Thai Journal of Mathematics
Volume 17 (2019) Number 1 : 155–164

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Automatic Continuity on Fundamental Locally Multiplicative Topological Algebras

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Abstract : In this paper we prove some results about a new class of topological algebras namely fundamental and fundamental locally multiplicative topological algebras (abbreviated by FLM). Then we present essential results about FLM algebras. We also obtain some results on the automatic continuity of homomorphisms between FLM algebras.

Keywords : fundamental topological algebra; FLM topological algebras; spectral radius; radius of boundedness; homomorphism. mappings
2010 Mathematics Subject Classification : 46H05.

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1 Introduction

In 1947, Some results concerning topological algebras had been published by R. Arens [1]. It was in 1952 that Arens and Michael independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras, see [1]. The notion of fundamental

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topological spaces (also algebras) has been introduced in [2] in 1990 extending the meaning of both local convexity and local boundedness. Also in [3] a topological structure is defined on the algebraic dual space of an FLM algebra to make it a normed space, and some of the famous theorems of Banach algebras are extended for complete metrizable FLM algebras. Continuity of the spectrum and spectral radius functions play a crucial role in automatic continuity. In this paper we present some results about radius of boundedness, FLM algebras and automatic continuity of homomorphisms between FLM algebras. We first recall some notions in topological algebras.

Definition 1.1. [2, 2.1] A topological linear space A is said to be a fundamental one if there exists $b > 1$ such that for every sequence $(x_n)_n$ of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that $(x_n)_n$ is Cauchy.

A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

Definition 1.2. [3, 4.2] A fundamental topological algebra is said to be locally multiplicative if there exists a neighborhood U_0 of zero such that for every neighborhood $V$of zero, the sufficiently large powers of U_0 lie in V. Such an algebra is known as an FLM algebra.

Example 1.3. Let $\mathbb{R}$ be equipped with its Euclidean topology. If $(x_n)_n$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} 2^n(x_n - x_{n-1}) = 0$, then $(x_n)_n$ is Cauchy sequence in $\mathbb{R}$.

Example 1.4. [2] Non-fundamental topological vector spaces

There are many topological vector spaces which are not fundamental. For example, the space $M([0,\infty))$ of all measurable functions on $[0,\infty)$, with the convergence in measure topology is not fundamental.

Definition 1.5. [4] Let x be an element of a topological algebra A. We say that x is bounded if there exists some $\lambda \in \mathbb{C} - \{0\}$ such that the sequence $(\frac{x^n}{\lambda^n})_n$ converges to zero. We denote

$$B_A(x) = \{\lambda \in \mathbb{C} - \{0\} : \quad \frac{x^n}{\lambda^n} \to 0\}.$$

The radius of boundedness of x with respect to A is denoted by $\beta_A(x)$ and defined by

$$\beta_A(x) = \inf\{|\lambda| : \lambda \in B_A(x)\},$$

with the convention, if $B_A(x) = \varnothing$, then $\beta_A(x) = +\infty$. We denoted

$$\mathfrak{B}(A) = \{x \in A : \beta_A(x) = 0\}.$$

We now define another important class of topological algebras, called Q-algebras. Let A be an algebra. An element $x \in A$ is called quasi-invertible if there exists $y \in A$ such that $x \diamond y = x + y - xy = 0$ and $y \diamond x = y + x - yx = 0$.

we denote the quasi inverse of x by $y = (x_\diamond)^{-1}$. The set of all quasi-invertible elements of A is denoted by $q - InvA$. A topological algebra A is called a *Q-algebra* if $q - InvA$ is open, or equivalently, if $q - InvA$ has an interior point in A [5, Lemma E2].
Note that, every complete metrizable FLM algebra is Q-algebra [2, Theorem 4.3]. For a unital algebra A with the unit e_A, the spectrum of an element $x \in A$, denoted by $\sigma_A(x)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e_A - x$ is not invertible in A. For a non-unital algebra A, the spectrum of $x \in A$ is $\sigma_A(x) = \{0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \frac{x}{\lambda} \notin q - InvA\}$. The spectral radius of an element $x \in A$ is $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$. If $\sigma_A(x) = \varnothing$, we define $r_A(x) = 0$. The element x is quasi-nilpotent if $r_A(x) = 0$, i.e., $\sigma_A(x) = \{0\}$ or $\sigma_A(x) = \varnothing$. The set of all quasi-nilpotents in A is denoted by

$$\mathfrak{Q}(A) = \{x \in A : r_A(x) = 0\}.$$

The *(Jacobson) radical* of an algebra A, $radA$, is the intersection of all maximal left (right) ideals in A. The algebra A is called *semisimple* if $radA = \{0\}$. In the case that A is a Banach algebra we have

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

We now recall a property of rad$A$, which will be used in the sequel.

Remark 1.6. *It is known that for any algebra A,*

$$radA = \{x \in A : r_A(xy) = 0, \text{ for every } y \in A\}.$$

Also, in the case where A is unital

$$radA = \{x \in A : e_A - xy \in InvA, \text{ for every } y \in A\},$$

in particular, $e_A + rad\, A \subset InvA$.

If A is a commutative Banach algebra, then $radA = \bigcap_{\varphi\in M(A)} ker\varphi$, where $M(A)$ is the continuous character space of A, i.e. the space of all continuous non-zero multiplicative linear functionals on A. See, for example, [6, Proposition 8.1.2].

We recall that automatic continuity of linear mappings, homomorphisms and almost multiplicative linear maps are very important in advanced studies on topological algebras and mathematical analysis. The following theorem is a well-known result, due to Šilov, concerning the automatic continuity of multiplicative linear maps (homomorphisms) between Banach algebras.

Theorem 1.7. [7, Theorem 2.3.3] *Let A and B be Banach algebras such that B is commutative and semisimple. Then, every homomorphism* $T : A \to B$ *is automatically continuous.*

By the above Theorem, every commutative semisimple Banach algebra has a unique complete norm. It was a historically important question, raised by Rickart in 1950, whether or not each semisimple Banach algebra has a unique complete norm.This was eventually proved in 1967 by B. E. Johnson [7, Corollary 5.1.6], and as a consequence of this result, it was shown that if $T : A \to B$ is a surjective homomorphism between a Banach algebra A and a semisimple Banach algebra B, then T is automatically continuous.

Many authors have investigated automatic continuity of homomorphisms between Banach algebras and Fréchet algebras, and there are many open questions in this area. For example, in 1952 [5], E. A. Michael posed the question as whether each multiplicative linear functional on a (commutative semisimple) Fréchet algebra is automatically continuous. This question, known as the Michael's problem, has been intensively studied, but only partial answers have been obtained so far. For further results on automatic continuity of homomorphisms between certain classes of Fréchet algebras, or partial answers to Michael's problem, one may refer, for example, to [8, 9] and the references therein.

Next, some results for automatic continuity in the area of Banach and Frechet algebras have been obtained by Aupetit [10], Ghasemi-Honary [9], and Omidi [11, 12].

2 Main Results

We now present the following result, which is essential and it plays a crucial role in this section. Also it is similar to [13, Theorem 10.3] and generalize some item of it. Therefore some of the famous theorems of Banach algebras are extended for complete metrizable FLM algebras.

Theorem 2.1. *Let A be a complete metrizable fundamental topological algebra and $x \in A$. Then*

(i) If $\beta_A(x) < 1$, then $x \in q - InvA$ and $(x_\diamond)^{-1} = -\sum_{n=1}^{\infty} x^n$.

(ii) $\beta_A(\alpha x) = |\alpha|\beta_A(x)$ and $\beta_A(x^N) = \beta_A(x)^N$, for $\alpha \in \mathbb{C}$ and $N \in \mathbb{N}$.

(iii) $r_A(x) \leq \beta_A(x)$.

Proof. (i) Let $\lambda > 1$, $\beta_A(x) < \frac{1}{\lambda} < 1$, then $\lambda^n x^n \to 0$ and $x^n \to 0$ as $n \to \infty$. Take $s_n = \sum_{k=1}^{n} x^k$, then $\lambda^n(s_n - s_{n-1}) \to 0$, as $n \to \infty$. By Definition 1.1, $(s_n)_n$ is a Cauchy sequence. Let $\lim_{n\to\infty} s_n = s$. Then

$$x \diamond s_n = x + s_n - xs_n = x + x - x^{n+1},$$

therefore $x + s - xs = x + x$ and so $x \diamond -s = 0$. Thus

$$(x_\diamond)^{-1} = -s = -\sum_{n=1}^{\infty} x^n.$$

(ii) The proof is straightforward.

(iii) Let $0 \neq \lambda \in Sp_A(x)$, then $\frac{x}{\lambda} \notin q - InvA$. Therefore $\beta_A(\frac{x}{\lambda}) \geq 1$, by applying (i). Hence $\beta_A(x) \geq |\lambda|$. We conclude that $r_A(x) \leq \beta_A(x)$. □

We recall that Continuity of the spectrum and spectral radius functions play a crucial role in automatic continuity. Now we have the following theorem.

Theorem 2.2. *Let A be a complete metrizable FLM algebra. Then $\beta_A : A \to [0, +\infty)$ is continuous at zero.*

Proof. Let $(a_n)_n$ be a sequence in A such that $\lim_{n\to\infty} a_n = 0$ and U_0 be a neighborhood of zero in A satisfying Definition 1.2. Then for $\varepsilon > 0$, $2\varepsilon^{-1}a_n \to 0$, so there exists $N_0 \in \mathbb{N}$ such that $2\varepsilon^{-1}a_n \in U_0$ for every $n \geq N_0$. Let V be an arbitrary neighborhood of zero in A. Hence there exists $K_0 \in \mathbb{N}$ such that $U_0^k \subseteq V$ for every $k \geq K_0$. Therefore for every $n \geq N_0$ and for each $k \geq K_0$, $(2\varepsilon^{-1}a_n)^k \in V$. This means that $(2\varepsilon^{-1}a_n)^k \to 0$, as $k \to \infty$ and so

$$|\beta_A(a_n) - \beta_A(0)| = |\beta_A(a_n)| \leqslant \frac{\varepsilon}{2} < \varepsilon,$$

for every $n \geq N_0$. □

Corollary 2.3. *Let A be a complete metrizable FLM algebra. Then $r_A : A \to [0, +\infty)$ is continuous at zero.*

Proof. By applying Theorem 2.1 and Theorem 2.2, we conclude that r_A is continuous at zero. □

Remark 2.4. *Let A be a topological linear normed algebra and $x \in A$. Then $B_A(x) \neq \varnothing$, that is, x is bounded. If $r > 0$ and $\|x\| < r$, then*

$$\|(\frac{x}{r})^n\| \leq \|\frac{x}{r}\|^n \to 0,$$

as $n \to \infty$. Hence $r \in B_A(x)$.

Example 2.1. The algebra $C(\mathbb{R})$ consisting of all continuous complex-valued functions on the real line $\mathbb{R}$ with the sequence $(p_n)_n$ of seminorms denoted by $p_n(f) = \sup_{|x|\leq n} |f(x)|$ is a complete metrizable fundamental topological algebra, but not a complete metrizable FLM algebra. It can be readily concluded that the spectral radius function is not continuous at zero. In general, this example shows that the spectral radius function may be discontinuous at zero.

We know that if A is a Banach algebra we have

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

we generalized it as follow.

Theorem 2.5. *Let A be a topological linear normed algebra. Then*

$$\beta_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}},$$

for every $x \in A$.

Proof. Let $\varepsilon > 0$ and $x \in A$. We know that $\frac{x^n}{(\beta_A(x)+\varepsilon)^n} \to 0$ as $n \to \infty$. Then $\|\frac{x^n}{(\beta_A(x)+\varepsilon)^n}\| \to 0$, so there exists $N_0 \in \mathbb{N}$ such that

$$\|\frac{x^n}{(\beta_A(x)+\varepsilon)^n}\| < 1,$$

for $n \geq N_0$. Then

$$\|x^n\|^{\frac{1}{n}} < \beta_A(x) + \varepsilon,$$

for $n \geq N_0$. Therefore

$$\limsup_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \beta_A(x).$$

On the other hand, take $\|x\| + \varepsilon = \lambda$, then $\|x\| < \lambda$ and

$$\|(\frac{x}{\lambda})^n\| \leq \|\frac{x}{\lambda}\|^n \to 0,$$

as $n \to \infty$. Thus $\beta_A(x) \leq \lambda$. Since ε is arbitrary, it follows that $\beta_A(x) \leq \|x\|$. By applying Theorem 2.1, we have

$$\beta_A(x) = \beta_A(x^n)^{\frac{1}{n}} \leq \|x^n\|^{\frac{1}{n}},$$

hence

$$\beta_A(x) \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

Therefore

$$\beta_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

This complete the proof. □

Now, by introducing the new notion of sub-multiplicative metrizable topological algebra, we generalize some well known properties of Banach algebras to complete metrizable FLM algebras.

Definition 2.6. Let (A, d) be a metrizable topological algebra. We say A is a sub-multiplicative metrizable topological algebra if

$$d(0, xy) \leq d(0, x)d(0, y)$$

for each $x, y \in A$.

Lemma 2.7. *Let (A, d) be a sub-multiplicative metrizable topological algebra and $x \in A$. Then*

$$\beta_A(x) = \lim_{n\to\infty} d(0, x^n)^{\frac{1}{n}}$$

Proof. The proof is similar to Theorem 2.5. □

The following theorem is proved for Banach algebras, it follows from [7, Proposition 2.3.27], and we give for the sake of reader, its proof for FLM algebras.

Theorem 2.8. *Let A be a topological linear normed algebra or sub-multiplicative metrizable topological algebra and $x, y \in A$, such that $xy = yx$. Then*

(i) $\beta_A(xy) \leq \beta_A(x)\beta_A(y)$.

(ii) $\beta_A(x+y) \leq \beta_A(x) + \beta_A(y)$.

Proof. (i) Let $x, y \in A$, then

$$\begin{aligned}\beta_A(xy) &= \lim_{n\to\infty} \|(xy)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|x^n y^n\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} = \beta_A(x)\beta_A(y).\end{aligned}$$

(ii) Let $x, y \in A$, $\beta_A(x) < \alpha$ and $\beta_A(y) < \beta$. Set $\frac{x}{\alpha} = a$ and $\frac{y}{\beta} = b$. By applying Theorem 2.5, there exists $N_0 \in \mathbb{N}$ such that

$$\|a^n\| \leq 1$$

and

$$\|b^n\| \leq 1,$$

for each $n \geq N_0$. Then

$$\begin{aligned}\|(x+y)^n\|^{\frac{1}{n}} = \|\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\|^{\frac{1}{n}} &\leq (\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} \|a\|^k \|b\|^{n-k})^{\frac{1}{n}} \\ &\leq (\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} c_n)^{\frac{1}{n}} \\ &= (\alpha+\beta)(c_n)^{\frac{1}{n}},\end{aligned}$$

for each $n \geq N_0$, where $c_n = \max_{0\leq k\leq n} \|a\|^k \|b\|^{n-k}$ and

$$\limsup_{n\to\infty} (c_n)^{\frac{1}{n}} = 1,$$

then

$$\beta_A(x+y) = \lim_{n\to\infty} \|(x+y)^n\|^{\frac{1}{n}} \leq \alpha + \beta.$$

Therefore $\beta_A(x+y) \leq \beta_A(x) + \beta_A(y)$. □

Corollary 2.9. *Let (A, d) be a commutative sub-multiplicative metrizable topological algebra, then β_A is continuous on A.*

Lemma 2.10. *Let (A, d) be a commutative sub-multiplicative metrizable topological algebra. Then*

$$\mathfrak{B}(A) \subset rad(A).$$

Proof. Let $x \in \mathfrak{B}(A)$ and y be an arbitrary element in A, then

$$r_A(xy) \leq \beta_A(xy) \leq \beta_A(x)\beta_A(y) = 0,$$

hence $r_A(xy) = 0$, so $x \in \text{rad}(A)$. □

In the sequel, we study the automatic continuity of linear maps or homomorphisms between complete metrizable FLM algebras. We note that if A and B are topological algebras and $T : A \to B$ is a continuous homomorphism, then

$$\beta_B(Tx) \leq \beta_A(x),$$

for every $x \in A$.

Theorem 2.11. *Let A be complete metrizable FLM algebra and B a commutative, semisimple sub-multiplicative complete metrizable FLM algebra. Let $T : A \to B$ be a linear map such that $\beta_B(Tx) \leq \beta_A(x)$, for every $x \in A$. Then T is continuous.*

Proof. Let $x_n \to 0$ in A and $T(x_n) \to b$ in B. Since β_A is continuous at zero, by Theorem 2.2, then $\beta_A(x_n) \to \beta_A(0) = 0$. Also β_B is continuous on B, by Corollary 2.9, then $\beta_B(T(x_n)) \to \beta_A(b)$. On the other hand

$$\beta_B(T(x_n)) \leq \beta_A(x_n) \to 0.$$

Then $\beta_B(b) = 0$ and by using Lemma 2.10, we conclude that

$$b \in \text{rad B} = \{0\}.$$

Therefore T is continuous. □

Corollary 2.12. *Let A be complete metrizable FLM algebra and B a commutative, semisimple sub-multiplicative complete metrizable FLM algebra. Let $T : A \to B$ be a linear map such that $r_B(Tx) \leq r_A(x)$, for every $x \in A$. Then T is continuous.*

Proof. Using a similar method as in Theorem 2.11, we conclude that T is continuous. □

Corollary 2.13. *Let A be complete metrizable FLM algebra, B a commutative, semisimple sub-multiplicative complete metrizable FLM algebra and $T : A \to B$ be a homomorphism. Then T is continuous.*

Proof. Let $T : A \to B$ be a homomorphism, then $r_B(Tx) \leq r_A(x)$, for every $x \in A$. By applying Corollary 2.12, T is continuous. □

Theorem 2.14. *Let A be a complete metrizable FLM algebra, $x \in A$, $\beta_A(x) < 1$ and $T : A \to \mathbb{C}$ be a homomorphism. Then $|T(x)| < 1$.*

Proof. Let $\beta_A(x) < 1$, so there exists $b > 1$ such that $b^n x^n \to 0$, as $n \to \infty$. First, we prove that $T(x) \neq 1$. Set $s_n = \sum_{k=1}^{n} x^k$, now $\lambda^n(s_n - s_{n-1}) \to 0$, as $n \to \infty$. By Definition 1.1, $(s_n)_n$ is a Cauchy sequence. Let $\lim_{n\to\infty} s_n = s$, using a similar method as in Theorem 2.1, we conclude that $x \diamond -s = x - s + xs = 0$. Thus $x = s - xs$. If $T(x) = 1$, then $1 = 0$, hence $T(x) \neq 1$. Let $|T(x)| > 1$, take $x_0 = \frac{x}{T(x)}$. We know that $T(x_0) = 1$, on the other hand

$$b^n x_0^n = \frac{1}{T(x)^n} b^n x^n \to 0,$$

by the above argument it follows that $T(x_0) \neq 1$, which is impossible. Therefore $|T(x)| < 1$. □

Theorem 2.15. [3, Theorem 4.5] *Let A be a complete metrizable FLM algebra and $T : A \to \mathbb{C}$ be a homomorphism. Then T is continuous.*

We now extend Theorem 2.15 as follows.

Theorem 2.16. *Let A be a complete metrizable FLM algebra, B be a commutative semi simple Banach algebra and $T : A \to B$ be a homomorphism. Then T is continuous.*

Proof. Let T be a homomorphism and $\varphi \in M(B)$. Then, $\varphi \circ T : A \to \mathbb{C}$ is a continuous homomorphism by Theorem 2.15. Now, suppose that $a_n \to 0$ in A and $Ta_n \to b$ in B. Then, by the continuity of $\varphi \circ T$, we have $(\varphi \circ T)(a_n) \to 0$. On the other hand, by the continuity of $\varphi : B \to \mathbb{C}$, $(\varphi \circ T)(a_n) = \varphi(Ta_n) \to \varphi(b)$. Consequently, $\varphi(b) = 0$ and since $\varphi \in M(B)$ was arbitrary, we get

$$b \in \bigcap_{\varphi \in M(B)} ker\varphi = radB = \{0\}.$$

Therefore, by the Closed Graph Theorem, T is continuous. □

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(Received 18 August 2018)
(Accepted 23 March 2019)