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Some L^s Inequalities for Polynomials not Vanishing Inside a Circle

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Abstract : In this paper, we establish some L^s-inequalities for polynomials not vanishing inside a circle, from which a variety of interesting results follows as special cases.

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1 Introduction and Statement of Results

Let P_n be the class of all polynomials

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

of degree at most n and $p'(z)$ its derivative. For $p \in P_n$, define

$$\| p(z) \|_s := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^s \right\}^{\frac{1}{s}} \quad , \quad 1 \leq s < \infty,$$

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$$\| p(z) \|_\infty := \max_{|z|=1} |p(z)| \text{ and } m = \min_{|z|=1} |p(z)|.$$

According to a famous result known as Bernstein's inequality [1], we have

$$\| p'(z) \|_\infty \leq n \| p(z) \|_\infty . \tag{1.1}$$

Also concerning the maximum modulus of $p(z)$ on $|z| = R > 1$, we have (for references see [2]),

$$\| p(Rz) \|_\infty \leq R^n \| p(z) \|_\infty . \tag{1.2}$$

Inequalities (1.1) and (1.2) can be obtained by letting $s \to \infty$ in the inequalities

$$\| p'(z) \|_s \leq n \| p(z) \|_s, \quad s \geq 1, \tag{1.3}$$

and

$$\| p(Rz) \|_s \leq R^n \| p(z) \|_s, \quad R > 1, \quad s > 0. \tag{1.4}$$

respectively. Inequality (1.3) was found by Zygmund [3] whereas inequality (1.4) is found in [4, Theorem 5.5].

Also, Arestov [5] proved that (1.3) remains true for $0 < s < 1$ as well. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, the inequalities (1.3) and (1.4) can be improved. In fact, it was shown by De-Bruijn [6] for $s \geq 1$ and Rahman and Schemeisser [7] extended it for $0 < s < 1$ that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, the inequality (1.3) can be replaced by

$$\| p'(z) \|_s \leq n \frac{\| p(z) \|_s}{\| 1 + z \|_s} , \quad s > 0. \tag{1.5}$$

Also Rahman and Shemeisser [7] proved for $0 < s < 1$ that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then inequality (1.4) can be replaced by

$$\| p(Rz) \|_s \leq \frac{\| R^n z + 1 \|_s}{\| 1 + z \|_s} \| p(z) \|_s, \quad R > 1. \tag{1.6}$$

Aziz and Rather [8] obtained generalizations of inequalities (1.3) and (1.5). In fact, they have shown that if $p \in P_n$, then for every $R > 1$ and $s > 0$,

$$\| p(Rz) - p(z) \|_s \leq (R^n - 1) \| p(z) \|_s, \tag{1.7}$$

whereas if $p(z) \neq 0$ in $|z| < 1$, then

$$\| p(Rz) - p(z) \|_s \leq \frac{(R^n - 1)}{\| 1 + z \|_s} \| p(z) \|_s . \tag{1.8}$$

Recently Aziz and Rather [9] considered the more general problem of investigating the dependence of

$$\| p(Rz) - \beta p(rz) \|_s \quad \text{on} \quad \| p(z) \|_s$$

for every $\beta \in C$ with $|\beta| \leq 1, R > r \geq 1, s > 0$ and proved the following:

Theorem A. [9] *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\beta \in C$ with $|\beta| \leq 1$ and $R > r \geq 1, s > 0$,*

$$\| p(Rz) - \beta p(rz) \|_s \leq \frac{\| (R^n - \beta r^n)z + (1 - \beta) \|_s}{\| 1 + z \|_s} \| p(z) \|_s \, . \tag{1.9}$$

The result is best possible and equality holds in (1.9) for $p(z) = az^n + b$, $|a| = |b| = 1$.

In this paper, we first prove the following more general result which among other things includes Theorem A as a special case.

Theorem 1.1. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\beta, \gamma \in C$ with $|\beta| \leq 1$, $|\gamma| \leq 1$ and $R > r \geq 1, s > 0$,*

$$\begin{aligned} \| p(Rz) - \beta p(rz) + \gamma \left(\frac{\left|R^n - \beta r^n\right| - \left|1 - \beta\right|}{2} \right) m \|_s & \\ \leq \frac{\| (R^n - \beta r^n)z + (1 - \beta) \|_s}{\| 1 + z \|_s} \| p(z) \|_s \, . \end{aligned} \tag{1.10}$$

The result is best possible and equality in (1.10) holds for $p(z) = z^n + 1$.

Remark 1.2. For $\gamma = 0$, Theorem 1.1 reduces to Theorem A. For $\gamma = 0$ and $r = \beta = 1$, Theorem 1.1 reduces to inequality (1.8). For $\beta = 0$ and $r = 1$, we get a result recently proved by Rather [10, Theorem 1.1].

A variety of interesting results can be easily deduced from Theorem 1.1. Here we mention a few of these. The following corollary immediately follows from Theorem 1.1 by taking $\beta = 1$.

Corollary 1.3. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\gamma \in C$ with $|\gamma| \leq 1$ and $R > r \geq 1, s > 0$,*

$$\| p(Rz) - p(rz) + \frac{\gamma(R^n - r^n)m}{2} \|_s \leq \frac{(R^n - r^n)}{\| 1 + z \|_s} \| p(z) \|_s \, . \tag{1.11}$$

If we divide the two sides of (1.11) bt $R - r$ and let $R \to r$, we get

Corollary 1.4. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\gamma \in C$ with $|\gamma| \leq 1$ and $r \geq 1, s > 0$,*

$$\| zp'(rz) + \frac{\gamma nmr^{n-1}}{2} \|_s \leq \frac{nr^{n-1}}{\| 1 + z \|_s} \| p(z) \|_s \, . \tag{1.12}$$

Remark 1.5. If we let $s \to \infty$ in (1.12) and choose argument of γ with $|\gamma| = 1$ suitably, we get for $|z| = 1$,

$$|p'(rz)| \leq \frac{nr^{n-1}}{2}\left(\max_{|z|=1}|p(z)| - \min_{|z|=1}|p(z)|\right). \tag{1.13}$$

For $r = 1$, inequality (1.13) reduces to a result of Aziz and Dawood [11].

Next we mention the following compact generalization of a result of Aziz and Dawood [11, Theorem 2] which immediately follows from Theorem 1.1 by letting $s \to \infty$ and choosing argument of γ with $|\gamma| = 1$ in (1.10).

Corollary 1.6. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\beta \in C$ with $|\beta| \leq 1, R > r \geq 1$, and $|z| = 1$,*

$$\begin{aligned}\big|p(Rz) - \beta p(rz)\big| \leq & \left\{\frac{\big|R^n - \beta r^n\big| + \big|1-\beta\big|}{2}\right\}\max_{|z|=1}|p(z)| \\ & - \left\{\frac{\big|R^n - \beta r^n\big| - \big|1-\beta\big|}{2}\right\}\min_{|z|=1}|p(z)|. \end{aligned} \tag{1.14}$$

If we take $\beta = 0$ in (1.14)*, we immediately get*

$$\| p(Rz) \|_\infty \leq \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|p(z)|, \quad R > 1.$$

The above inequality is due to Aziz and Dawood [11, Theorem 2].

Finally, as an application of Theorem 1.1, we prove the following generalization and refinement of (1.6) for $s \geq 1$.

Theorem 1.7. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\beta,\ \gamma \in C$ with $|\beta| \leq 1,\ |\gamma| \leq 1$ and $R > r \geq 1, s \geq 1$,*

$$\begin{aligned}\| p(Rz) + & \gamma\left(\frac{\big|R^n - \beta r^n\big| - \big|1-\beta\big|}{2}\right)m \|_s \\ & \leq \left\{|\beta| r^n + \frac{\| (R^n - \beta r^n)z + (1-\beta) \|_s}{\| 1+z \|_s}\right\} \| p(z) \|_s . \end{aligned} \tag{1.15}$$

Remark 1.8. For $\beta = \gamma = 0$, Theorem 1.7 reduces to (1.6).

The following corollary which is a compact generalization of a result of Aziz and Dawood [11, Theorem 2] to L^s norm is obtained from Theorem 1.7 by letting $\beta = r = 1$.

Corollary 1.9. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every $\gamma \in C$ with $|\gamma| \leq 1,\ R > 1$ and $s \geq 1$,*

$$\| p(Rz) + \gamma\left(\frac{R^n-1}{2}\right)m \|_s \leq \left\{1 + \frac{(R^n-1)}{\| 1+z \|_s}\right\} \| p(z) \|_s . \tag{1.16}$$

2 Lemmas

Lemma 2.1. [9] *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every complex β with $|\beta| \leq 1$, $R > r \geq 1$ and $|z| = 1$,*

$$\big|p(Rz) - \beta p(rz)\big| \leq \big|q(Rz) - \beta q(rz)\big| \tag{2.1}$$

where $q(z) = z^n\overline{p\big(\frac{1}{\bar{z}}\big)}$.

Lemma 2.2. *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1, m = \min_{|z|=1} |p(z)|$, then for every complex β with $|\beta| \leq 1$, $R > r \geq 1$ and $|z| = 1$,*

$$\big|p(Rz) - \beta p(rz)\big| \leq \big|q(Rz) - \beta q(rz)\big| - \Big\{\big|R^n - \beta r^n\big| - \big|1 - \beta\big|\Big\}m, \tag{2.2}$$

where $q(z) = z^n\overline{p\big(\frac{1}{\bar{z}}\big)}$.

Proof. Since $m \leq |p(z)|$ for $|z| = 1$, it follows by Rouche's theorem that for $m > 0$ and for every complex number α with $|\alpha| < 1$, the polynomial $h(z) = p(z) + \alpha m z^n$ has no zeros in $|z| < 1$.

Applying Lemma 2.1 to the polynomial $h(z)$, we get for every complex number α with $|\alpha| < 1$,

$$\big|p(Rz) - \beta p(rz) + \alpha m(R^n - \beta r^n)z^n\big| \;\leq\; \big|q(Rz) - \beta q(rz) + \overline{\alpha} m(1 - \beta)\big|,$$

for $|z| = 1$ and for every β with $|\beta| \leq 1$ and $R > r \geq 1$.
If we now choose the argument of α in the left hand side of inequality (2.3) such that

$$\begin{aligned}\big|p(Rz) - \beta p(rz) &+ \alpha m(R^n - \beta r^n)z^n\big| \\ &= \big|p(Rz) - \beta p(rz)\big| + m|\alpha|\big|R^n - \beta r^n\big||z|^n,\end{aligned}$$

we get for $|z| = 1$ and $R > r \geq 1$,

$$\big|p(Rz) - \beta p(rz)\big| + m|\alpha|\big|R^n - \beta r^n\big| \leq \big|q(Rz) - \beta q(rz)\big| + m|\alpha||1 - \beta|.$$

Now, if in (2.4), we make $|\alpha| \to 1$, we get for $|z| = 1$,

$$\big|p(Rz) - \beta p(rz)\big| \leq \big|q(Rz) - \beta q(rz)\big| - \Big\{\big|R^n - \beta r^n\big| - \big|1 - \beta\big|\Big\}m,$$

with $R > r \geq 1$ and for every β with $|\beta| \leq 1$. □

Lemma 2.3. [9] *If $p \in P_n$ and $p(z)$ does not vanish in $|z| < 1$, then for every complex β with $|\beta| \leq 1$, $R > r \geq 1$, $s > 0$ and α real,*

$$\begin{aligned}\int_0^{2\pi} \bigg|\Big(p(Re^{i\theta}) - \beta p(re^{i\theta})\Big) &+ e^{i\alpha}\Big(R^n p\big(\frac{e^{i\theta}}{R}\big) - \overline{\beta} r^n p\big(\frac{e^{i\theta}}{r}\big)\Big)\bigg|^s d\theta \\ &\leq \Big|\big(R^n - \beta r^n\big) + e^{i\alpha}\big(1 - \overline{\beta}\big)\Big|^s \int_0^{2\pi} |p(e^{i\theta})|^s d\theta.\end{aligned}$$

Lemma 2.4. [12] *If A, B and C are non-negative real numbers such that $B+C \leq A$, then for every real number α,*

$$\left|(A-C)e^{i\alpha}+(B+C)\right| \leq \left|Ae^{i\alpha}+B\right|.$$

3 Proof of Theorems

Proof of Theorem 1.1. Since $p(z) \neq 0$ in $|z|<1$, it follows by Lemma 2.2, for each $\theta,\ 0 \leq \theta < 2\pi,\ R > r \geq 1$ and for every complex number β with $|\beta| \leq 1$,

$$\left|p(Re^{i\theta})-\beta p(re^{i\theta})\right| \leq \left|R^n p(\frac{e^{i\theta}}{R})-\overline{\beta}r^n p(\frac{e^{i\theta}}{r})\right| - \left\{\left|R^n-\beta r^n\right|-\left|1-\beta\right|\right\}m,$$

which implies,

$$\left|p(Re^{i\theta})-\beta p(re^{i\theta})\right| + \left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m \leq \left|R^n p(\frac{e^{i\theta}}{R})-\overline{\beta}r^n p(\frac{e^{i\theta}}{r})\right| - \left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m. \tag{3.1}$$

Taking $A=\left|R^n p(\frac{e^{i\theta}}{R})-\overline{\beta}r^n p(\frac{e^{i\theta}}{r})\right|,\quad B=\left|p(Re^{i\theta})-\beta p(re^{i\theta})\right|$ and $C=\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m$ in Lemma 2.4, we see with the help of (3.1) that

$$B+C \leq A-C \leq A,$$

we get for every real α,

$$\left|\left\{\left|R^n p(\frac{e^{i\theta}}{R})-\overline{\beta}r^n p(\frac{e^{i\theta}}{r})\right| - \left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m\right\}e^{i\alpha} + \left\{\left|p(Re^{i\theta})-\beta p(re^{i\theta})\right| + \left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m\right\}\right| \leq \left|\left|R^n p(\frac{e^{i\theta}}{R})-\overline{\beta}r^n p(\frac{e^{i\theta}}{r})\right|e^{i\alpha} + \left|p(Re^{i\theta})-\beta p(re^{i\theta})\right|\right|.$$

This implies for each $s>0$,

$$\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^s d\theta \leq \int_0^{2\pi}\left|\left|R^n p(\frac{e^{i\theta}}{R})-\overline{\beta}r^n p(\frac{e^{i\theta}}{r})\right|e^{i\alpha} + \left|p(Re^{i\theta})-\beta p(re^{i\theta})\right|\right|^s d\theta, \tag{3.2}$$

where $\quad F(\theta) = \left|p(Re^{i\theta}) - \beta p(re^{i\theta})\right| + \left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m$

and $\quad G(\theta) = \left|R^n p(\frac{e^{i\theta}}{R}) - \overline{\beta} r^n p(\frac{e^{i\theta}}{r})\right| - \left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m.$

Integrating both sides of (3.2) with respect to α from 0 to 2π, we get with the help of Lemma 2.3, for each $s > 0$, $R > r \geq 1$ and α real,

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta) + e^{i\alpha}G(\theta)\right|^s d\theta d\alpha$$

$$\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|R^n p(\frac{e^{i\theta}}{R}) - \overline{\beta} r^n p(\frac{e^{i\theta}}{r})\right|e^{i\alpha} + \left|p(Re^{i\theta}) - \beta p(re^{i\theta})\right|\right|^s d\alpha\right\}d\theta$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(R^n p(\frac{e^{i\theta}}{R}) - \overline{\beta} r^n p(\frac{e^{i\theta}}{r})\right)e^{i\alpha} + \left(p(Re^{i\theta}) - \beta p(re^{i\theta})\right)\right|^s d\alpha\right\}d\theta$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(R^n p(\frac{e^{i\theta}}{R}) - \overline{\beta} r^n p(\frac{e^{i\theta}}{r})\right)e^{i\alpha} + \left(p(Re^{i\theta}) - \beta p(re^{i\theta})\right)\right|^s d\theta\right\}d\alpha$$

$$\leq \left\{\int_0^{2\pi}\left|\left(R^n - \beta r^n\right) + e^{i\alpha}\left(1-\overline{\beta}\right)\right|^s d\alpha\right\}\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^s d\theta\right\}. \tag{3.3}$$

Now for every real α, $t \geq 1$ and $s > 0$, we have

$$\int_0^{2\pi}\left|t + e^{i\alpha}\right|^s d\alpha \geq \int_0^{2\pi}\left|1 + e^{i\alpha}\right|^s d\alpha.$$

If $F(\theta) \neq 0$, we take $t = \left|\frac{G(\theta)}{F(\theta)}\right|$, then by (3.1), $t \geq 1$ and we get

$$\int_0^{2\pi}\left|F(\theta) + e^{i\alpha}G(\theta)\right|^s d\alpha = \left|F(\theta)\right|^s\int_0^{2\pi}\left|1 + \frac{G(\theta)}{F(\theta)}e^{i\alpha}\right|^s d\alpha$$

$$= \left|F(\theta)\right|^s\int_0^{2\pi}\left|\frac{G(\theta)}{F(\theta)} + e^{i\alpha}\right|^s d\alpha = \left|F(\theta)\right|^s\int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right| + e^{i\alpha}\right|^s d\alpha$$

$$\geq \left|F(\theta)\right|^s\int_0^{2\pi}\left|1 + e^{i\alpha}\right|^s d\alpha$$

$$= \left\{\left|p(Re^{i\theta}) - \beta p(re^{i\theta})\right| + \left(\frac{\left|R^n - \beta r^n\right| - \left|1-\beta\right|}{2}\right)m\right\}^s\int_0^{2\pi}\left|1 + e^{i\alpha}\right|^s d\alpha.$$

If $F(\theta)=0$, this inequality is trivially true. Using this in (3.3), we conclude that for every complex β with $|\beta|\leq 1,\ R>r\geq 1$ and α real,

$$\int_0^{2\pi}\left|1+e^{i\alpha}\right|^s d\alpha\int_0^{2\pi}\left\{\left|p(Re^{i\theta})-\beta p(re^{i\theta})\right|+\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m\right\}^s d\theta$$
$$\leq\left\{\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\overline{\beta}\right)\right|^s d\alpha\right\}\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^s d\theta\right\}. \tag{3.4}$$

Since

$$\begin{aligned}\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\overline{\beta}\right)\right|^s d\alpha&=\int_0^{2\pi}\left|\left|R^n-\beta r^n\right|+e^{i\alpha}\left|1-\overline{\beta}\right|\right|^s d\alpha\\&=\int_0^{2\pi}\left|\left|R^n-\beta r^n\right|+e^{i\alpha}\left|1-\beta\right|\right|^s d\alpha\\&=\int_0^{2\pi}\left|\left|R^n-\beta r^n\right|e^{i\alpha}+\left|1-\beta\right|\right|^s d\alpha\\&=\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)e^{i\alpha}+\left(1-\beta\right)\right|^s d\alpha,\end{aligned} \tag{3.5}$$

and for every complex γ with $|\gamma|\leq 1$, we have

$$\left|p(Re^{i\theta})-\beta p(re^{i\theta})+\gamma m\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)\right|$$
$$\leq\left|p(Re^{i\theta})-\beta p(re^{i\theta})\right|+m\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right), \tag{3.6}$$

the desired result follows by using (3.5) and (3.6) in (3.4). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.7. We have by Minkowski's inequality, for every $s\geq 1$,

$$\begin{aligned}\| p(Rz)&+\gamma\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m\ \|_s\\&=\| p(Rz)-\beta p(rz)+\gamma\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m+\beta p(rz)\ \|_s\\&\leq\| p(Rz)-\beta p(rz)+\gamma\left(\frac{\left|R^n-\beta r^n\right|-\left|1-\beta\right|}{2}\right)m\ \|_s+\|\ \beta p(rz)\ \|_s\ .\end{aligned} \tag{3.7}$$

Using inequalities (1.4) and (1.10) in (28), we get for every $\beta,\ \gamma \in C$ with $|\beta| \leq 1,\ |\gamma| \leq 1$ and $R > r \geq 1,\ s \geq 1$,

$$\begin{aligned}\| p(Rz) + \gamma\left(\frac{\left|R^n - \beta r^n\right| - \left|1-\beta\right|}{2}\right) m \|_s &\leq \frac{\| (R^n - \beta r^n)z + (1-\beta) \|_s}{\| 1+z \|_s} \| p(z) \|_s + |\beta| r^n \| p(z) \|_s \\ &= \left\{ |\beta| r^n + \frac{\| (R^n - \beta r^n)z + (1-\beta) \|_s}{\| 1+z \|_s} \right\} \| p(z) \|_s,\end{aligned}$$

which is inequality (1.16) and Theorem 1.7 is completely proved. □

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