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Coincidence and Common Fixed Points for Generalized Multivalued Nonexpansive Relative Maps

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Abstract : In this paper first, we prove the existence coincidence points under certain generalized contraction and nonexpansive relative maps, where f, g and S, T are single-valued and multi-valued mappings, respectively. As applications, related common fixed point and coincidence point results are established. Our results unify, and generalize various know results existing in the literature.

Keywords : coincidence point, common fixed point, multi-valued mapping, weakly commuting.
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1 Introduction and Preliminaries

Let (X, d) be a complete metric space. We denote by $CL(X)(resp.CB(X), K(X), KC(X))$ the family of all nonempty closed (resp. nonempty closed bounded, nonempty compact, nonempty compact convex) subset of X, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for $A, B \in CB(X)$, where $d(x, E) = \inf\{d(x, y) : y \in E\}$ is the distance from x to $E \subset X$.

Let M be a subset of a normed space X. Let $f : M \to M$. A mapping $T : M \to CL(M)$ is called *f-nonexpansive* if $H(Tx, Ty) \leq \|fx - fy\|$ hold for all $x, y \in M$. A point $x \in M$ is called a *coincidence point* (respectively *common fixed point*) of f and T if $fx \in Tx$ (respectively $x = fx \in Tx$). The set of coincidence points of f and T is denoted by $C(T, f)$. The set of fixed points of T (respectively f) is represented by $F(T)$ (respectively f). The maps $f, g : X \to X$ and S, T :

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$X \to CL(X)$ are called *nonexpansive relative* to f and g if $H(Sx, Ty) \leq \|fx - gy\|$ for all $x, y \in M$.

The maps $T : X \to CL(X)$ and $f : X \to X$ are *reciprocally continuous* on X if $fTx \in CL(X)$ for each $x \in X$ and $\lim_n fTx_n = fM, \lim_n Tfx_n = ft$ where $\{x_n\}$ is a sequence in X such that

$$\lim_n Tx_n = M \in CL(M), \quad \lim_n fx_n = t \in M.$$

For self-maps $f, g : X \to X$, this definition due to Pant [5] reads: f and g are reciprocal continuous if and only if $\lim_n gfx_n = gt$ and $\lim_n fg_n = ft$ where $\{x_n\} \subset X$ is such that $\lim_n gx_n = \lim_n fx_n = t \in X$. Clearly, any continuous pair is reciprocally continuous but, the converse in not true (see [7, Example 2.2]).

The pair (T, f) is call *compatible* if $fTx \in CL(X)$ for each $x \in X$ and $\lim_n H(Tfx_n, fTx_n) = 0$ where $\{x_n\}$ is a sequence in X such that $\lim_n Tx_n = M \in CL(X)$ and $\lim_n fx_n = t \in M$.

The pair (T, f) is called *commuting* if $Tfx = fTx$ for all $x \in M$. The pair (T, f) is called *weakly compatible* if f and T commute at there coincidence point. The mapping f is called T-weakly commuting if for all $x \in M$, $ffx \in Tfx$. A subset M of X is said to be star-shaped with respect to $q \in M$ if $\{(1-t)x + tq : 0 \leq t \leq 1\} \subset M$.

A Banach space X is said to satisfy *Opial's condition* if for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| \leq \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \neq x$. The map $T : M \to CL(X)$ is said to be *demiclosed* at 0 if for every sequence $\{x_n\}$ in M and $\{y_n\}$ in X with $y_n \in Tx_n$ such that $\{x_n\}$ converging weakly to x and $\{y_n\}$ converges to $0 \in X$, then $0 \in Tx$.

Lemma 1.1 Latif cf.[4] *Let M be a nonempty weakly compact subset of a Banach space X satisfying Opial's condition. Let $f : M \to M$ be a weakly continuous mapping and $T : M \to K(M)$ an f-nonexpansive map. Then $(f-T)$ is demiclosed.*

Lemma 1.2 [7, Corollary 3.3] *Let (X, d) be a metric space and $S, T : X \to CL(X), f, g : X \to X$ such that*

(i) $S(X) \subset g(X), T(X) \subset f(X)$, and the pair (S, f) is compatible and reciprocally continuous.
If there exists $q \in (0, 1)$ such that

(ii) $H(Sx, Ty) \leq qm(x, y)$ for $x, y \in X$,
where $m(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\}$,

then $C(S, f)$ and $C(T, g)$ are nonempty.

Latif and Tweddle [4] established some coincidence point theorems for f-nonexpansive mappings using the commutativity condition of maps. Afterwards, Shahzad and Hussain [6] extended and improved the above mentioned results.

The aim of this paper we consider a very general type of condition involving two multi-valued mappings and two single-valued mappings and establish coincidence and fixed point theorems which improve, extend and unify some coincidence theorems. Our results are generalization and improvement of the corresponding results of Shahzad and Hussian [6], Khan, et al. [3], and many authors.

2 Coincidence and Common Fixed Point Theorems

Let M be a subset of a normed space X. The set M is call *generalized q-starshaped* if there exists $q \in M$ and a fixed sequence $\{k_n\}$ with $0 < k_n < 1$ converging to 1 such that $(1-k_n)q + k_nTx \subset M$ for each $x \in M$.

Theorem 2.1 *Let M be a nonempty complete and generalized q-starshaped subset of a normed space X, and $f, g : M \to M$ such that surjective. Assume that $S, T : M \to CL(M)$ are generalized nonexpansive relative i.e.,*

$$H(Sx,Ty) \leq \max\Big\{ \|fx-gy\|, dist(fx,S_\lambda x), dist(gy,T_\lambda y), \frac{1}{2}[dist(fx,T_\lambda y)+dist(gy,S_\lambda x)]\Big\} \tag{2.1}$$

for all $x, y \in M$, where $T_\lambda x = (1-\lambda)q + \lambda Tx = [q, Tx]$ and $S_\lambda x = (1-\lambda)q + \lambda Sx = [q, Sx]$ for all $\lambda \in [0,1]$. Suppose that $T(M)$ is bounded and $(f-S)(M), (g-T)(M)$ is closed, and the pair (S, f) is reciprocally continuous and nonvacuously compatible, then $C(S,f) \neq \emptyset$ and $C(T,g) \neq \emptyset$.

Proof Take $q \in M$ and define $S_n, T_n : M \to CL(M)$ by

$$S_nx = k_nSx + (1-k_n)q \quad \text{and} \quad T_nx = k_nTx + (1-k_n)q \tag{2.2}$$

for all $x \in M$ and fixed sequence of real numbers $k_n, 0 < k_n < 1$ converging to 1. Then, for each $n, T_n(M) \subset M = f(M)$ and $S_n(M) \subset M = g(M)$ and we have

$$H(S_nx,T_ny) = k_nH(Sx,Ty) \leq \max\Big\{ \|fx-gy\|, dist(fx,S_nx), dist(gy,T_ny), \frac{1}{2}[dist(fx,T_ny)+dist(gy,S_nx)]\Big\} \tag{2.3}$$

for all $x, y \in M$, and $) < k_n < 1$. By Theorem 1.2, for each $n \in \mathbb{N}$, there exist $x_n \in M$ such that $fx_n \in Sx_n$. This implies that there is a $y_n \in Sx_n$ such that

$$fx_n - y_n = (1-k_n)(q-y_n).$$

Since $S(M)$ is bounded and $k_n \to 1$ as $n \to \infty$, it follow that $fx_n - y_n \to o$ as $n \to \infty$. Since $(f-S)(M)$ is closed it follow that $0 \in (f-S)(M)$ and so $fx_0 \in Sx_0$ for some $x_0 \in M$. Hence $C(S,f) \neq \emptyset$. Similarly, we have $C(T,g) \neq \emptyset$.

Theorem 2.2 *Let M be a nonempty complete and generalized q-starshaped subset of a normed space X, and $f, g : M \to M$ such that surjective. Assume that $S, T : M \to CL(M)$ are generalized nonexpansive relative (satisfies (2.1)) for all $x, y \in M$ and $\lambda \in [0,1]$. Suppose that $T(M)$ is bounded and $(f-S)(M), (g-T)(M)$ is closed, and the pair (S, f) is continuous, then*

(i) $F(S) \cap F(f) \neq \emptyset$,
provided f is S-weakly commuting at v and $ffv = fv$ for some $v \in C(S, f)$;

(ii) $F(T) \cap F(g) \neq \emptyset$,
provided g is T-weakly commuting at u and $ggu = fu$ for some $u \in C(T, g)$.

Proof Since f is S-weakly commuting at $v \in C(S, f)$, then $ffv \in Sfv$ and hence $fv = ffv \in Sfv$. Thus $F(f) \cap F(S) \neq \emptyset$, (and since g is T-weakly commuting at $u \in C(T, g)$, similarly, we obtain $F(T) \cap F(g) \neq \emptyset$.)

If $S = T$ and $f = g$ in Theorem 2.1 and Theorem 2.2, we can drop conditions the pair (S, f) is reciprocally continuous and nonvacuously compatible, we obtain the following corollaries:

Corollary 2.3 ([3, Theorem 2.1]) *Let M be a nonempty complete and generalized q-starshaped subset of a normed space X, and $f : M \to M$ such that surjective. Assume that $T : M \to CL(M)$ is generalized f-nonexpansive mapping i.e.,*

$$H(Tx, Ty) \leq \max\Big\{ \|fx - fy\|, dist(fx, T_\lambda x), dist(fy, T_\lambda y), \frac{1}{2}[dist(fx, T_\lambda y) + dist(fy, T_\lambda x)]\Big\} \tag{2.4}$$

for all $x, y \in M$ and $\lambda \in [0,1]$. Suppose that $T(M)$ is bounded and $(f - S)(M)$ is closed, then $C(T, f) \neq \emptyset$. If, in addition, f is T- weakly commuting at v and $ffv = fv$ for $v \in C(T, f)$, then $F(f) \cap F(T) \neq \emptyset$.

Corollary 2.4 ([6, Theorem 2.1]) *Let M be a nonempty complete and generalized q-starshaped subset of a norm space X, and $f : M \to M$ with $f(M) = M$. Assume that $T : M \to CL(M)$ is f-nonexpansive map such that $T(M)$ is bounded and $(f - T)(M)$ is closed, then $C(T, f) \neq \emptyset$.*

Theorem 2.5 *Let M be a nonempty complete and generalized q-starshaped subset of a normed space X, and $f : M \to M$ with $f(M) = M$. Assume that $T : M \to CL(M)$ satisfies (2.4) for all $x, y \in M$ and $\lambda \in [0,1]$, If $T(M)$ is bounded and $(f - T)(M)$ demiclosed at 0, then $C(T, f) \neq \emptyset$.*

Proof As in the proof of Theorem 2.1 and Corollary 2.3.
Since $(f - T)$ is demiclosed at 0, we have $0 \in (f - T)y$. Thus $C(T, f) \neq \emptyset$.

Theorem 2.6 *Let M be a nonempty complete and generalized q-starshaped subset of a normed space X and $f : M \to M$ such that $f(M) = M$. Assume that $T : M \to CL(M)$ is a generalized f-nonexpansive map $T(M)$ is bounded, and $(f-T)(M)$ is closed. If, in addition, f is T- weakly commuting at v and $ffv = fv$ for $v \in C(T, f)$, then $F(f) \cap F(T) \neq \emptyset$.*

Proof By Theorem 2.5, $C(T,f) \neq \emptyset$. Suppose $v \in C(T,f)$. Then $fv = ffv \in Tfv$. Thus $F(f) \cap F(T) \neq \emptyset$.

Corollary 2.7 ([3, Theorem 2.4]) *Let M be a nonempty weakly compact and generalized q-starshaped subset of a Banach space X, and $f : M \to M$ with $f(M) = M$. Assume that $T : M \to CL(M)$ satisfies (2.4) for all $x, y \in M$ and* $\lambda \in [0,1]$, *If $T(M)$ is bounded and $(f - T)(M)$ demiclosed at* 0, *then $C(T,f) \neq \emptyset$. Moreover, f is T- weakly commuting at v and $ffv = fv$ for $v \in C(T,f)$, then $F(f) \cap F(T) \neq \emptyset$.*

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