Thai Journal of Mathematics
Volume 17 (2019) Number 1 : 107–113

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Complete Solution of 3-step Hamiltonian Grids and Torus Graphs

Gee-Choon Lau[†,1], Sin-Min Lee[‡], Karl Schaffer[§] and Siu-Ming Tong[£]

[†]Faculty of Computer & Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus) Johor, Malaysia
e-mail : geeclau@yahoo.com

[‡]1304, North First Avenue, Upland, CA 91786 USA
email : sinminlee@gmail.com

[§]Department of Mathematics, De Anza College, Cupertino CA 95014, USA
email : karl_schaffer@yahoo.com

[£]Department of Computer Science, Northwestern Polytechnic University Fremont, CA 94539, USA
e-mail : tjadis@aol.com

Abstract : For a (p, q)-graph G, if the vertices of G can be arranged in a sequence $v_1, v_2, \ldots, v_p$ such that for each $i = 1, 2, \ldots, p-1$, the distance from v_i to v_{i+1} equal to k, then the sequence is called an $AL(k)$-step traversal. Furthermore, if $d(v_p, v_1) = k$, the sequence $v_1, v_2, \ldots, v_p, v_1$ is called a k-step Hamiltonian tour and G is k-step Hamiltonian. In this paper we completely determine which rectangular grid graphs are 3-step Hamiltonian and show that the torus graph $C_m \times C_n$ is 3-step Hamiltonian for all $m \geq 3, n \geq 5$.

Keywords : k-step Hamiltonian; grid graphs; torus graphs; NP-complete problem.

2010 Mathematics Subject Classification : 05C78; 05C25.

---

[1]Corresponding author.

1 Introduction

All graphs considered are simple and loopless. For terms used but not defined here, we refer to [1]. For a (p, q)-graph G, if the vertices of G can be arranged in a sequence $v_1, v_2, \ldots, v_p$ such that for each $i = 1, 2, \ldots, p-1$, the distance from v_i to v_{i+1} is equal to $k \geq 1$, then the sequence is called an $AL(k)$-step traversal. If $d(v_p, v_1) = k$, the sequence $v_1, v_2, \ldots, v_p, v_1$ is called a k-step Hamiltonian tour and we say G is k-step Hamiltonian (see [2]). Clearly, a 1-step Hamiltonian graph is also Hamiltonian. The problem of Hamiltonian graphs has application in the traveling salesman problem. The readers may refer to [3, 4] for a survey on the developments of Hamiltonian graphs and the traveling salesman problem. In [2,5], the authors showed that all bipartite graphs are not k-step Hamiltonian for all even k and also determined the k-step Hamiltonicity of many families of graphs.

The classical closed knight's tour chessboard problem asks whether a knight on a chessboard can visit every square and returned to its starting position. The closed knight's tour problem is the problem of constructing such a tour on a given chessboard. It is clear that every rectangular chessboard of size $m \times n$ corresponds to a rectangular grid graphs of order mn and size $2mn-m-n$ denoted $G(m, n)$ for $n \geq m \geq 1$. Hence, a closed knight's tour of a rectangular chessboard always corresponds to a 3-Hamiltonian tour in a rectangular grid graph. However, the converse may not true. For simplicity, we label vertices of $G(m, n)$ by (i, j) counting from the upper left corner of the grid in matrix fashion. In this paper we completely determine which rectangular grid graphs are 3-step Hamiltonian and show that the torus graph $C_m \times C_n$ is 3-step Hamiltonian for all $m \geq 3, n \geq 5$.

2 Main Results

Definition 2.1. For a graph G, let $D_k(G)$ denote the graph generated from G such that $V(D_k(G)) = V(G)$ and $E(D_k(G)) = \{uv | d(u, v) = k \text{ in } G.\}$.

Lemma 2.2. *A graph G is k-step Hamiltonian or admits an $AL(k)$-step Hamiltonian traversal if and only if $D_k(G)$ is Hamiltonian or admits a Hamiltonian path.*

Lemma 2.3. *If G is a bipartite graph with bipartition (X, Y) and it is 3-step Hamiltonian then $|X| = |Y|$.*

Proof. $D_3(G)$ is also a bipartite graph with bipartition (X, Y). Since $D_3(G)$ is Hamiltonian, therefore $|X| = |Y|$. □

Remark 2.4. *A bipartite graph with $|X| = |Y|$ in bipartition need not be 3-step Hamiltonian. The simplest example is $K(n, n)$, for $n > 3$.*

In 1750s, Euler presented solutions for the standard 8×8 board (see [6, 7]) and the knight's tour problem is easily generalized to rectangular boards. In 1991 Schwenk [7] completely answered the question: Which rectangular chessboards have a knight's tour?

Theorem 2.5. ([7], Schwenk) *An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:*

(a) *m and n are both odd;*

(b) *$m \in \{1, 2, 4\}$;*

(c) *$m = 3$ and $n \in \{4, 6, 8\}$.*

We now present our complete solution to the question: Which rectangular boards have a 3-step Hamiltonian tour?

Theorem 2.6. *For $n \geq m \geq 1$, $G(m, n)$ is 3-step Hamiltonian except for the following four conditions:*

(a) *m, n are odd;*

(b) *$m = 1$;*

(c) *$m = 2, n = 2, 3, 4, 6, 7, 8, 9, 12, 14$;*

(d) *$m = 4, n = 7$.*

Proof. By Theorem 2.5, it is clear that we need only consider the grid graphs that do not admit a closed knight's tour. We first show in Figure 1 that $G(3, n)$ is 3-step Hamiltonian for $n = 4, 6, 8$.

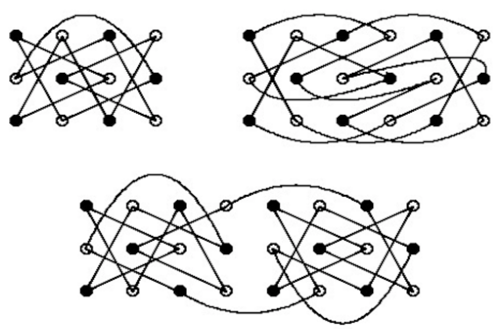


Figure 1: A 3-step Hamiltonian tour in $G(3, n), n = 4, 6, 8$

We now consider the following 4 cases.

(a) Suppose $m = 2s+1$ and $n = 2t+1$. We see that $|X| = (s+1)(t+1)+ts = 2st + s + t + 1$ and $|Y| = s(t + 1) + (s + 1)t = 2st + s + t$.

As $|X| \neq |Y|$, thus by Lemma 2.2. $G(m, n)$ cannot be 3-step Hamiltonian

(b) If $m = 1$, it is clear that $D_3(G(1, n))$ is disconnected. Hence, $G(1, n)$ cannot be 3-step Hamiltonian.

(c) If $m = 2$, we first show that $G(2, n)$ is not 3-step Hamiltonian for $n = 2, 3, 4, 6, 7, 8, 9, 12, 14$. For $n = 2, 3, 4, 6, 7, 8, 9$, it is routine to show that no 3-step Hamiltonian tour exists.

We now consider the remaining values of n. In Figure 2, we give a 3-step Hamiltonian tour in $G(2,n)$ for $n=5,10,11$. Note that a 3-step Hamiltonian tour in $G(2,13)$ (see Figure 3) can be obtained from the 3-step Hamiltonian tour in $G(2,11)$ in Figure 2. In a similar way, we can construct a 3-step Hamiltonian tour in $G(2,n)$ for odd $n \geq 13$.

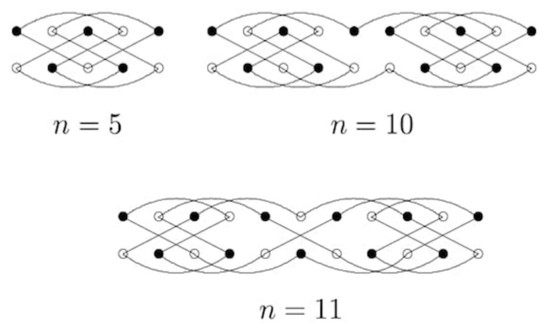


Figure 2: A 3-step Hamiltonian tour in $G(2,n), n=5,10,11$

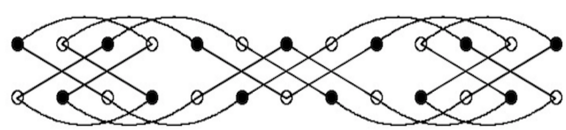


Figure 3: A 3-step Hamiltonian tour in $G(2,13)$

Using Maple software, we found that for $n=12,14$, $D_3(G(2,n))$ is not Hamiltonian and hence $G(2,n)$ is not 3-step Hamiltonian. In Figure 4, we give a 3-step Hamiltonian tour in $G(2,16)$ and $G(2,18)$. We can then construct a 3-step Hamiltonian tour in $G(2,n)$ for even $n \geq 18$ in a similar way.

(d) If $m=4$, we can construct a 3-step Hamiltonian tour in $G(4,n)$ from each 3-step Hamiltonian tour in $G(2,n)$ in part (c). In Figure 5, we show an extension of a 3-step Hamiltonian tour in $G(2,5)$ to a 3-step Hamiltonian tour in $G(4,5)$.

We now only need to consider $G(4,n)$ for $n=4,6,7,8,9,12,14$ in which $G(2,n)$ is not 3-step Hamiltonian. In Figure 6, we give a 3-step Hamiltonian tour for $n=4,6,8,9,12,14$.

Using Maple software, we found that $D_3(G(4,7))$ is not Hamiltonian. This completes the proof.

□

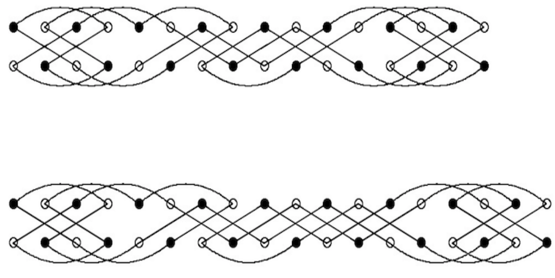


Figure 4: A 3-step Hamiltonian tour in $G(2, n), n = 16, 18$

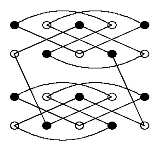


Figure 5: Extension of a 3-step Hamiltonian tour in $G(2, 5)$ to one in $G(4, 5)$

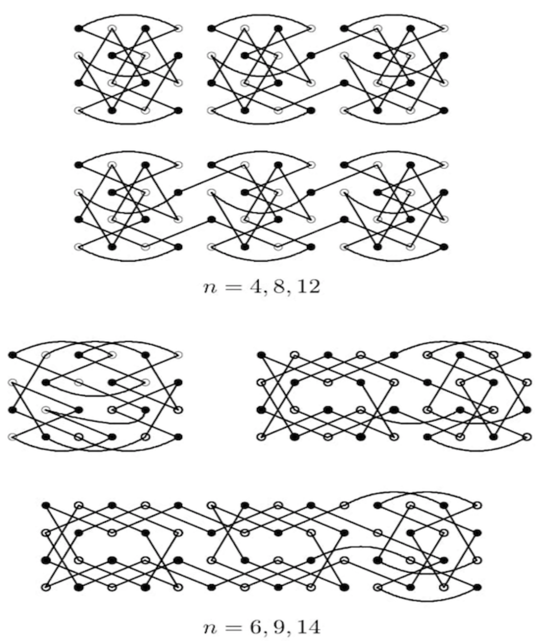
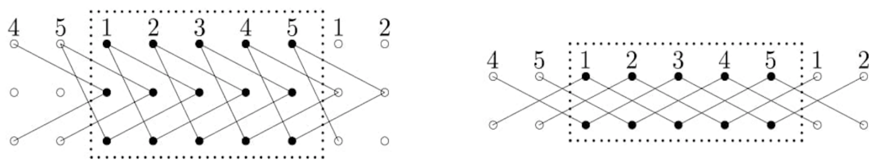


Figure 6: A 3-step Hamiltonian tour in $G(4, n), n = 4, 6, 8, 9, 12, 14$

As a natural extension, we have the following results.

Theorem 2.7. *The cylinder graph $P_m \times C_n$ is 3-step Hamiltonian for all $m \geq 3, n \geq 5$.*

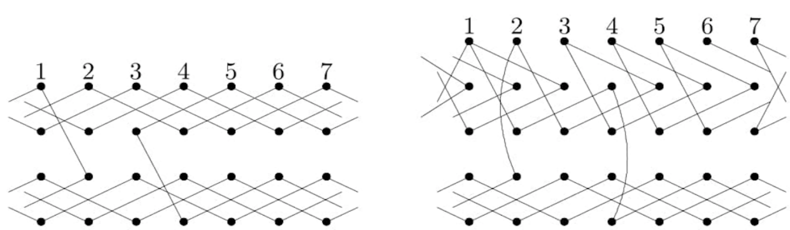
Proof. $P_m \times C_n$ contains $G(m, n)$ as a subgraph, so if $G(m, n)$ is 3-step Hamiltonian, then so is $P_m \times C_n$. This leaves the cases in which m and n are both odd, as well as $m = 4, n = 7$. Figure 7 shows a 3-step Hamiltonian tour for $P_3 \times C_5$ with the same pattern may be extended for all $n > 5$ and $m = 3$ as well as a 3-step Hamiltonian tour for $P_2 \times C_5$ with the same pattern may be extended for any odd n.



$P_3 \times C_5$ $\qquad$ $P_2 \times C_5$

Figure 7: A 3-step Hamiltonian tour $P_m \times C_5, m = 2, 3$

These two patterns may be combined to create 3-step Hamiltonian tours for other odd m and n. In Figure 8, we give a 3-step Hamiltonian tour for $P_4 \times C_7$ and one for $P_5 \times C_7$. The pattern may be extended to other odd m by adding other additional rows of $P_2 \times C_n$ and linking them similarly.



$P_4 \times C_7$ $\qquad$ $P_5 \times C_7$

Figure 8: A 3-step Hamiltonian tour for $P_4 \times C_7$ and $P_m \times C_n, m \geq 5, n \geq 7$ both odd

□

Corollary 2.8. *The torus graph $C_m \times C_n$ is 3-step Hamiltonian for all $m \geq 4, n \geq 5$.*

Proof. The torus graph contains $P_m \times C_n$ as a subgraph. Observe that the distance between any 2 vertices of a C_3 is 1. This means the 3-step Hamiltonian tour of $P_3 \times C_m$ is not a 3-step Hamiltonian tour of $C_3 \times C_m$. Hence, the theorem holds. □

Since determining whether a bipartite graph is Hamiltonian is NP-complete [8, 9], we would like to end the paper with the following conjecture.

Conjecture 2.9. *The problem of 3-step Hamiltonian bipartite graphs is NP-complete.*

References

[1] G. Chartrand, P. Zhang, Introduction to Graph Theory, Walter Rudin Student Series in Advanced Mathematics, McGraw-Hill, 2004.

[2] G.C. Lau, S.M. Lee, K. Schaffer, S.M. Tong, S. Lui, On k-step Hamiltonian graphs, J. Combin. Math. Combin. Comp. 90 (2014) 145-158.

[3] R. Gould, Advances on the Hamiltonian problem, a survey, Graphs and Combinatorics 19 (2003) 7-52.

[4] S. Goyal, A Survey on Travelling Salesman Problem, (2010).

[5] G.C. Lau, S.M. Lee, K. Schaffer, S.M. Tong, On k-step Hamiltonian Bipartite and Tripartite Graphs, Malaya J. Mat. 2 (3) (2014) 180-187.

[6] E. Sandifer, How Euler Dit It?, (2012).

[7] A.J. Schwenk, Which rectangular chessboards have a knight's tour?, Mathematics Magazine (1991) 325-332.

[8] T.S.N. Akiyama, T. Nishizeki, NP-completeness of the Hamiltonian cycle problem for bipartite graphs, Journal of Information Processing 3 (1980) 73-76.

[9] A. Itai, C.H. Papadimitriou, J.L. Szwarcfiter, Hamilton paths in grid graphs, SIAM Journal on Computing 11 (4) (1982) 676-686.

(Received 28 November 2012)
(Accepted 26 December 2018)