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Some Properties of Valuation Ideals and Primary Ideals in Additive Semigroups

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Abstract : This paper deals with the connections and interdependencies among prime ideals, primary ideals, valuation ideals, valuation semigroups and semigroups. R. Gilmer and J. Ohm [1] studied primary ideals and valuation ideals for integral domains. In this article, we generalize this concept for semigroups. It is proved that if T is a prime ideal of a semigroup S, and $\{Q_\alpha\}$ is the set of primary ideals that belongs to T. Further, if $I = \bigcap Q_\alpha$ and every Q_α is a valuation ideal, then I is prime.

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1 Introduction

Throughout this paper, S will represent commutative cancellation nontrivial additive semigroup with zero 0. A nonempty subset I of a semigroup S is called an ideal of S if $I + S \subset I$. A proper ideal P of S is called a prime ideal of S if $a + b \in P$ with $a, b \in S$ implies either $a \in P$ or $b \in P$. The supremum of lengths n of chains of prime ideals $0 \subseteq P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$ of a semigroup is called the dimension (dim) of S. So, we write dim $(S) = 1$ if S has one and only one prime ideal. For any $y \in S$, $(y) = y + S = \{y + a \mid a \in S\}$. Then (y) is called a principal ideal of S. For $x_1, x_2, \cdots, x_n \in S$, $I = (x_1, x_2, \cdots, x_n) = \bigcup_{i=1}^{n}(x_i + S)$ is called an ideal generated by $x_1, x_2, \cdots, x_n \in S$. Moreover, an ideal M of S is

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said to be a maximal ideal of S if $M \neq S$ and there exists no ideal I of S where, $M \subseteq I \subseteq S$. The maximal ideal is a prime ideal of S. An element $x \in S$ is called a unit if $x + y = 0$ for some $y \in S$. Also, if $M = \{m \in S \mid \text{m is a non-unit element of } S\}$ is a non-empty set, then M is the unique maximal ideal of S. If I is an ideal of S then the radical of I, denoted by rad(I) or $\sqrt{I}$, is defined to be rad$(I) = \{s \in S \mid ns \in I \text{ for some } n \in N\}$, where N is the set of positive integers. We know that rad(I) is an ideal of S.

A proper ideal Q of S is said to be a primary ideal of S if $x + y \in Q$, where x, $y \in S$ and $x \notin Q \Rightarrow y \in rad(Q)$. It is a well known fact that the concept of primary ideals generalizes the notion of prime ideals. For a primary ideal Q of S, $P = \sqrt{Q}$ is a prime ideal of S, and we term Q a primary ideal belonging to P or a P-primary ideal. Moreover, Q is a P-primary if and only if $\sqrt{Q} = P$ and if $x + y \in Q$ with $x \notin P$ then $y \in Q$. For $t \in T$, let $S[t] = \{s + nt \mid s \in S, n \in N$ or n=0$\}$, where N is the set of natural numbers. An element $t \in T$ is called integral over S if $nt \in S$ for some $n \in N$. The set S^* of elements $t \in T$ that are integral over S is said to be the integral closure of S in T. If $S = S^*$, we say that S is integrally closed in T.

We recall some notations and definitions [2–4] used in the subsequent section of this paper.

(1) A non-empty subset A_s of a semigroup S is called an additive system of S if x, $y \in A_s \Rightarrow x + y \in A_s$ and $0 \in A_s$. Moreover, let $S_{A_s} = \{s - t \mid s \in S, t \in A_s\}$. Then, A_s is an oversemigroup of S and is called an additive system of S and the quotient semigroup S_T is denoted by S_P. Each element $t \in A_s$ is a unit of S.

(2) Let $G = \{s - s' \mid s, s' \in S\}$, then G is a torsion-free abelian group and S is a subsemigroup of G. G is called the quotient of S and is denoted by $q(S)$. T is called an oversemigroup of S if T is a subsemigroup of $q(S)$ containing S.

(3) Let S be a semigroup and H a totally ordered additive group, that is, H is totally ordered set and additive group such that if $a \leq b$ then $a + c \leq b + c$ for each a, b, $c \in H$. Let $v : q(S) \to H$ be a mapping. We say that v is a valuation on $q(S)$ if $v(\alpha + \beta) = v(\alpha) + v(\beta)$ for any α, $\beta \in q(S)$.

(4) A semigroup S is a valuation semigroup if and only if $S = \{a \in q(S) \mid v(a) \geq 0\}$ for each $a \in q(S)$.

(5) A semigroup S is a valuation semigroup if and only if either $\alpha \in S$ or $-\alpha \in S$ for each $\alpha \in G$.

(6) An ideal I_v of a semigroup S is called a valuation ideal if there exists a valuation semigroup $S_v \supseteq S$ and an ideal I of S_v with $I \cap S = I_v$. If we like to specify the particular valuation semigroup S_v, we say that I_v is a v-ideal. If I_v is a v-ideal, then $(I_v + S_v) \cap S = I_v$.

2 Primary Ideals and Valuation Ideals

To start with, we need the following:

Definition 2.1. A semigroup S (with unit) is called almost semigroup if for any maximal ideal M of S, S_M is a semigroup.

Definition 2.2. A commutative semigroup (with unit) which has only finitely many maximal ideals is called quasi-local semigroup.

Definition 2.3. Let S be a semigroup and $S \subseteq V$ with V a quasi-local semigroup having maximal ideal M. Then $M \cap S$ is called the centre of V on S.

Definition 2.4. A semigroup S is defined to have the finite rank r if every finitely generated subsemigroup of S is generated by $\leq r$ elements. The smallest number r endowed with this property is called the rank of S.

Lemma 2.5. *Let I_v be a valuation ideal of the semigroup S, and P, Q be any subsets of S. Then, we have the following:*

(i) $P + Q \subseteq I_v \Rightarrow \{2p \mid p \in P\} \subseteq I_v$ *or* $\{2q \mid q \in Q\} \subseteq I_v$.

(ii) $2P + 2Q \subseteq I_v \Rightarrow P + Q \subseteq I_v$.

Proof. (i) Let there be $q \in Q$ such that $2q \notin I_v$, and suppose S_v is a valuation semigroup such that $(I_v + S_v) \cap S = I_v$. So for any $p \in P$, $v(2p) \geq v(p+q)$ or $v(2q) \geq v(p+q)$; and so we have $2p \in I_v + S_v$ or $2q \in I_v + S_v$. Since $2q \notin I_v$, $2q \notin I_v + S_v$ and therefore $2p \in I_v + S_v$. Hence, $\{2p \mid p \in P\} \subseteq (I_v + S_v) \cap S = I_v$.

(ii) Let $p \in P$, $q \in Q$ and S_v be a valuation semigroup such that $(I_v + S_v) \cap S = I_v$. Suppose that $v(p) \leq v(q)$. Then $v(p+q) \geq v(2p) \geq v(2p+2q)$, therefore, $p + q \in (2p + 2q) + S_v \subseteq I_v + S_v$. Hence, $p + q \in (I_v + S_v) \cap S = I_v$. □

Theorem 2.6. *A semigroup S is a valuation semigroup if every principal ideal of S is a valuation ideal.*

Proof. By Lemma 2.5(i), we have that $2l \in (l+m)$ or $2m \in (l+m)$, for every nonzero elements l, $m \in S$. Then $l - m$ or $m - l \in S$ and hence S is a valuation semigroup. □

Theorem 2.7. *Suppose Q is a primary ideal of a semigroup S and A_s is an additive system in S so that $Q \cap A_s = \varnothing$. Further, assume S_0 is a semigroup that contains S and $(Q + S_0) \cap S = Q$, and $S_0^* = (S_0)_{A_s}$, $S^* = S_{A_s}$, $Q^* = S_{A_s} + Q$. Then we have that $(S_0^* + Q^*) \cap S^* = Q^*$.*

Proof. Obviously, $Q^* \subseteq (S_0^* + Q^*) \cap S^*$. Let $y \in (S_0^* + Q^*) \cap S^* = (S_0^* + Q) \cap S^*$,

$$y = k - l = m - n, k \in S_0 + Q, l, n \in A_s, m \in S.$$

Thus, $n + k = l + m$. However, $n + k \in S_0 + Q$ and $m + l \in S$, therefore, $m + l \in (S_0 + Q) \cap S = \varnothing$. Since $Q \cap A_s = \varnothing$, $l \in A_s$ implies $l \notin Q$. Hence $m \in Q$ and $y = m - n \in Q^*$. □

Corollary 2.8. *Suppose A_s is an additive system, S is a semigroup and Q is a primary ideal. Then $Q^* = Q + S_{A_s}$ is also a valuation ideal if Q is a valuation ideal.*

Proof. Let $(S_v+Q)\cap S=Q$, where S_v is a valuation semigroup, and if $S_v^*=(S_v)_{A_s}$, then $(S_v^*+Q^*)\cap S^*=Q^*$ by Theorem 2.7. Hence Q^* is a valuation ideal. □

Lemma 2.9. *Suppose V is a semigroup, and $V_1,\cdots,V_n$ are v-ideals of S for some fixed v. Let $s_i\in S$ and $s_i\notin V_i$, $i=1,\cdots,n$, then*

$$s=s_1+\cdots+s_n\notin V_1+V_2+\cdots+V_n.$$

Proof. As $(V_i+S_v)\cap S=V_i$, $s_i\notin V_i\Rightarrow s_i\notin V_i+S_v$. So, $v(s_i)<v(q_i)$ for all $q_i\in V_i+S_v$. However, then $v(s)=v(s_1+\cdots+s_n)<v(q_1+\cdots+q_n)$ for all q_1 $,\cdots,q_n$, $q_i\in V_i+S_v$. This shows that $s\notin(V_1+S_v)+\cdots+(V_n+S_v)$; and because $V_1+\cdots+V_n\subseteq(V_1+S_v)+\cdots+(V_n+S_v)$, we have that $s\notin V_1+\cdots+V_n$. □

Theorem 2.10. *Suppose I_1, I_2 are ideals of the semigroup S, where I_1 is a valuation ideal and $nI_2\subseteq nI_1$. Then, $I_2\subseteq I_1$.*

Proof. There exists $x\in I_2$, $x\notin I_1$ if $I_2\nsubseteq I_1$. Hence, by Lemma 2.9, $nx\notin nI_1$; therefore, $nI_2\nsubseteq nI_1$. □

Lemma 2.11. *Suppose S is a semigroup, and I is an ideal of S such that nI is a valuation ideal for all n. Then, we have that $\chi=\bigcap_{n=1}^{\infty}nI$ is prime.*

Proof. Let $a+b\in\chi$. This implies that $a+b\in 2nI=2(nI)$, for all n. So, by Lemma 2.9, $a\in nI$ or $b\in nI$. Hence, $a\in\chi$ or $b\in\chi$. □

Lemma 2.12. *Suppose P is a prime ideal of a valuation semigroup S, and further assume that T is the intersection of the primary ideals that belongs to P. Then, T is prime, and, moreover, there exists no prime ideal P_1 such that $T\subset P_1\subset P$.*

Proof. Suppose that $S=S_P$ so that S is quasi-local and P is maximal in S. The lemma is established if $T=P$. Further, given $y\in P$, $y\notin Q$, $Q\subset(y)\subseteq P$ if $T\subset P$ so that there exists a P-primary ideal $Q\subset P$. So if Q_α is any P-primary ideal of S, then $iy\in Q_\alpha$ for some i so that $(iy)\subseteq Q_\alpha$. Moreover, $\sqrt{(iy)}=\sqrt{(y)}=P$, and so (iy) is P-primary. Then by Lemma 2.11 this implies that $T=\bigcap_{i=1}^{\infty}(iy)$ is prime. Again, if I is an ideal of S such that $T\subset I\subset P$, then $I\nsubseteq(ny)$ for some n, so that $(ny)\subset I$. Hence, $I\subset P=\sqrt{(ny)}\subseteq\sqrt{I}$ and I is not prime. □

Lemma 2.13. *Suppose $\{I_\alpha\}=V(S)$ is a set of valuation ideals of a semigroup S, and assume for any $I_1,I_2\in V(S)$ there exists an $I_3\in V(S)$ such that $I_3\subseteq I_1\cap I_2$. If $I=\bigcap I_\alpha$, then $\sqrt{I}$ is prime.*

Proof. Let $a+b\in\sqrt{I}$. This implies that $n(a+b)\in I$ for some n. Then $na+nb\in I_\alpha$ for all α; therefore by Lemma 2.5(i), $2na\in I_\alpha$ or $2nb\in I_\alpha$. If $2na\notin I_1$ and $2nb\notin I_2$ for some $I_1,I_2\in V(S)$, then there exists $I_3\in V(S)$ such that $I_3\subseteq I_1\cap I_2$; and then $2na\notin I_3$, $2nb\notin I_3$, which is a contradiction. Therefore, we suppose $2na\in I_\alpha$ for all α. However, then $2na\in I$ and hence $a\in\sqrt{I}$. □

Theorem 2.14. *Suppose T is a prime ideal of a semigroup S, and $\{Q_\alpha\}$ is the set of primary ideals that belongs to T. If $I = \bigcap Q_\alpha$ and every Q_α is a valuation ideal, then I is prime.*

Proof. Suppose Q is a primary ideal of S, and let S_v be a valuation semigroup such that $Q_v \cap S = Q$, $Q_v = Q + S_v$. If $T_v = \sqrt{Q_v}$, T_v is prime and $T_v \cap S = T$. The prime ideals of a valuation semigroup are linearly ordered so that every ideal of a valuation semigroup has prime radical. Suppose T_v^* is the intersection of the T_v-primary ideals of S_v, and suppose $T^* = T_v^* \cap S$. By Lemma 2.12, T_v^* is prime, therefore T^* is also prime. So, $I \subseteq T^* \subseteq Q$, and consequently $\sqrt{I} \subseteq T^* \subseteq Q$. Because this is valid for any T_v-primary ideal Q, $\sqrt{I} \subseteq I$. Hence, $\sqrt{I} = I$. Moreover, if Q_1, Q_2 are T-primary ideals, then $Q_3 = Q_1 \cap Q_2$, is also T-primary. So, by using Lemma 2.13, we infer that $I = \sqrt{I}$ is prime. □

Lemma 2.15. *Suppose L is a prime ideal of a semigroup S, and assume there is a prime ideal $P \subset L$ such that there exists no prime ideal P_1 such that $P \subset P_1 \subset L$. Then, P is the intersection of the L-primary ideals of S containing P.*

Proof. We prove the lemma under the hypothesis that S is a one-dimensional and quasi-local semigroup with maximal ideal L and $P = (0)$. Since every nonzero ideal is L-primary, and intersection of all nonzero ideals of S is (0), the lemma is established. □

Theorem 2.16. *Suppose L is a prime ideal of a semigroup S, and let every L-primary ideal is a valuation ideal. If we have a prime ideal $P \subset L$ and there exists no prime ideal P_1 such that $P \subset P_1 \subset L$, then P is unique.*

Proof. Suppose L_0 is the intersection of the L-primary deals. L_0 is prime and is properly contained in L by Theorem 2.14 and Lemma 2.15. We now prove $P \subseteq L_0$. Then it implies that $P = L_0$ and so P is unique. By the 1-1 correspondence between primary(prime) ideals of $S \subset L$ and primary(prime) ideals of S_L and Corollary 2.8, we substitute S by S_L and therefore suppose that S is quasi-local with maximal ideal L. Suppose then Q is any L-primary ideal of S, and then we prove $P \subseteq Q$. Let $y \in Q$, $y \notin P$ and assume $N = Q + P + (4y)$. Then $N \subseteq Q$ and $\sqrt{N} \supseteq (P, y) \supset P$; therefore $\sqrt{N} = L$, and so N is L-primary. By our assumption, N is a valuation ideal, therefore there exists a valuation semigroup S_v and an ideal N_v of S_v such that $N_v \cap S = N$; and now let $N_v = N + S_v$. Further suppose $P_v = P + S_v$, $Q_v = Q + S_v$. Now on the contrary assume $2y \in P_v \Rightarrow y + 2y \in (Q_v + P_v) \cap S \subseteq N$. Then $3y = m + n + 4y$, $m \in Q + P$, $n \in S$. So, $y + (m + n) = 0 \in P$. Since $m + n$ is a unit of S, this shows $y \in P$, a contradiction. So, $2y \notin P_v$. Since S_v is a valuation semigroup, the ideals of S_v are linearly ordered; therefore $2y \notin P_v \Rightarrow P_v \subseteq 2y + S_v$. So, $2P_v \subseteq (2y + S_v) + P_v$. However, $2P + (2y)$ is a valuation ideal, so by Lemma 2.5(ii), $y + P \subseteq 2P + (2y)$. Therefore, $y + P \subseteq 2P + (2y) + P$ because $y \notin P$. Then $(y + S_v) + P_v \subseteq (2P_v) + (2y + S_v) + P_v = (2y + S_v) + P_v$. So, $(y + S_v) + P_v = (2y + S_v) + P_v \Rightarrow P_v = (y + S_v) + P_v =$

$(2y + S_v) + P_v = (3y + S_v) + P_v = \cdots$. So, $P_v \subseteq \bigcap_{i=1}^{\infty} (iy + S_v) = P_1$, where P_1 is prime by Lemma 2.11, and $y \notin P_1 \Rightarrow P_1 \cap S \subset L$. So, $P \subseteq P_v \cap S \subseteq P_1 \cap S \subset L$, so by our assumption, $P = P_1 \cap S$. This implies N and P are v-ideals for the same v. Because $N \nsubseteq P$, we get $P = P_v \cap S \subseteq N_v \cap S = N$. Hence, $P \subseteq N \subseteq Q$. □

A semigroup satisfies the ascending chain condition(a.c.c.) for prime ideals provided any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset P_3 \subset \cdots$ terminates after a finite number of steps. Equivalently, every nonempty family of prime ideals contains a maximal element.

Lemma 2.17. *Suppose S is a quasi-local semigroup, and for any nonzero prime ideal P of S there exists a prime ideal $N(P) \subset P$ such that if P_1 is a prime ideal $\subset P$, then $P_1 \subseteq N(P)$. Then S satisfies the a.c.c. for prime ideals and the prime ideals of S are linearly ordered.*

Proof. Let $P_1 \subset P_2 \subset P_3 \subset \cdots$ be an ascending chain of prime ideals of S, then $P = \bigcup P_i$ is also prime; therefore if $P \neq P_i$ for all i, then $P_i \subseteq N(P)$ for all i; and thus we get $P = \bigcup P_i \subseteq N(P) \subset P$, which is a contradiction. So, S has the a.c.c. for prime ideals. Now let there be prime ideals P_1, P_2 of S such that $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Because S satisfies the a.c.c. for prime ideals, there exists a prime ideal L, which is maximal in view of $P_1 \subseteq L$, $P_2 \nsubseteq L$. Since $P_2 \nsubseteq L$, L is not the maximal ideal of S and there exists a prime ideal $L_\beta \supset L$. If $\{L_\beta\}$ is the set of all such prime ideals, then $L \neq \bigcap L_\beta$ since $P_2 \subseteq \bigcap L_\beta$ and $P_2 \nsubseteq L$. So, by Zorn's lemma, there exists a prime ideal L_0 minimal to $L_0 \supset L$. So, $L \subseteq N(L_0) \subset L_0 \Rightarrow L = N(L_0)$. However, then $P_2 \subset L_0$ implies $P_2 \subseteq N(L_0) = L$, and that is a contradiction to our hypothesis. □

Theorem 2.18. *Suppose S is a quasi-local semigroup such that S has the a.c.c. for prime ideals. If $Q(S) \subseteq V(S)$, then the prime ideals of S are linearly ordered, where $Q(S)$ is the set of primary ideals and $V(S)$ is the set of valuation ideal.*

Proof. Let P be any nonzero prime ideal of S. Since S has the a.c.c. for prime ideals, the set of all prime ideals $P_1 \subset P$ contains a maximal element $N(P)$. So, $N(P)$ is unique by Theorem 2.16 and so contains every prime ideal $P_1 \subset P$. Hence, by Lemma 2.17, the prime ideals of S are linearly ordered. □

Theorem 2.19. *Suppose S is a semigroup which satisfies the a.c.c. for prime ideals. Then the following conditions are equivalent:*

(i) There exists a finite set $S_{v_1}, \cdots, S_{v_n}$ of valuation semigroups such that every primary ideal of S is a v_i-ideal for some i.

(ii) S is valuation semigroup with maximal ideals.

Proof. $(i) \Rightarrow (ii)$ S is a valuation semigroup by [1, Theorem 4]. If M is a maximal ideal of S, M is a v_i-ideal for some v_i and hence M is the centre of S_{v_i} on S. We have that there exists at most n such distinct centres.

$(ii) \Rightarrow (i)$ Suppose $M_1, \cdots, M_t$, $t \leq n$ are the maximal ideals of S. Because S is valuation semigroup, S_{M_i} is a valuation semigroup, and then $S_{M_1}, \cdots, S_{M_t}$ are the desired valuation semigroup. □

Corollary 2.20. *Suppose S is a semigroup which has a.c.c. for prime ideals, and assume that every primary ideal of S is a v-ideal for some fixed v. Then S is a valuation semigroup.*

Proof. S is a semigroup having one maximal ideal M by Theorem 2.19, and hence $S = S_M$ is a valuation semigroup. □

Theorem 2.21. *Suppose S is a semigroup and $T \supset U$ prime ideals of S such that T is a minimal prime of $U + I$ for some finitely generated ideal I and such that every T-primary ideal is a valuation ideal. Suppose P is the intersection of T-primary ideals. Then P is a prime ideal satisfying the relation $U \subseteq P \subseteq T$ and there exists no prime ideal P_1 having $P \subset P_1 \subset T$.*

Proof. Because T is a minimal prime of $U + I$, T is not a union of prime ideals properly between U and T, we use Zorn's lemma to obtain that there is a prime ideal P with $U \subseteq P \subset T$ and there is no prime ideal P_1 satisfying $P \subset P_1 \subset T$. Hence, using Theorem 2.16, we get that P is the intersection of all T-primary ideals. □

Corollary 2.22. *Suppose S is a semigroup such that $Q(S) \subseteq V(S)$, and for every ideal P of S there is a valuation semigroup S_v of rank 1 such that P is a v-ideal. Then dim $S \leq 1$ and S is a valuation semigroup.*

Proof. Let there be prime ideals $T \subset L$ in S. Let $y \in L$, $y \notin T$ and L_0 be a minimal prime of $T + (y)$. Then $T \subseteq P$, where P is the intersection of the L_0-primary ideals and by Theorem 2.21 $T \subset L_0$. There exists a rank 1 valuation semigroup S_v such that $(L_0 + S_v) \cap S = L_0$; therefore if L_v is the maximal ideal of S_v, then $L_v \cap S = L_0$. So, every L_v-primary ideal of S_v gives in to an L_0-primary ideal of S. Because S_v has rank 1, the intersection of the L_v-primary ideals of S_v is (0). So, the intersection P of the L_0-primary ideals of S is also (0). Therefore, $T \subseteq P = (0)$, so, dim $S = 1$. Hence by [1, Theorem 4], S is a valuation semigroup. □

Corollary 2.23. *A semigroup S with quotient group G is almost semigroup if $Q(S) \subseteq V(S)$ and for every prime ideal P of S there is a rank 1, discrete valuation semigroup $S_v \subset G$ with P is a v-ideal.*

Proof. Let there be a proper prime ideal P of S. Then dim $S = 1$ and S is a valuation semigroup by Corollary 2.22. So, S_p is of rank 1 valuation semigroup and therefore S_p is a maximal subsemigroup of G. However, if P is a v-ideal, then $S_p \subseteq S_v \subset G$; therefore, $S_p = S_v$. Hence, S_p is , discrete, rank 1 and so, S is almost semigroup. □

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