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Common Fixed Point Theorems in Complex Valued Metric Space and Application

R. K. Verma†,1 and H. K. Pathak‡

†Department of Mathematics, Govt. C. L. C. College Patan
Dist.-Durg (C. G.) 491111, India
e-mail : rohitverma1967@rediffmail.com

‡School of Studies in Mathematics, Pt. Ravishankar Shukla University
Raipur (C. G.) 492010, India
e-mail : hkpathak05@gmail.com

Abstract : In this paper, we prove some common fixed point results for two pairs of weakly compatible mappings in a complex valued metric space by using a new property. Our result is then applied to prove a common fixed point theorem in GV-fuzzy metric space. These results are the generalization of various theorems of ordinary and fuzzy metric spaces.

Keywords : complex-valued metric space; contractive condition; (CLR)-property; (CLCS)-property; (E.A)-property; partial order; weakly compatible mappings.
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1 Introduction

In 2011, Azam et. al. [1] introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The theorems proved by Azam et. al. [1] and Bhatt et. al. [2] uses the rational inequality in a complex valued metric space as contractive condition. We use more general contractive condition.

[1]Corresponding author.

The study of existence of common fixed point grown from weakly commutativity [3] to compatibility [4] and weakly compatibility [5]. Similarly, the noncommutativity of mappings grown from noncompatibility [6] to property (E.A) [7]. The concept of (E.A) property allows us to replace the completeness requirement of the space by a more natural condition of closeness of range. Pathak, Lopez and Verma [8] proved a common fixed point theorem in metric space for an integral type implicit relation using the property (E.A). By using the idea of property (E.A), Sintunavarat and Kumam [9] introduced the concept of 'common limit range property' or (CLR)-property, for a pair of mappings. They proved some common fixed point results for a pair of weakly compatible mappings in a fuzzy metric space satisfying (CLR)-property. Sintunavarat, Cho and Kumam [10] proved a common fixed point theorem in complex valued metric space using Urysohn integral equation approach. Also, Sintunavarat and Kumam [11] proved common fixed point theorem for R-weakly commuting mappings in fuzzy metric space. More results on common fixed point theorems using the (CLR) property can be find in [12–21] etc.

In this paper, we prove some common fixed point theorems for two pairs of weakly compatible mappings in a complex-valued metric space by using new property, known as common limit converging in the subset or (CLCS)-property. Using this idea we will prove a common fixed point theorem for two pairs of weakly compatible mappings in a complex valued metric space. We apply this result to prove a common fixed point theorem in GV-fuzzy metric space.

An ordinary metric d is a real-valued function from a set $X \times X$ into $\mathbb{R}$, where X is a nonempty set. That is, d : $X \times X \to \mathbb{R}$. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second coordinate is called $Im(z)$. Thus a complex valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, where X is a nonempty set and $\mathbb{C}$ is the set of complex number. That is, $d : X \times X \to \mathbb{C}$. Let $z_1, z_2 \in \mathbb{C}$, define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), \ \ Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2), \ \ Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), \ \ Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), \ \ Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), \ \ Im(z_1) = Im(z_2)$.

In (i), (ii) and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So $|z_1| \leq |z_2|$.

In particular, $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), (iii) is satisfy. In this case $|z_1| < |z_2|$. We will write $z_1 \prec z_2$ if only (iii) satisfy. Further,

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2 \ \text{ and } \ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Azam et. al. [1] defined the complex-valued metric space (X, d) in the following way:

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:
(C1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(C3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a *complex valued metric* on X, and (X, d) is called a *complex valued metric space.*

A point $x \in X$ is called an *interior point* of $A \subseteq X$ if there exists $r \in \mathbb{C}$, where $0 \prec r$, such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a *limit point* of $A \subseteq X$, if for every $0 \prec r \in \mathbb{C}$, we have

$$B(x, r) \cap (A\ X) \neq \phi.$$

The set A is called *open* whenever each element of A is an interior point of A. A subset B is called *closed* whenever each limit point of B belongs to B.

The family $\mathcal{F} := \{B(x, r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is called *convergent.* Also, $\{x_n\}$ *converges* to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$); and x is the *limit point* of $\{x_n\}$. The sequence $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} |d(x_n, x)| = 0$.

If for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called *Cauchy sequence* in (X, d). If every Cauchy sequence converges in X, then X is called a *complete complex valued metric space.* The sequence $\{x_n\}$ is called Cauchy if and only if $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.

Definition 1.2. [2, 5] A pair of self-mappings $A, S : X \to X$ is called *weakly-compatible* if they commute at their coincidence points. That is, if there be a point $u \in X$ such that $Au = Su$, then $ASu = SAu$, for each $u \in X$.

Example 1.3. Let $X = \mathbb{C}$. Define complex-metric $d : X \times X \to \mathbb{C}$ by: $d(z_1, z_2) := e^{ia}|z_1 - z_2|$, where $a \in [0, \frac{\pi}{2})$. Then (X, d) is a complex-valued metric space. Suppose $A, S : X \to X$ be defined as:

$$Az = 2e^{i\pi/4}, \text{ if Re(z)}\neq 0, \quad Az = 3e^{i\pi/3}, \text{ if Re(z)=0}, \quad \text{and}$$

$$Sz = 2e^{i\pi/4}, \text{ if Re(z)}\neq 0, \quad Sz = 4e^{i\pi/6} \text{ if Re(z)=0}.$$

Then observe that $Az = Sz = 2e^{i\pi/4}$, when Re(z)$\neq 0$, and so $ASz = SAz = 2e^{i\pi/4}$. Hence (A, S) is weakly compatible at all $z \in \mathbb{C}$ with Re(z)$\neq$0.

Definition 1.4. We define the '' function, call 'selection of one co-ordinate among these options', for the partial order relation $\precsim$ by:
(1) $\{z_1, z_2\} = z_2$ if $z_1 \precsim z_2$.
(2) $z_1 \precsim \{z_2, z_3\} \Rightarrow z_1 \precsim z_2$, or $z_1 \precsim z_3$.
(3) $\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$, or $|z_1| \leq |z_2|$.

Using Definition 1.4 we have the following Lemma:

Lemma 1.5. *Let $z_1, z_2, z_3.... \in \mathbb{C}$ and the partial order relation $\precsim$ is defined on $\mathbb{C}$. Then following statements are easy to prove:*
(i) If $z_1 \precsim z_2$ and $z_2 \precsim \{z_2, z_3\}$, then $z_1 \precsim z_2$;
(ii) If $\{z_1, z_2\} \precsim z_3$ and $z_4 \precsim \{z_1, z_3, z_4\}$ then $z_1 \precsim z_4$ if ;
(iii) If $z_1 \precsim \{z_2, z_3, z_4, z_5\}$ and $\{z_3, z_4, z_5\} \precsim z_2$, then $z_1 \precsim z_2$ and so on.

Definition 1.6. (Sintunavarat and Kumam [9]) Suppose that (X, d) is a metric space and $f, g : X \to X$. Two mappings f and g are said to satisfy the *common limit in the range of g* property, in short, (CLR_g)-property if:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx, \tag{1.1}$$

for some $x \in X$.

In a complex-valued metric space (X, d), Definition 1.6 will be same but the space X will be a complex-valued metric space.

Remark 1.7. If the mapping pair $f, g : X \to X$ satisfy (CLR_g)-property, then it also satisfy (CLR_f)-property and vice-versa. Examples 1.8 and 1.9 below, verify this fact. Keeping this view in mind, we are going to unify the (CLR_g) and (CLR_f)-properties, in our notion of (CLCS) property.

Example 1.8. Let $X = \mathbb{C}$ and d be any complex valued metric on X. Define $f, g : X \to X$ by $fz = z + 2i$ and $gz = 3z$, $\forall z \in X$. Consider a sequence $\{z_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ in X, then

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} z_n + 2i = \lim_{n\to\infty} (i + \frac{1}{n}) + 2i = 3i,$$

and

$$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} 3(i + \frac{1}{n}) = 3i = g(0 + i).$$

Hence, the pair (f, g) satisfies property (CLR_g) in X with $x = 0 + i \in X$.

Example 1.9. Let $X = \mathbb{C}$ and $d(z_1, z_2) = e^{ia}|z_1 - z_2|$ be any complex-valued metric on X. Define $f, g : X \to X$ by: $fz = 2z - 4$ and $gz = z + 2i$, $\forall z \in X$. Consider a sequence $\{z_n\} = \{4 + 2i + \frac{1}{n}\}_{n\geq 1}$ in X, then

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} 2z_n - 4 = 4 + 4i = \lim_{n\to\infty} gz_n = \lim_{n\to\infty} z_n + 2i = 4 + 4i = g(4 + 2i).$$

Hence, the pair (f, g) satisfies property (CLR_g) in X with $x = 4+2i \in X$. Observe that, (f, g) satisfy (CLR_f) property also, in X.

In Definition 1.6, the notion of (CLR) property does not require the condition of closeness of the range (sub)space but the common limit t goes to different sets (in fX for (CLR_f) and in gX for (CLR_g)). By unifying above definition of (CLR_f) and (CLR_g) properties and by generalizing the (E.A) property, we give the following notion:

Definition 1.10. [21] Suppose that (X, d) be a complex valued metric space and $f, g : X \to X$. Let $Y \subseteq X$. The mappings f, g are said to satisfy the *property of common limit in the subset* Y, in brief (CLCS) property, if there exist a sequence $\{z_n\}$ in X such that

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n \in Y \tag{1.2}$$

for some sequence $\{z_n\}$ in X.

Remark 1.11. The (CLR_g) and (CLR_f) properties unify if $Y = fX \cap gX$.

Following are some examples of (CLCS) property in metric and complex-valued metric spaces:

Example 1.12. Let (X, d) be any complex valued metric space and $f, g : X \to X$ be defined as $fz = \frac{z}{4}$, $gz = \frac{3z}{4}$, $\forall z \in X$. Then for the sequence $\{z_n\} = \{\frac{1}{n}\}$, we have

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n = 0 \in fX \cap gX.$$

Hence (f, g) is (CLCS) in $fX \cap gX$.

Example 1.13. Let (X, d) be any complex valued metric space and $f, g : X \to X$ be defined as $fz = z+1$, $gz = 2z$, $\forall z \in X$. Then for the sequence $\{z_n\} = \{1+\frac{1}{n}\}$, we have

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n = 2 \in fX \cap gX.$$

Hence (f, g) is (CLCS) in $fX \cap gX$.

Example 1.14. Let (X, d) be a usual metric space and $f, g : X \to X = [0, \infty)$ be defined as $fx = x^2 + 2$, $gz = 2x + 10$, $\forall x \in X$. Then for the sequence $\{x_n\} = \{4 - \frac{1}{n}\}$, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 18 \in fX \cap gX = [10, \infty).$$

Hence (f, g) is (CLCS) in $fX \cap gX$.

Example 1.15. Let (X, d) be a metric space with $d(x, y) = \frac{1}{4}|x - y|$; and $f, g : X \to X = [5, 50]$ be defined as $fx = \frac{x+5}{2}$, $gz = \frac{x+15}{3}$, $\forall x \in X$. Then for the sequence $\{x_n\} = \{15 + \frac{1}{n}\}$, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 10 \in fX \cap gX = [5, \frac{65}{3}].$$

Hence (f, g) is (CLCS) in $fX \cap gX$.

2 Preliminaries

Let $\phi : \mathbb{C} \to \mathbb{C}$ be a continuous function with the partial order relation '$\prec$' in $\mathbb{C}$ such that $\phi(0) = 0$ and $|\phi(p)| < |p|, \ \forall p \in \mathbb{C}$.

Example 2.1. If $\phi(z) = \frac{1}{2}z \ \forall z \in \mathbb{C}$, then ϕ is continuous, $\phi(0) = 0$ and $|\phi(p)| < |p|$.

Theorem 2.2. *Let (X, d) be a complex valued metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying:*

(i) *(A, S) satisfy (CLCS) property in $T(X)$ or $B(X)$, and (B, T) satisfy (CLCS) property in $S(X)$ or $A(X)$,*

(ii) $d(Ax,By) \precsim \phi\Big(\max\big\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(By,Sx), d(Ax,Ty)\big\}\Big)$ *for all $x, y \in X$. If (A, S) and (B, T) are weakly compatible then mappings A, B, S and T have a unique common fixed point in X.*

Proof. We take condition (i), one by one.

Case I. First suppose that the pair (A, S) satisfy (CLCS) property in $T(X)$. Then, according to Definition 1.10, there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n \in TX$. So, there exist $t \in T(X)$ such that $t = Tv$ for some $v \in X$, where $t = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$. We claim that $Bv = t$, i.e. $d(Bv, t) = 0$. If not, then putting $x = x_n, \ y = v$ in (ii) we have

$$d(Ax_n,Bv) \precsim \phi\Big(\max\big\{d(Sx_n,Tv), d(Ax_n,Sx_n), d(Bv,Tv), d(Bv,Sx_n), d(Ax_n,Tv)\big\}\Big),$$

since ϕ is continuous, letting $n \to \infty$, we have

$$d(t, Bv) \precsim \phi\Big(\max\big\{0, 0, d(Bv, t), d(Bv, t), 0\big\}\Big). \tag{2.1}$$

• If we choose $\max\big\{0, 0, d(Bv, t), d(Bv, t), 0\big\} = 0$ in eq.(2.1), then $d(t, Bv) \precsim \phi(0)$, from which $|d(t, Bv)| \le |\phi(0)| < 0$, which is a contradiction.

• If we choose $\max\big\{0, 0, d(Bv, t), d(Bv, t), 0\big\} = d(Bv, t)$ in eq.(2.1), then we have

$$d(t, Bv) \precsim \phi\big(d(t, Bv)\big),$$

whence, $|d(t, Bv)| \le |\phi\big(d(t, Bv)\big)| < |d(t, Bv)|$, as $|\phi(t)| < |t|$, which is again a contradiction.

Thus, in both cases, the assumption of $d(Bv, t) \neq 0$ is wrong. So, $Bv = t$. It shows that v is a coincidence point of (B, T). The weakly compatibility of the pair (B, T) yields $BTv = TBv$, or $Bt = Tt$.

Now, we claim that t is a common fixed point of (B, T). If not, then $Bt = Tt \neq t$. So put $x = y_n, \ y = t$ in (ii) we have

$$d(Ay_n, Bt) \precsim \phi\Big(\max\big\{d(Sy_n, Tt), d(Ay_n, Sy_n), d(Bt, Tt), d(Bt, Sy_n), d(Ay_n, Tt)\big\}\Big),$$

since ϕ is continuous, letting $n \to \infty$, we have

$$d(t, Bt) \precsim \phi\Big(\max\big\{d(t, Bt), 0, 0, d(Bt, t), d(t, Bt)\big\}\Big). \tag{2.2}$$

• If we choose $\max\big\{d(t, Bt), 0, 0, d(Bt, t), d(t, Bt)\big\} = 0$ in eq.(2.2), then $d(t, Bt) \precsim \phi(0)$, from which $|d(t', Bt)| \le |\phi(0)| < 0$, which is a contradiction.

• If we choose $\max\big\{d(t, Bt), 0, 0, d(Bt, t), d(t, Bt)\big\} = d(Bt, t)$ in eq.(2.2), then we have

$$d(t, Bt) \precsim \phi\big(d(t, Bt)\big),$$

whence, $|d(t, Bt)| \le |\phi\big(d(t, Bt)\big)| < |d(t, Bt)|$, as $|\phi(t)| < |t|$, which is again a contradiction. Thus, in both cases, the assumption of $d(Bt, t) \ne 0$ is wrong. So, $Bt = t$. It shows that $t \in T(X)$ is a common fixed point of (B, T).

Case II. Similar argument arises if the pair (A, S) satisfy (CLCS) property in $B(X)$. In this case $t \in B(X)$ is a common fixed point of (B, T).

Case III. Next, suppose that the pair (B, T) satisfy (CLCS) property in $S(X)$. Then, according to Definition 1.10, there exist a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n \in S(X)$. So, there exist $t' \in S(X)$ such that $t' = Su$ for some $u \in X$, where $t' = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$. The claim $Au = t'$ follows exactly as in case I. It shows that u is a coincidence point of (A, S). The weakly compatibility of (A, S) implies that $ASu = SAu = At' = St'$. It shows that t' is a coincidence point of (A, S) and $t \in S(X)$.

Now, we claim that t' is a common fixed point of (A, S). This follows exactly as in case I, by putting $x = t'$, $y = y_n$ in condition (ii), making $n \to \infty$, and using $At' = St'$. Hence $At' = t'$. It shows that $t' \in S(X)$ is a common fixed point of (A, S).

Case IV. Similar argument arises if the pair (B, T) satisfy (CLCS) property in $A(X)$. In this case $t' \in A(X)$ is a common fixed point of (A, S).

Further, we claim that the common fixed point t' of (A, S), and t of (B, T) are same, i.e., $t = t'$. If not, then put $x = t'$, $y = t$ in condition (ii), we have

$$d(At', Bt) \precsim \phi\Big(\max\big\{d(St', Tt), d(At', St'), d(Bt, Tt), d(Bt, St'), d(At', Tt)\big\}\Big),$$

or, $$d(t', t) \precsim \phi\Big(\max\big\{d(t', t), 0, 0, d(t, t'), d(t', t)\big\}\Big),$$

• If we choose $\max\big\{d(t', t), 0, 0, d(t, t'), d(t', t)\big\} = 0$, then $d(t, t') \precsim \phi(0)$, from which $|d(t, t')| \le |\phi(0)| < 0$, which is a contradiction.

• If we choose $\max\big\{d(t', t), 0, 0, d(t, t'), d(t', t)\big\} = d(t, t')$, then we have

$$d(t, t') \precsim \phi\big(d(t, t')\big),$$

whence, $|d(t,t')| \leq |\phi\big(d(t,t')\big)| < |d(t,t')|$, as $|\phi(t)| < |t|$, which is again a contradiction. Thus, in both cases, the assumption of $d(t,t') \neq 0$ is wrong. So, $t = t'$. It shows that t' is a common fixed point of A, B, S, T. The existence of common fixed point is easy to prove. This completes the proof. □

Remark 2.3. The subset in which $t = t'$ exist is either $Y_1 = TX \cap SX$, or $Y_2 = TX \cap AX$, $Y_3 = BX \cap SX$, or $Y_4 = BX \cap AX$. Note that, $Y_1, ..., Y_4$ need not be closed.

Putting $A = B = f$ and $S = T = g$ in Theorem 2.2, we obtain the following common fixed point result for one pair of weakly compatible mappings:

Corollary 2.4. *Let (X, d) be a complex valued metric space and $f, g : X \to X$ be two self-mappings satisfying:*
(i) (f, g) *satisfy (CLCS) property in* $g(X)$,

$$(ii) \qquad d(fx, gy) \precsim \phi\Big(\max\big\{d(gx, gy), d(fx, gx), d(fy, gy), d(fy, gx), d(fx, gy)\big\}\Big),$$

for all $x, y \in X$. If (f, g) is weakly compatible then mappings f and g have a unique common fixed point in X.

If the (CLCS) property of one pair lies in the common range-subspace of the other pair, and vice-versa, we have the following theorem:

Theorem 2.5. *Let (X, d) be a complex valued metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying:*
(i) (A, S) satisfy (CLCS) property in $T(X) \cap B(X)$, and (B, T) satisfy (CLCS) property in $S(X) \cap A(X)$,

$$(ii)\ d(Ax, By) \precsim \phi\Big(\max\big\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)\big\}\Big),$$

for all $x, y \in X$. If (A, S) and (B, T) are weakly compatible then mappings A, B, S and T have a unique common fixed point in X.

Proof. In the proof, we merge cases I and II of Theorem 2.2 for the pair (A, S) to satisfy (CLCS) property. Similarly, we merge cases III and IV for the pair (B, T) to satisfy (CLCS) property. The proof exactly runs as that of Theorem 2.2. □

Putting $A = B = f$ and $S = T = g$ in Theorem 2.5, we obtain the following result for a pair of weakly compatible mappings:

Corollary 2.6. *Let (X, d) be a complex valued metric space and $f, g : X \to X$ be two self-mappings satisfying:*
(i) (f, g) *satisfy (CLCS) property in* $f(X) \cap g(X)$,

$$(ii) \qquad d(fx, gy) \precsim \phi\Big(\max\big\{d(gx, gy), d(fx, gx), d(fy, gy), d(fy, gx), d(fx, gy)\big\}\Big),$$

for all $x, y \in X$. If (f, g) is weakly compatible then mappings f and g have a unique common fixed point in $fX \cap gX$.

3 Application in Fuzzy Metric Space

In 1994, George and Veeramani [22], introduced and studied a notion of fuzzy metric space which constitute a modification of the one due to Kramosil and Michalek [23].

Definition 3.1. (George and Veeramani [22]) A fuzzy metric space is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, 1]$ and the following conditions are satisfied for all $x, y \in X$ and $\alpha, \beta > 0$;
(GV-1) $M(x, y, \alpha) > 0$;
(GV-2) $M(x, y, \alpha) = 1 \Leftrightarrow x = y$;
(GV-3) $M(x, y, \alpha) = M(y, x, \alpha)$;
(GV-4) $M(x, y, .) : (0, \infty) \to (0, 1)$ is continuous;
(GV-5) $M(x, y, \alpha + \beta) \geq M(x, y, \alpha) * M(x, y, \beta)$.

From (GV-1) and (GV-2), it follows that if $x \neq y$, then $0 < M(x, y, \alpha) < 1$ for all $\alpha > 0$. In what follows, fuzzy metric spaces in the sense of George and Veeramani will be called GV-fuzzy metric spaces.

Definition 3.2. (Schweizer and Sklar [24]) A continuous t-norm is a binary operation $*$ on [0,1] satisfying the following conditions:
(t-1) $*$ is commutative and associative;
(t-2) $a * 1 = a$;
(t-3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$;
(t-4) the mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous.

Example 3.1. $T_{Max}(a, b) = \{a+b-1, 0\}$, $T_{min}(a, b) = \min\{a, b\}$, $T_{product}(a, b) = a.b$ are some t-norms.

Throughout this section, let Φ be class of all mappings $\varphi : [0, 1] \to [0, 1]$ satisfying the following properties:
(φ.1) φ is continuous and non-decreasing in [0,1];
(φ.2) $\varphi(x) > x, \ \forall x \in (0, 1)$.
Sintunavarat and Kumam [25] proved a common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space.

Following theorem is the GV-fuzzy metric version of our main Theorem 2.2:

Theorem 3.3. *Let $(X, M, *)$ be a GV-fuzzy metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying:*
(i) (A, S) satisfy (CLCS) property in $T(X)$ or $B(X)$, and (B, T) satisfy (CLCS) property in $S(X)$ or $A(X)$,

$$(ii)\ M(Ax, By, \alpha) \geq \varphi\Big(\min\big\{M(Sx, Ty, \alpha), M(Ax, Sx, \alpha), M(By, Ty, \alpha), M(By, Sx, \alpha), M(Ax, Ty, \alpha)\big\}\Big),$$

for all $x, y \in X$, where $\alpha > 0$ and $\varphi \in \Phi$. If (A, S) and (B, T) are weakly compatible then mappings A, B, S and T have a unique common fixed point in X.

Proof. We take condition (i), one by one.

Case I. First suppose that the pair (A, S) satisfy (CLCS) property in $T(X)$. Then, according to Definition 1.10, there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n \in TX. \tag{3.1}$$

So, there exist $t \in T(X)$ such that $t = Tv$ for some $v \in X$, where $t = \lim_{n\to\infty} Ax_n$ $= \lim_{n\to\infty} Sx_n$. Let α be a continuity point of $(X, M, *)$. Then

$$M(Ax_n, Bv, \alpha) \geq \varphi\Big(\min\big\{M(Sx_n, Tv, \alpha), M(Ax_n, Sx_n, \alpha), M(Bv, Tv, \alpha), M(Bv, Sx_n, \alpha), M(Ax_n, Tv, \alpha)\big\}\Big), \tag{3.2}$$

for all $n \in \mathbb{N}$. By taking the limit as $n \to \infty$ in (3.2) and using (3.1), we have

$$\begin{aligned} M(t,Bv,\alpha) &\geq \varphi\Big(\min\big\{M(t,t,\alpha), M(t,t,\alpha), M(Bv,t,\alpha), M(Bv,t,\alpha), M(t,t,\alpha)\big\}\Big), \\ &= \varphi\Big(\min\big\{1, 1, M(Bv, t, \alpha), M(Bv, t, \alpha), 1\big\}\Big) = \varphi\big(M(t, Bv, \alpha)\big). \end{aligned}$$

Now, we claim that $t = Bv$. If not, then from (GV-1) and (GV-2),

$$0 < M(t, Bv, \alpha) < 1, \tag{3.3}$$

for all $\alpha > 0$. From conditions of $(\varphi.2)$, $\varphi\big(M(t, Bv, \alpha)\big) > M(t, Bv, \alpha)$, $\forall \alpha \in (0, 1)$. This is a contradiction. Thus $t = Bv$. Therefore, we have

$$Ax_n = \lim_{n\to\infty} Sx_n = t = Tv = Bv \in TX. \tag{3.4}$$

Eq. (3.4) shows that v is a coincidence point of the pair (B, T). Now, the weakly compatibility of (B, T) gives, $BTv = TBv = Bt = Tt$. Hence t is a coincidence point of (B, T).

Now, we show that t is a common fixed point of the pair (B, T). If not, then $Bt \neq t$. By (GV-1) and (GV-2), it implies that $0 < M(t, Bt, \alpha) < 1$ for all $\alpha > 0$. By $(\varphi.2)$, we know that $\varphi\big(M(t, Bt, \alpha)\big) > M(t, Bt, \alpha)$. Let α be a continuity point of $(X, M, *)$, then from condition (ii) we have

$$M(Ax_n, Bt, \alpha) \geq \varphi\Big(\min\big\{M(Sx_n, Tt, \alpha), M(Ax_n, Sx_n, \alpha), M(Bt, Tt, \alpha), M(Bt, Sx_n, \alpha), M(Ax_n, Tt, \alpha)\big\}\Big),$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, putting $Bt = Tt$ and using (3.1) we have

$$M(t,Bt,\alpha)\geq\varphi\Big(\min\big\{M(t,Bt,\alpha),M(t,t,\alpha),M(Bt,Bt,\alpha),M(Bt,t,\alpha),M(t,Bt,\alpha)\big\}\Big)$$
$$=\varphi\Big(\min\big\{M(t,Bt,\alpha),1,1,M(Bt,t,\alpha),M(t,Bt,\alpha)\big\}\Big)=\varphi\big(M(t,Bt,\alpha)\big),$$

a contradiction. Thus $Bt = t$. Hence $t \in T(X)$ is a common fixed point of (B, T).

Case II. Similar argument arises if the pair (A, S) satisfy (CLCS) property in $B(X)$. In this case $t \in B(X)$ is a common fixed point of (B, T).

Case III. Similarly, if the pair (B, T) satisfy (CLCS) property in $S(X)$, then as in Theorem 2.2, the common convergence point t' of (B, T) for a sequence $\{y_n\} \subseteq X$, is a common fixed point of (A, S) and $t' \in S(X)$.

Case IV. Similarly, if the pair (B, T) satisfy (CLCS) property in $A(X)$, then as in Theorem 2.2, the common convergence point (here t') of (B, T) for a sequence $\{y_n\} \subseteq X$, is a common fixed point of (A, S) and $t' \in A(X)$.

Unifying Case I (or II), and Case III (or IV), we can say that $t = Bt = Tt$ and $t' = At' = Tt'$. We claim that $t = t'$. If not, then for each $\alpha > 0$, by (GV-1) and (GV-2) it implies that $0 < M(t, t', \alpha) < 1$. By (φ.2), we know that $\varphi\big(M(t, t', \alpha)\big) > M(t, t', \alpha)$. It follows from condition (ii) that

$$M(t,t',\alpha)=M(At,Bt',\alpha)\geq\varphi\Big(\min\big\{M(St,Tt',\alpha),M(At,St,\alpha),M(Bt',Bt',\alpha),$$
$$M(Bt',St,\alpha),M(At,Bt',\alpha)\big\}\Big)$$
$$=\varphi\Big(\min\big\{M(t,t',\alpha),1,1,M(t',t,\alpha),M(t,t',\alpha)\big\}\Big)=\varphi\big(M(t,t',\alpha)\big),$$

which is a contradiction. Thus $t = t'$. Hence t is a common fixed point of A, B, S and T. The uniqueness of common fixed point t is easy to show. For, if there be another common fixed point of A, B, S, T with $w \neq t$; then putting $x = t$, $y = w$ in condition (ii) and using the fact that $\varphi\big(M(t, w, \alpha)\big) > M(t, w, \alpha)$, we obtain a contradiction. This completes the proof. □

Putting $A = B = f$ and $S = T = g$ in Theorem 3.3, we obtain the following common fixed point result for one pair of weakly compatible mappings:

Corollary 3.4. *Let $(X, M, *)$ be a GV-fuzzy metric space and $f, g : X \to X$ be two self-mappings satisfying:*
(i) (f, g) *satisfy (CLCS) property in* $g(X)$,

(ii) $$M(fx,fy,\alpha)\geq\varphi\Big(\min\big\{M(gx,gy,\alpha),M(fx,gx,\alpha),M(fy,gy,\alpha),$$
$$M(fy,gx,\alpha),M(fx,gy,\alpha)\big\}\Big),$$

for all $x, y \in X$, where $\alpha > 0$ and $\varphi \in \Phi$. If (f, g) is weakly compatible then mappings f and g have a unique common fixed point in X.

Remark 3.5. Corollary 3.4 is same as the main Theorem 2.8 Sintunavarat and Kumam [9], because the (CLCS) property in $g(X)$ is identical to the (CLR) property.

If the (CLCS) property of one pair lies in the common range-subspace of the other pair, and vice-versa, we have the following theorem:

Theorem 3.6. *Let $(X, M, *)$ be a GV-fuzzy metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying:*
(i) (A, S) satisfy (CLCS) property in $T(X) \cap B(X)$, and (B, T) satisfy (CLCS) property in $S(X) \cap A(X)$,

$$(ii) \qquad M(Ax, By, \alpha) \geq \varphi\Big(\min\big\{M(Sx, Ty, \alpha), M(Ax, Sx, \alpha), M(By, Ty, \alpha), M(By, Sx, \alpha), M(Ax, Ty, \alpha)\big\}\Big),$$

for all $x, y \in X$, where $\alpha > 0$ and $\varphi \in \Phi$. If (A, S) and (B, T) are weakly compatible then mappings A, B, S and T have a unique common fixed point in X.

Putting $A = B = f$ and $S = T = g$ in Theorem 3.3, we obtain the following common fixed point result for one pair of weakly compatible mappings:

Corollary 3.7. *Let $(X, M, *)$ be a GV-fuzzy metric space and $f, g : X \to X$ be two self-mappings satisfying:*
(i) (f, g) satisfy (CLCS) property in $f(X) \cap g(X)$,

$$(ii) \qquad M(fx, fy, \alpha) \geq \varphi\Big(\min\big\{M(gx, gy, \alpha), M(fx, gx, \alpha), M(fy, gy, \alpha), M(fy, gx, \alpha), M(fx, gy, \alpha)\big\}\Big),$$

for all $x, y \in X$, where $\alpha > 0$ and $\varphi \in \Phi$. If (f, g) is weakly compatible then mappings f and g have a unique common fixed point in $fX \cap gX$.

Remark 3.8. This Corollary 3.7 is free from the condition of different types of (CLR) properties, because the (CLCS) property and the unique common fixed point lies in $fX \cap gX$. Thus property (CLR_f) and property (CLR_g) is unified here.

References

[1] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Numerical Functional Analysis and Optimization 3 (3) (2011) 243-253.

[2] S. Bhatt, S. Chaukiyal, R.C. Dimri, A common fixed point theorem for weakly compatible maps in complex valued metric spaces, Int. J. Math. Sci. Appl. 1 (3) (2011) 1385-1389.

[3] S. Sessa, On a weak commutative condition in fixed point consideration, Publ. Inst. Math. (Beograd) 32 (1982) 146-153.

[4] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (4) (1986) 771-779.

[5] G. Jungck, Common fixed points for non-continuous non-self mapping on a non-numeric spaces, Far East J. Math. Sci. 4 (2) (1996) 199-212.

[6] R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (1994) 436-440.

[7] M. Aamri, D.El. Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. (2002) 181-188.

[8] H.K. Pathak, R.R. Lopez, R.K. Verma, A common fixed point theorem of integral type using implicit relation, Nonlinear funct. Anal. Appl. 15 (1) (2009) 1-12.

[9] W. Sintunavarat, P. Kumam, Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space, Jour. Appl. Math. 2011 (2011) http://dx.doi.org/10.1155/2011/637958.

[10] W. Sintunavarat, Y.J. Cho, P. Kumam, Urisohn integral equations approach by common fixed points in complex valued metric spaces, Advances in Differ. Eqns. (2013) https://doi.org/10.1186/1687-1847-2013-49.

[11] W. Sintunavarat, P. Kumam, Common fixed point theorem for R-weakly commuting in fuzzy metric space, Annali. dell' Univ di Ferrara. 58 (2012) 389-406.

[12] M. Jain, S. Kumar, A common fixed point theorems in fuzzy metric spaces using the property (CLR_g), Thai Journal of Mathematics 14 (3) (2016) 627-636.

[13] G.V.R. Babu, P. Shubhasini, Coupled common fixed point theorems of Ciric type g-weak contractions with CLR_g property, J. Nonlinear Anal. Optim. Theory & Appl. 4 (1) (2013) 133-145.

[14] M. Imdad, B.D. Pant, S. Chouhan, Fixed point theorems in Menger spaces using the (CLR_{ST}) property and applications, J. Nonlinear Anal. Optim. Theory Appl. 3 (2) (2012) 225-237.

[15] M. Jain, K. Tas., S. Kumar, N. Gupta, Coupled fixed point theorems for a pair of weakly compatible mappings alongwith CLR_g property in fuzzy metric spaces, Jour. Anal. Math. (2012) Article ID 961210.

[16] M. Kumar, P. Kumar, S. Kumar, Some common fixed point theorems using (CLR_g) property in cone metric spaces, Advances in Fixed Point Theory 2 (3) (2012) 340-356.

[17] A. Rani, S. Kumar, N. Kumar, S.K.Garg, Common fixed point theorem for compatible and weakly compatible maps on G-metric spaces, Applied Math. 3 (10) (2012) 1128-1134.

[18] S.L. Singh, B.D. Pant, S. Chouhan, Fixed point theorems in non-archimedean Menger PM-spaces a weak commutative condition in fixed point consideration, J. Nonlinear Anal. Optim. Theory Appl. 3 (2) (2012) 153-160.

[19] W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, J. Inequal. Appl. 84 (2012) https://doi.org/10.1186/1029-242X-2012-84.

[20] R.K. Verma, H.K. Pathak, Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai. J. Math. 11 (2) (2013) 347-355.

[21] R.K. Verma, Fixed point theorems using (CLCS) property in complex valued b-metric spaces, Facta Univer. (Nis), Ser. Math. inf. 32 (3) (2017) 269-292.

[22] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (3) (1997) 365-368.

[23] I. Kramosil, J. Michalek, Fuzzy metrices and statistical metric spaces, Kybernetika 11 (5) (1975) 326-334.

[24] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 313-334.

[25] W. Sintunavarat, P. Kumam, Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space, Jour. Appl. Math. (2011) http://dx.doi.org/10.1155/2011/637958.

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