to Y$ satisfies (Q) for all $x, y \in G$, then it is obvious that f satisfies (AQ) for all $x, y \in G$.

($\Rightarrow$) Suppose that a function $f : G \to Y$ satisfies (AQ) for all $x, y \in G$.
We will prove that f satisfies (Q) for all $x, y \in G$ by a contradiction.
Suppose there exist $x_0, y_0 \in G$ such that

$$f(x_0 + y_0) + f(x_0 - y_0) \neq 2f(x_0) + 2f(y_0). \tag{3.1}$$

The assumption (3.1) will be used to eliminate an alternative from $\mathcal{AQ}_f(x, y)$ for many suitable choices of x and y.

In order to better understand the ideas, we will divide the proof into a few steps.

Step 1: Determine $f(x_0 + y_0), f(x_0 - y_0), f(y_0)$ in terms of $f(x_0)$.
$\mathcal{AQ}_f(x_0, y_0)$ with (3.1) gives

$$f(x_0 + y_0) + f(x_0 - y_0) = 2f(x_0) - 2f(y_0). \tag{3.2}$$

$\mathcal{AQ}_f(x_0 + y_0, x_0 - y_0)$ with $f(2x) = 4f(x)$ simplifies to

$$\begin{aligned} & 2f(x_0) + 2f(y_0) = f(x_0 + y_0) + f(x_0 - y_0) \\ \text{or} \quad & 2f(x_0) + 2f(y_0) = f(x_0 + y_0) - f(x_0 - y_0). \end{aligned}$$

Taking the assumption (3.1) into account, we are left with

$$f(x_0 + y_0) - f(x_0 - y_0) = 2f(x_0) + 2f(y_0). \tag{3.3}$$

$\mathcal{AQ}_f(y_0, x_0)$ with $f(-x) = f(x)$ as well as the assumption (3.1) gives

$$f(x_0 + y_0) + f(x_0 - y_0) = 2f(y_0) - 2f(x_0). \tag{3.4}$$

For convenience, we will let

$$a := f(x_0).$$

Equating (3.2) and (3.4) yields

$$f(y_0) = a.$$

From (3.2) and (3.3), we get that

$$f(x_0 + y_0) = 2a \quad \text{and} \quad f(x_0 - y_0) = -2a.$$

The values of $f(x_0+y_0), f(x_0-y_0), f(y_0)$ and $f(x_0)$ in terms of a will be regarded as known and will be used in the subsequent steps.

Step 2: Determine all possible values of $f(x_0 + 2y_0)$ and $f(x_0 - 2y_0)$.
$\mathcal{AQ}_f(x_0 + 2y_0, x_0 - 2y_0)$ with $f(2x) = 4f(x)$ and $f(4x) = 16f(x)$ gives

$$10a = f(x_0 + 2y_0) \pm f(x_0 - 2y_0). \tag{3.5}$$

$\mathcal{AQ}_f(x_0 - y_0, -y_0)$ with $f(-x) = f(x)$ gives

$$f(x_0 - 2y_0) \in \{-3a, -7a\}. \tag{3.6}$$

Substituting $f(x_0-2y_0)$ from (3.5) into (3.6) gives 4 possible values for $f(x_0+2y_0)$:

$$f(x_0 + 2y_0) \in \{3a, 7a, 13a, 17a\}. \tag{3.7}$$

$\mathcal{AQ}_f(x_0 + y_0, y_0)$ gives 2 possible values for $f(x_0 + 2y_0)$

$$f(x_0 + 2y_0) \in \{a, 5a\}. \tag{3.8}$$

Step 3: Put the jigsaw together to conclude the value of a.
From the values of $f(x_0 + 2y_0)$ in (3.7) and (3.8), we can conclude that $a = 0$, which in turn implies that

$$f(x_0 + y_0) = f(x_0 - y_0) = f(x_0) = f(y_0) = 0$$

and eventually contradict the assumption (3.1).

Therefore, f satisfies (Q) as desired. □

The following example will show that the 2-divisiblity of G is crucial to the equivalence of (AQ) and (Q). It should be emphasize that the 2-divisibility has been used only is the proof of Lemma. 2.2.

Example 3.2. Consider a function $f : \mathbb{Z} \to \mathbb{R}$ defined by

$$f(n) = \begin{cases} 0 & \text{if n is even,} \\ 1 & \text{if n is odd} \end{cases}$$

for all $n \in \mathbb{Z}$.

For any $m, n \in \mathbb{Z}$, we can see that $m+n$ and $m-n$ are always of the same parity. Therefore,

$$f(m+n) + f(m-n) = \begin{cases} 0 & \text{if} \quad m \equiv n \pmod 2, \\ 2 & \text{if} \quad m \not\equiv n \pmod 2. \end{cases}$$

Moreover, if $m \equiv n \pmod 2$, then $f(m) - f(n) = 0$,
and if $m \not\equiv n \pmod 2$, then $f(m) + f(n) = 1$. Hence,

$$f(m+n) + f(m-n) = 2f(m) \pm 2f(n);$$

that is f satisfies (AQ) for all $m, n \in \mathbb{Z}$. However, one can easily verify that

$$f(2) + f(0) \neq 2f(1) + 2f(1).$$

Hence, f does not satisfy (Q).

Therefore, the alternative quadratic functional equation (AQ) is not equivalent to the quadratic functional equation (Q) for this particular function f defined on the group $(\mathbb{Z}, +)$ which is not 2-divisible.

References

[1] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, 2002.

[2] M. Kuczma, On some alternative functional equations, Aequations Math. 17 (1978) 182-198.

[3] R. Ger, On an alternative functional equation, Aequationes Math. 15 (1977) 145-162.

[4] F. Skof, On Some Alternative Quadratic Equation, Results in Mathematics 27 (1995) 402-411.

[5] G.L. Forti, A note on an alternative quadratic equation, Annales Univ. Sci. Budapest., Sect. Comp. 40 (2013) 223-232.

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