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Fixed Points of Demicontinuous ϕ-Nearly Lipschitzian Mappings in Banach Spaces

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Abstract : In this study, we introduce the classes of ϕ-nearly contraction mappings, ϕ-nearly nonexpansive mappings, ϕ-nearly asymptotically nonexpansive mappings, ϕ-nearly uniformly k-Lipschitzian mappings and ϕ-nearly uniform k-contraction mappings. These classes include those classes studied by Sahu [1] as special cases. We study the existence of fixed points and the structure of their fixed point sets of mappings in Banach spaces. Our results improve and generalize many celebrated results of fixed point theory in the context of demicontinuity.

Keywords : demicontinuity, ϕ-nearly Lipschitzian mappings; ϕ-nearly contraction mappings; ϕ-nearly nonexpansive mappings; Banach spaces.
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1 Introduction

Let E be a real Banach space, C a nonempty subset of E and $\phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We associate a ϕ-normalized duality mapping $J_\phi : E \to 2^{E^*}$ to

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the function ϕ defined by

$$J_\phi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\phi(\|x\|) \text{ and } \|f^*\| = \phi(\|x\|)\}, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle ., . \rangle$ denotes the duality pairing. We shall denote a single-valued duality mapping by j_ϕ. If $\phi(t) = t$, then J_ϕ reduces to the usual duality mapping J.

The following relationship exists between J_ϕ and J, which was shown in [2].

$$J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|} J(x) \ \ \forall \ x \neq 0. \tag{1.2}$$

The concepts of *nearly Lipschitzian mappings, nearly contraction mappings, nearly nonexpansive mappings, nearly asymptotically nonexpansive mappings, nearly uniformly k-Lipschitzian mappings* and *nearly uniform k-contraction mappings* were introduced by Sahu [1] in 2005. Our interest in this paper is to develop asymptotic fixed point theory for the class of ϕ-nearly Lipschitzian mappings in Banach spaces. It is shown in this study that Banach contraction mapping principle [3] can be improved for demicontinuous ϕ-nearly contraction mappings in Banach spaces. It is also shown that the continuity of nonexpansive mappings from the results of Browder [4] and Kirk [5] can be weakened to demicontinuity for ϕ-nearly nonexpansive mappings in uniformly convex Banach spaces. The results of this paper generalize the results of Sahu [1] and the references therein.

Definition 1.1. Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \ \ \forall \ \ x, y \in C. \tag{1.3}$$

The infimum of constants k_n for which (1.3) holds will be denoted by $\eta(T^n)$ and called *nearly Lipschitz constant*. Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in C, x \neq y\right\}. \tag{1.4}$$

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
(ii) *nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,
(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,
(iv) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,
(v) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Sahu [1] provided the following example of a map which is nearly nonexpansive but not continuous and non-Lipschitzian.

Example 1.2. [1] Let $E = \mathbb{R}$, $C = [0,1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Definition 1.3. [2] Let $T : C(T) \subset E \to E$ be a mapping with domain $C(T)$, $\phi : \mathbb{R}^+ = [0,\infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$ and $F(T) := \{x \in C(T) : Tx = x\}$ be the nonempty set of fixed points of T. T is said to be *ϕ-uniformly L-Lipschitzian* if there exists $L > 0$ such that for all $x, y \in C(T)$

$$\|T^n x - T^n y\| \leq L.\phi(\|x - y\|). \tag{1.5}$$

Inspired by the facts above, we now introduce the following classes of nonlinear mappings.

Definition 1.4. Let C be a nonempty subset of a Banach space E, $\phi : \mathbb{R}^+ = [0,\infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$ and fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$. A mapping $T : C \to C$ will be called *ϕ-nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n.\phi(\|x - y\| + a_n) \ \ \forall \ \ x, y \in C. \tag{1.6}$$

The infimum of constants k_n for which (1.6) holds will be denoted by $\eta(T^n)$ and called *ϕ-nearly Lipschitz constant.* Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\phi(\|x - y\| + a_n)} : x, y \in C, x \neq y\right\}. \tag{1.7}$$

A ϕ-nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *ϕ-nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,

(ii) *ϕ-nearly nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$,

(iii) *ϕ-nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,

(iv) *ϕ-nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(v) *ϕ-nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Observe that if ϕ is identity in Definition 1.4, then we obtain the concepts introduced by Sahu [1] (see Definition 1.1 above).

The following definition will be needed in this study.

Definition 1.5. [1] Let C be a nonempty subset of a Banach space E and $T : C \to C$ a mapping. T is said to be *demicontinuous* if whenever a sequence $\{x_n\}$ in C converges strongly to $x \in C$, then $\{Tx_n\}$ converges weakly to Tx.

Let C be a nonempty closed convex subset of a uniformly convex Banach space E, $\{x_n\}$ a bounded sequence in E and $r : C \to [0, \infty)$ a functional defined by

$$r(x) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in C. \tag{1.8}$$

The infimum of $r(.)$ over C is called *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r(C, \{x_n\})$. A point $z \in C$ is said to be an *asymptotic centre* of the sequence $\{x_n\}$ with respect to C if

$$r(z) = \inf\{r(x) : x \in C\}. \tag{1.9}$$

The set of all asymptotic centres is denoted by $A(C, \{x_n\})$.

It is known that every bounded sequence $\{x_n\}$ in a uniformly convex Banach space E has a unique asymptotic centre with respect to any closed convex subset C of E (see, for example [1]), i.e.,

$$A(C, \{x_n\}) = \{z\}$$

and

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - x\| \quad \forall x \neq z.$$

A convex subset C of a Banach space E is said to have the *approximate fixed point property* ($AFPP$) for a nonexpansive mapping $T : C \to C$ if

$$\inf\{\|x - Tx\| : x \in C\} = 0, \tag{1.10}$$

i.e., there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Such a sequence $\{x_n\}$ is called *approximate fixed point sequence* of T.

It is well known that every closed convex bounded subset C of E has approximating fixed point property ($AFPP$) for nonexpansive mappings (see Sahu [1]).

Recently, Olaleru and Okeke [6] introduced the classes of asymptotically demicontractive mappings in the intermediate sense and asymptotically hemicontractive mappings in the intermediate sense. We established some fixed point results for these classes of nonlinear mappings. Qin *et al.* [7] obtained interesting fixed point results for the class of asymptotically pseudocontractive mappings in the intermediate sense. It is our purpose in this study to prove some existence results for the classes of nonlinear mappings introduced in this study.

The following lemmas will be needed in this study.

Lemma 1.6. [8] *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, $\{x_n\}$ a bounded sequence in E and $A(C, \{x_n\}) = \{z\}$. Then we have*

$$(\{y_n\} \subset C \text{ and } r(y_m) \to r(C, \{x_n\}) \text{ as } m \to \infty) \implies (y_m \to z \text{ as } m \to \infty).$$

Lemma 1.7. [1] *Let C be a nonempty subset of a Banach space and let $T : C \to C$ be demicontinuous. Suppose that $T^n u \to x^*$ as $n \to \infty$ for some $u, x^* \in C$. Then x^* is an element of $F(T)$.*

We hereby give an example of some of the nonlinear mappings introduced in this study.

Example 1.8. Let $X = \mathbb{R}$, $C = [0,1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{\frac{1}{2^{2n}}}{1+\frac{1}{2^n}} & \text{if } x \in [0, \frac{1}{2}] \\ \\ 0 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{1+t}$. Clearly, ϕ is strictly increasing and $\phi(0) = 0$. Observe that T is not continuous and non-Lipschitzian. We now show that T is ϕ-nearly nonexpansive, with sequence $\{a_n\} = \frac{1}{2^n} \to 0$ as $n \to \infty$, we have for each $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\| + a_n$$

and for each $x, y \in C$ and $n \in [2, \infty)$, we obtain

$$\|T^n x - T^n y\| \leq \frac{\frac{1}{2^{2n}}}{1+\frac{1}{2^n}} + a_n \leq \frac{\frac{1}{2^n}\cdot\frac{1}{2^n}}{1+\frac{1}{2^n}} + a_n \leq \phi(\|x - y\| + a_n).$$

Hence, T is ϕ-nearly nonexpansive.

2 Main Results

We shall adopt the method of proof of Sahu [1]. The following lemma will be needed in our main results.

Lemma 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) \leq 1$. If $\{y_n\}$ is a bounded sequence in C such that*

$$\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0 \text{ and } A(C, \{y_n\}) = \{x^*\},$$

then x^* is a fixed point of T.

Proof. Define a sequence $\{x_n\}$ in C by

$$x_m = T^m x^*, \quad m \in \mathbb{N}. \tag{2.1}$$

Using (1.6), we have that for each $m, n \in \mathbb{N}$,

$$\begin{aligned} \|x_m - y_n\| &\leq \|T^m x^* - T^m y_n\| + \|T^m y_n - y_n\| \\ &\leq \eta(T^m).\phi(\|x^* - y_n\| + a_m) + \|T^m y_n - y_n\|. \end{aligned} \tag{2.2}$$

We define a functional $r : C \to \mathbb{R}^+$ by

$$r(y) = \limsup_{n\to\infty} \|y_n - y\|, \quad y \in C. \tag{2.3}$$

Using (2.2), we obtain

$$\begin{aligned} r(x_m) &= \limsup_{n\to\infty} \|y_n - x_m\| \\ &= \limsup_{n\to\infty} \|y_n - T^m y_n + T^m y_n - T^m x^*\| \\ &\leq \limsup_{n\to\infty} \|T^m y_n - T^m x^*\| + \limsup_{n\to\infty} \|y_n - T^m y_n\| \\ &\leq \eta(T^m).\phi(r(x^*) + a_m) + \limsup_{n\to\infty} \|y_n - T^m y_n\| \\ &\longrightarrow \phi(r(x^*)) \;\text{ as }\; m\to\infty, \end{aligned} \tag{2.4}$$

since ϕ is a strictly increasing function with $\phi(0) = 0$. Using Lemma 1.6, we have $T^m x^* \to x^*$. From Lemma 1.7, we conclude that $x^* \in F(T)$. □

2.1 The Concept of ϕ-Nearly Contraction Mappings for Existence of Fixed Points

In this subsection, we obtain existence and uniqueness of fixed points of demicontinuous ϕ-nearly Lipschitzian mappings in a general Banach space.

Theorem 2.2. *Let C be a nonempty closed subset of a Banach space E and let $T : C \to C$ be a demicontinuous ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$. Suppose $\eta_\infty(T) = \limsup_{n\to\infty}[\eta(T^n)]^{\frac{1}{n}} < 1$. Then we have the following:*

(a) T has a unique fixed point $x^* \in C$;

(b) for each $x_0 \in C$, the sequence $\{T^n x_0\}$ converges strongly to x^*;

(c) $\|T^n x_0 - x^*\| \leq \phi(\|x_0 - Tx_0\| + M)\sum_{i=n}^{\infty}\eta(T^i)$ for all $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}} a_n$.

Proof. (a) Fix $x_0 \in C$ and let $x_n = T^n x_0$, $n \in \mathbb{N}$. We set $d_n := \|x_n - x_{n+1}\|$. Hence, from (1.6) we obtain

$$d_n = \|T^n x_0 - T^{n+1}x_0\| \leq \eta(T^n).\phi(\|x_0 - Tx_0\| + a_n). \tag{2.5}$$

From (2.5), we have

$$\sum_{n=1}^{\infty} d_n \leq \phi(d_0 + M)\sum_{n=1}^{\infty}\eta(T^n), \tag{2.6}$$

for $M > 0$, since $\lim_{n\to\infty} a_n = 0$ and ϕ is a strictly increasing function with $\phi(0) = 0$. Using the Root Test for convergence of series, if $\eta_\infty(T) = \limsup_{n\to\infty}[\eta(T^n)]^{\frac{1}{n}} < 1$, then we have $\sum_{n=1}^{\infty}\eta(T^n) < \infty$. It follows that $\sum_{n=1}^{\infty} d_n < \infty$ and so we see that $\{x_n\}$ is a Cauchy sequence. Thus, $\lim_{n\to\infty} x_n$ exists (say $x^* \in C$). Using Lemma 1.7, we have that x^* is a fixed point of T. Next, we prove the uniqueness of x^*. Let w be another fixed point of T such that $x^* \neq w$. Then we have

$$\begin{aligned} \infty &= \sum_{n=1}^{\infty}\|x^* - w\| \\ &= \sum_{n=1}^{\infty}\|T^n x^* - T^n w\| \\ &\leq \sum_{n=1}^{\infty}\eta(T^n).\phi(\|x^* - w\| + a_n) \\ &\leq \phi(\|x^* - w\| + M)\sum_{n=1}^{\infty}\eta(T^n) \\ &< \infty, \end{aligned} \tag{2.7}$$

a contradiction. Hence, T has a unique fixed point $x^* \in C$.
(b) It follows easily from (a).
(c) For each $m \in \mathbb{N}$, we obtain

$$\begin{aligned} \|x_n - x_{n+m}\| &= \|T^n x_0 - T^{n+m} x_0\| \\ &\leq \textstyle\sum_{i=n}^{n+m-1} \|T^i x_0 - T^{i+1} x_0\| \\ &\leq \textstyle\sum_{i=n}^{n+m-1} \eta(T^i).\phi(\|x_0 - Tx_0\| + a_i) \\ &\leq \phi(\|x_0 - Tx_0\| + M) \textstyle\sum_{i=n}^{n+m-1} \eta(T^i). \end{aligned} \tag{2.8}$$

Letting $m \to \infty$, (2.8) gives

$$\|x_n - x^*\| \leq \phi(\|x_0 - Tx_0\| + M) \sum_{i=n}^{\infty} \eta(T^i). \tag{2.9}$$

Hence, the proof of Theorem 2.2 is completed. □

Remark 2.3. Theorem 2.2 is a generalization of Banach contraction principle [3]. It includes Theorem 3.1 of Sahu [1] as a special case.

Theorem 2.4. *Let C be a nonempty closed subset of a Banach space and $T : C \to C$ a ϕ-nearly contraction map with sequence $\{(a_n, \eta(T^n))\}$ such that*

$$\lim_{n\to\infty} \frac{a_n}{\eta(T^n)^{-1} - 1} = 0.$$

Then $F(T)$ has at most one element.

Proof. Suppose that x and y are two distinct elements of $F(T)$. Then

$$\|x - y\| = \|T^n x - T^n y\| \leq \eta(T^n).\phi(\|x - y\| + a_n). \tag{2.10}$$

Since ϕ is a strictly increasing function, then from (2.10) we obtain

$$\|x - y\| \leq \frac{a_n}{\eta(T^n)^{-1} - 1} \longrightarrow 0 \text{ as } n \to \infty. \tag{2.11}$$

Hence, T has at most one fixed point. □

Corollary 2.5. *Let C be a nonempty closed subset of a Banach space and $T : C \to C$ a ϕ-nearly uniform k-contraction. Then $F(T)$ has at most one element.*

Proposition 2.6. *If $T : C \to C$ is a ϕ-nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$, then*

$$\|T^n x - T^n y\| \leq \max\left\{\phi(\|x - y\|), \tfrac{\phi(a_n)}{\eta(T^n)^{-1} - 1}\right\} \textit{ for all } x, y \in C \textit{ and } n \in \mathbb{N}.$$

Proof. Observe that

$$\eta(T^n).\phi(\|x-y\|+a_n) \leq \phi(\|x-y\|) \Leftrightarrow \phi(\|x-y\|) \geq \frac{\phi(a_n)}{\eta(T^n)^{-1}-1}. \tag{2.12}$$

If $\phi(\|x-y\|) \geq \frac{\phi(a_n)}{\eta(T^n)^{-1}-1}$, then

$$\|T^n x - T^n y\| \leq \eta(T^n).\phi(\|x-y\|+a_n) \leq \frac{\phi(a_n)}{\eta(T^n)^{-1}-1}. \tag{2.13}$$

Since ϕ is a strictly increasing function, then Proposition 2.6 follows. □

The following results show that Banach spaces have $AFPP$ for the class of ϕ-nearly contraction mappings.

Theorem 2.7. *Let C be a nonempty closed convex bounded subset of a Banach space E and $T : C \to C$ a ϕ-nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \frac{a_n}{\eta(T^n)^{-1}-1} = 0$. Then*

$$\lim_{n\to\infty}(\inf\{\|x-T^n x\| : x \in C\}) = 0,$$

i.e., there exists a sequence $\{x_m\}$ in C such that

$$\lim_{n\to\infty}(\lim_{m\to\infty}\|x_m - T^n x_m\|) = 0.$$

Proof. Suppose that $\{t_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 1$. We have that for each $n \in \mathbb{N}$ and some $u \in C$, define

$$T_n x = (1-t_n)u + t_n T^n x, \quad x \in C. \tag{2.14}$$

Clearly,

$$\|T_n x - T_n y\| \leq t_n\|T^n x - T^n y\| \leq \max\left\{t_n.\phi(\|x-y\|), \frac{t_n a_n}{\eta(T^n)^{-1}-1}\right\}, \tag{2.15}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
\|T_n^\ell x - T_n^\ell y\| &\leq \max\left\{t_n\|T_n^{\ell-1}x - T_n^{\ell-1}y\|, \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\
&\leq \max\left\{t_n.\phi(\|x-y\|+a_n), \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\
&\leq \max\left\{t_n \max\left\{t_n\|T_n^{\ell-2}x - T_n^{\ell-2}y\|, \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\}, \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\
&= \max\left\{t_n^2\|T_n^{\ell-2}x - T_n^{\ell-2}y\|, \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\
&\leq \max\left\{t_n^2.\phi(\|x-y\|+a_n), \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\} \\
&\;\vdots \\
&\leq \max\left\{t_n^\ell.\phi(\|x-y\|+a_n), \tfrac{t_n a_n}{\eta(T^n)^{-1}-1}\right\}.
\end{aligned} \tag{2.16}$$

Since ϕ is a strictly increasing function, C is bounded and $\lim_{\ell\to\infty} t_n^\ell = 0$ for each $n, \ell \in \mathbb{N}$,

$$\|T_n^\ell x - T_n^{\ell+1} x\| \leq \max\left\{t_n^\ell.\phi(diam\ C), \frac{t_n a_n}{\eta(T^n)^{-1} - 1}\right\} \longrightarrow \frac{t_n a_n}{\eta(T^n)^{-1} - 1} \qquad (2.17)$$

as $\ell \to \infty$. From (2.17), we obtain

$$\inf\{\|x - T_n x\| : x \in C\} \leq \frac{t_n a_n}{\eta(T^n)^{-1} - 1}. \qquad (2.18)$$

Hence, using (2.14) we have

$$\begin{aligned} \|x - T^n x\| &\leq t_n^{-1}\|x - T_n x\| + t_n^{-1}(1 - t_n)\|x - u\| \\ &\leq t_n^{-1}\|x - T_n x\| + t_n^{-1}(1 - t_n).\phi(diam(C)). \end{aligned} \qquad (2.19)$$

From (2.19), we obtain

$$\inf\{\|x - T^n x\| : x \in C\} \leq \frac{a_n}{\eta(T^n)^{-1} - 1} + t_n^{-1}(1 - t_n).\phi(diam(C)). \qquad (2.20)$$

Hence, the proof of Theorem 2.7 is completed. □

Next, we obtain a more general result by establishing the fact that every demicontinuous ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) \leq 1$ has a fixed point in a uniformly convex Banach space.

Theorem 2.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) \leq 1$. Then the following statements are equivalent:*

(a) T has a fixed point;

(b) there exists a bounded sequence $\{T^n x_0\}$ in C;

(c) there exists a bounded sequence $\{y_n\}$ in C such that

$$\lim_{m\to\infty}(\lim_{n\to\infty}\|y_n - T^m y_n\|) = 0.$$

Proof. It is easy to show that (a) ⇒ (b) since $T^n x_0 = x_0$ for each $n \in \mathbb{N}$ and (a) ⇒ (c) since

$$\begin{aligned} \lim_{m\to\infty}(\lim_{n\to\infty}\|y_n - T^m y_n\|) &= (\lim_{n\to\infty}\|y_n - y_n\|) \\ &= 0 \ \text{ for each } \ n \in \mathbb{N}. \end{aligned}$$

We now show that
(b) ⇒ (a). Suppose that the sequence $\{T^n x_0\}$ is bounded and $A(C, \{T^n x_0\}) = \{z\}$. Since, for $n \geq m \geq 1$

$$\|T^n x_0 - T^m z\| \leq \eta(T^m).\phi(\|T^{n-m} x_0 - z\| + a_m), \qquad (2.21)$$

From (2.21), we obtain

$$\limsup_{n\to\infty} \|T^n x_0 - T^m z\| \leq \eta(T^m).\phi(\limsup_{n\to\infty} \|T^n x_0 - z\| + a_m). \tag{2.22}$$

Therefore, $r(T^m z) \to r(C, \{T^n x_0\})$ as $m \to \infty$. Using Lemma 1.1 we obtain $T^m z \to z$ as $m \to \infty$. It follows from Lemma 1.2 that z is a fixed point of T, meaning that (b) $\Rightarrow$ (a).
(c) $\Rightarrow$ (a). Let $\{y_n\}$ be a bounded sequence in C such that

$$\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0.$$

The result follows from Lemma 2.1. □

We now combine Theorems 2.7 and 2.8 and establish that the condition $\frac{a_n}{\eta(T^n)^{-1}-1} \to 0$ also guarantees the existence of fixed points of ϕ-nearly contraction mappings in uniformly convex Banach spaces.

Theorem 2.9. *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \frac{a_n}{\eta(T^n)^{-1}-1} = 0$. Then T has a unique fixed point.*

Proof. Using Theorem 2.7, it follows that there exists a sequence $\{y_n\}$ in C such that $\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0$. The result follows from Theorem 2.8. □

The following corollary is a generalization of the results of Goebel and Kirk [9].

Corollary 2.10. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly asymptotically nonexpansive mapping. If there is a point $x_0 \in C$ such that $\{T^n x_0\}$ is bounded, then T has a fixed point in C.*

Remark 2.11. Theorems 2.4, 2.7, 2.8 and 2.9 includes Theorems 3.4, 3.7, 3.8 and 3.9 of Sahu [1] as special cases.

2.2 Structure of Set of Fixed Points of Demicontinuous ϕ-Nearly Lipschitzian Mappings

Theorem 2.12. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) = 1$. Then $F(T)$ is closed and convex.*

Proof. Closedness of $F(T)$: Let $\{z_n\}$ be a sequence in $F(T)$ such that $z_n \to z$. We are to show that $z \in F(T)$. Observe that

$$\|z_n - T^n z\| = \|T^n z_n - T^n z\| \leq \eta(T^n).\phi(\|z_n - z\| + a_n), \tag{2.23}$$

from (2.23), we obtain

$$\limsup_{n\to\infty} \|z_n - T^n z\| = 0. \tag{2.24}$$

Since ϕ is a strictly increasing function and

$$\phi(\|z - T^n z\|) \leq \phi(\|z - z_n\|) + \phi(\|z_n - T^n z\|), \tag{2.25}$$

we obtain

$$\limsup_{n\to\infty} \|z - T^n z\| = 0. \tag{2.26}$$

Using Lemma 1.7, we obtain $z \in F(T)$, meaning that $F(T)$ is closed.

Convexity of $F(T)$: Let $x, y \in F(T)$ such that $x \neq y$. Let $z = \frac{1}{2}(x+y)$. Then we obtain

$$\|T^n z - x\| = \|T^n z - T^n x\| \leq \eta(T^n).\phi(\|z - x\| + a_n) \leq \eta(T^n).\phi(\frac{1}{2}\|x - y\| + a_n). \tag{2.27}$$

and

$$\|T^n z - y\| \leq \eta(T^n).\phi(\frac{1}{2}\|x - y\| + a_n). \tag{2.28}$$

Hence,

$$\begin{aligned} \|T^n z - z\| &= \|\tfrac{1}{2}(T^n z - x) + \tfrac{1}{2}(T^n z - y)\| \\ &\leq \eta(T^n).\phi(\tfrac{1}{2}\|x - y\| + a_n)\left\{1 - \delta\left(\frac{\phi(\|x-y\|)}{\eta(T^n).\phi(\frac{1}{2}\|x-y\|+a_n)}\right)\right\} \end{aligned}$$

for every $n \in \mathbb{N}$, where δ is modulus of convexity of E. It follows that

$$\lim_{n\to\infty} \|T^n z - z\| = 0,$$

hence $z \in F(T)$ using Lemma 1.7. □

Corollary 2.13. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly contraction mapping with sequence $\{(a_n, \eta(T^n))\}$. Then $F(T)$ is closed and convex.*

Corollary 2.14. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a demicontinuous ϕ-nearly asymptotically nonexpansive mapping. Then $F(T)$ is closed and convex.*

A subset $F \subseteq C$ is said to be a *1-local retract* of C [10] if every family $\{B_i : i \in I\}$ of closed balls centered at points of F has the property:

$$\left(\bigcap_{i\in I} B_i\right) \cap C \neq \emptyset \implies \left(\bigcap_{i\in I} B_i\right) \cap F \neq \emptyset.$$

It is easy to see that a 1-local retract of a convex set is metrically convex and 1-local retract of a closed set must itself be closed.

The next result shows that the fixed point set of a ϕ-nearly Lipschitzian mapping is a 1-local retract of its domain.

Theorem 2.15. *Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ a ϕ-nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\limsup_{n\to\infty} \eta(T^n) \leq 1$. Suppose that each closed convex subset D of C has the fixed point property for T. Then $F(T)$ is a (nonempty) 1-local retract of C.*

Proof. By assumption $F(T) \neq \emptyset$. Let $\{B(x_i, r_i) : i \in I\}$ be a family of closed balls centered at points $x_i \in F(T)$. Suppose

$$S_0 = \left(\bigcap_{i\in I} B(x_i, r_i)\right) \cap C \neq \emptyset, r(\{T^n x\}, x_i) = \limsup_{n\to\infty} \|T^n x - x_i\|,\ x \in C \quad (2.29)$$

and

$$S_1 = \{x \in C : r(\{T^n x\}, x_i) \leq r_i\}.$$

Let $x \in S_0$; then we obtain

$$\begin{aligned} r(\{T^n x\}, x_i) &= \limsup_{n\to\infty} \|T^n x - x_i\| \\ &= \limsup_{n\to\infty} \|T^n x - T^n x_i\| \\ &\leq \limsup_{n\to\infty} \eta(T^n).\phi(\|x - x_i\| + a_n) \\ &\leq \phi(\|x - x_i\|) \\ &\leq \phi(r_i). \end{aligned} \quad (2.30)$$

It follows that $x \in S_1$, i.e., $S_0 \subseteq S_1 \neq \emptyset$.

Let $x, y \in S_1$ such that $z = (1-t)x + ty$, $t \in [0, 1]$. We obtain

$$\begin{aligned} r(\{T^n z\}, x_i) &\leq \limsup_{n\to\infty} \|T^n z - x_i\| \\ &\leq \|z - x_i\| \\ &\leq (1-t)\|x - x_i\| + t\|y - x_i\| \\ &\leq r_i, \end{aligned} \quad (2.31)$$

hence $z \in S_1$. So we obtain that S_1 is convex.

Now let $\{y_m\}$ be a sequence in S_1 such that $y_m \to y$ as $m \to \infty$. We claim that $y \in S_1$. So we have

$$r(\{T^n y_m\}, x_i) \leq r_i \ \ \forall \ \ i \in I.$$

Hence for each $m \in \mathbb{N}$,

$$\begin{aligned} \limsup_{n\to\infty} \|T^n y - x_i\| &\leq \limsup_{n\to\infty} \|T^n y - T^n y_m\| \\ &\quad + \limsup_{n\to\infty} \|T^n y_m - x_i\| \\ &\leq \limsup_{n\to\infty} \eta(T^n).\phi(\|y - y_m\| + a_n) + r_i \\ &\leq \phi(\|y - y_m\|) + r_i \\ &\leq r_i \end{aligned} \quad (2.32)$$

as $m \to \infty$, it follows that $y \in S_1$. Hence S_1 is a nonempty closed and convex subset of C. Moreover, T is self-mapping on S_1. Hence $S_1 \cap F(T)$ is nonempty by assumption. Suppose that p is an element of $S_1 \cap F(T)$. Then, we obtain

$$r(\{T^n p\}, x_i\}) = r(\{p\}, x_i) = \|p - x_i\| \leq \phi(r_i),$$

this implies that

$$S_1 \cap F(T) \subseteq S_0 \cap F(T).$$

We observe that $S_0 \cap F(T) \subset S_1 \cap F(T)$ since $S_0 \subseteq S_1$. Hence, $S_0 \cap F(T) = S_1 \cap F(T) \neq \emptyset$. □

3 Numerical Example

In this section, we give numerical example to illustrate our results.

Example 3.1. Let $X = \mathbb{R}$, $C = [0, 1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases} \tag{3.1}$$

It was shown in [1] that T is nearly nonexpansive.

Let D be a nonempty convex subset of a Banach space. Let $T : D \to D$ be a mapping, the Mann iteration scheme, is defined by

$$\begin{cases} x_0 \in D \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \end{cases} \tag{3.2}$$

where $\alpha_n \in [0, 1]$, $n \geq 0$. Choose $\alpha_n = 1 - \frac{1}{n^2}$, we obtain

$$x_{n+1} = \frac{x}{n^2} - \frac{1}{n^2} + \frac{1}{2}. \tag{3.3}$$

Hence, it converges to $p = \frac{1}{2}$, which is the unique fixed point of T.

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